Adjoint Functors and Heteromorphisms

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Abstract

Category theory has foundational importance because it provides conceptual lenses to characterize what is important in mathematics. Originally the main lenses were universal mapping properties and natural transformations. In recent decades, the notion of adjoint functors has moved to center-stage as category theory's primary tool to characterize what is important in mathematics. Our focus here is to present a theory of adjoint functors. The basis for the theory is laid by first showing that the object-to-object "heteromorphisms" between the objects of different categories (e.g., insertion of generators as a set to group map) can be rigorously treated within category theory. The heteromorphic theory shows that all adjunctions arise from the birepresentations of the heteromorphisms between the objects of different categories.

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1 The Importance of Adjoints

Category theory is of foundational importance in mathematics but it is not "foundational" in the sense normally claimed by set theory. It does not try to provide some basic objects (e.g., sets) from which other mathematical objects can be constructed. Instead, the foundational role of category theory lies in providing conceptual lenses to characterize what is universal and natural in mathematics.[1] Two of the most important concepts are universal mapping properties and natural transformations. These two concepts are combined in the notion of adjoint functors. In recent decades, adjoint functors have emerged as the principal lens through which category theory plays out its foundational role of characterizing what is important in mathematics.[2]

The developers of category theory, Saunders MacLane and Samuel Eilenberg, famously said that categories were defined in order to define functors, and functors were defined in order to define *natural* transformations. Their original paper [6] was entitled not "General Theory of Categories" but *General Theory of Natural Equivalences*. Adjoints were defined more than a decade later by Daniel Kan [14] but the realization of their foundational importance has steadily increased over time [18, 16]. Now it would perhaps not be too much of an exaggeration to see categories, functors, and natural transformations as the prelude to defining adjoint functors. As Steven Awodey put it in his recent text:

> The notion of adjoint functor applies everything that we've learned up to now to unify and subsume all the different universal mapping properties that we have encountered, from free groups to limits to exponentials. But more importantly, it also captures an important mathematical phenomenon that is invisible without the lens of category theory. Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics. [2]

Other category theorists have given similar testimonials.

> To some, including this writer, adjunction is the most important concept in category theory. [24, p. 6]

> The isolation and explication of the notion of *adjointness* is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas.[13, p. 438]

[1] For summary statements, see [1], [17], or [9].

[2] Some familiarity with basic category theory is assumed but the paper is written to be accessible to a broader audience than specialists. Whenever possible, I follow MacLane [20] on notation and terminology.

> Nowadays, every user of category theory agrees that [adjunction] is the concept which justifies the fundamental position of the subject in mathematics. [23, p. 367]

Given the importance of adjoint functors in category theory and in mathematics as a whole, it would seem worthwhile to further investigate the concept of an adjunction. Where do adjoint functors come from; how do they arise? In this paper we will present a theory of adjoint functors to address these questions [10, 11]. One might well ask: "Where could such a theory come from?"

Category theory groups together in *categories* the mathematical objects with some common structure (e.g., sets, partially ordered sets, groups, rings, and so forth) and the appropriate morphisms between such objects. Since the morphisms are between objects of similar structure, they are ordinarily called "homomorphisms" or just "morphisms" for short. But there have always been other morphisms which occur in mathematical practice that are between objects with different structures (i.e., in different categories) such as the insertion-of-generators map from a set to the free group on that set. In order to contrast these morphisms with the homomorphisms between objects within a category, they might be called *heteromorphisms* or, more colorfully, *chimera morphisms* (since they have a tail in one category and a head in another category). The usual machinery of category theory (bifunctors, in particular) can be adapted to give a rigorous treatment of heteromorphisms.

With a precise notion of heteromorphisms in hand, it can then be seen that adjoint functors arise as the functors giving the representations, using homomorphisms *within* each category, of the heteromorphisms *between* two categories. And, conversely, given a pair of adjoint functors, then heteromorphisms can be defined between (isomorphic copies of) the two categories so that the adjoints arise out of the representations of those heteromorphisms. Hence this heteromorphic theory shows where adjoints "come from" or "how they arise." It would seem that this theory showing the origin of adjoint functors was not developed in the conventional treatment of category theory since heteromorphisms, although present in mathematical practice, are not part of the usual machinery of category theory.

2 Overview of the Theory of Adjoints

The cross-category object-to-object morphisms $c : x \Rightarrow a$, called *heteromorphisms* (*hets* for short) or *chimera morphisms,* will be indicated by double arrows ($\Rightarrow$) rather than single arrows ($\rightarrow$). The first question is how do heteromorphisms compose with one another? But that is not necessary. Chimera do not need to 'mate' with other chimera to form a 'species' or category; they only need to mate with the intra-category morphisms on each side to form other chimera. The appropriate mathematical machinery to describe that is the generalization of a group acting on a set to a generalized monoid or category acting on a set (where each element of the set has a "domain" and a "codomain" to determine when composition is defined). In this case, it is two categories acting on a set, one on the left and one on the right. Given a chimera morphism $c : x \Rightarrow a$ from an object in a category $\mathbf{X}$ to an object in a category $\mathbf{A}$ and morphisms $h : x' \to x$ in $\mathbf{X}$ and $k : a \to a'$ in $\mathbf{A}$, the composition $ch : x' \to x \Rightarrow a$ is another chimera $x' \Rightarrow a$ and the composition $kc : x \Rightarrow a \to a'$ is another chimera $x \Rightarrow a'$ with the usual identity, composition, and associativity properties. Such an action of two categories acting on a set on the left and on the right is exactly described by a bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ where $\mathrm{Het}(x, a) = \{x \Rightarrow a\}$ and where $\mathbf{Set}$ is the category of sets and set functions. Thus the natural machinery to treat object-to-object chimera morphisms *between* categories are het-bifunctors $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ that generalize the hom-bifunctors $\mathrm{Hom} : \mathbf{X}^{op} \times \mathbf{X} \to \mathbf{Set}$ used to treat object-to-object morphisms *within* a category.

How might the categorical properties of the heteromorphisms be expressed using homomorphisms? Represent the het-bifunctors using hom-functors on the left and on the right (see any category theory text such as [20] for Alexander Grothendieck's notion of a *representable functor*). Any bifunctor $\mathcal{D} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ is *represented on the left*[3] if for each x in $\mathbf{X}$ there is an object Fx in $\mathbf{A}$ and an isomorphism $\mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathcal{D}(x, a)$ natural in a. It is a standard result that the assignment $x \mapsto Fx$ extends to a functor F and that the isomorphism is also natural in x. Similarly, $\mathcal{D}$ is *represented on the right* if for each a there is an object Ga in $\mathbf{X}$ and an isomorphism $\mathcal{D}(x, a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga)$ natural in x. And similarly, the assignment $a \mapsto Ga$ extends to a functor G and that the isomorphism is also natural in a.

If a het-bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ is represented on both the left and the right, then we have two functors $F : \mathbf{X} \to \mathbf{A}$ and $G : \mathbf{A} \to \mathbf{X}$ and the isomorphisms are natural in x and in a:

$$\mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Het}(x, a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga).$$

It only remains to drop out the middle term $\mathrm{Het}(x, a)$ to arrive at the *pas de deux* of the 'official' definition of a pair of adjoint functors which does not mention heteromorphisms.

While a birepresentation of a het-bifunctor gives rise to an adjunction, do all adjunctions arise in this manner? To round out the theory, we give an "adjunction representation theorem" which shows how, given any adjunction $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$, heteromorphisms can be defined between (isomorphic copies of) the categories $\mathbf{X}$ and $\mathbf{A}$ so that (isomorphic copies of) the adjoints arise from the representations on the left and right of the het-bifunctor. Given any set function $f : X \to A$ from the set X to a set A, the graph $\mathrm{graph}\,(f) = \{(x, f(x)) : x \in X\} \subseteq X \times A$ of the function is set-isomorphic to the domain of the function X. The embedding $x \longmapsto$

[3] This terminology "represented on the left" or "on the right" is used to agree with the terminology for left and right adjoints.

$(x, f(x))$ maps X to the set-isomorphic copy of X, namely $\mathrm{graph}(f) \subseteq X \times A$. That isomorphism generalizes to categories and to functors between categories. Given any functor $F : \mathbf{X} \to \mathbf{A}$, the domain category $\mathbf{X}$ is embedded in the product category $\mathbf{X} \times \mathbf{A}$ by the assignment $x \mapsto (x, Fx)$ to obtain the isomorphic copy $\widehat{\mathbf{X}}$ (which can be considered as the graph of the functor F). Given any other functor $G : \mathbf{A} \to \mathbf{X}$, the domain category $\mathbf{A}$ is embedded in the product category by $a \mapsto (Ga, a)$ to yield the isomorphic copy $\widehat{\mathbf{A}}$ (the graph of the functor G). If the two functors are adjoints, then the properties of the adjunction can be nicely expressed by the commutativity within the one category $\mathbf{X} \times \mathbf{A}$ of "hom-pair adjunctive squares" where morphisms are pairs of homomorphisms (in contrast to a "het adjunctive square" defined later).

$$
\begin{array}{ccccc}
 & (x, Fx) & \stackrel{(f,Ff)}{\longrightarrow} & (Ga, FGa) & \\
(\eta_x, 1_{Fx}) & \downarrow & \searrow^{(f,g)} & \downarrow & (1_{Ga}, \varepsilon_a) \\
 & (GFx, Fx) & \stackrel{(Gg,g)}{\longrightarrow} & (Ga, a) &
\end{array}
$$

Hom-pair adjunctive square

The main diagonal (f, g) in a commutative hom-pair adjunctive square pairs together maps that are images of one another in the adjunction isomorphism $\mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga)$. If $f \in \mathrm{Hom}_{\mathbf{X}}(x, Ga)$, $g = f^* \in \mathrm{Hom}_{\mathbf{A}}(Fx, a)$ is the corresponding homomorphism on the other side of the isomorphism between hom-sets called its *adjoint transpose* (or later "adjoint correlate") and similarly $f = g^*$. Since the maps on top are always in $\widehat{\mathbf{X}}$ and the maps on the bottom are in $\widehat{\mathbf{A}}$, the main diagonal pairs of maps (including the vertical maps)—which are ordinary morphisms in the product category—have all the categorical properties of heteromorphisms from objects in $\mathbf{X} \cong \widehat{\mathbf{X}}$ to objects in $\mathbf{A} \cong \widehat{\mathbf{A}}$. Hence the heteromorphisms are abstractly defined as the *pairs of adjoint transposes*, $\mathrm{Het}(x, a) = \{(x, Fx) \stackrel{(f,f^*)}{\longrightarrow} (Ga, a)\}$, and the adjunction representation theorem is that (isomorphic copies of) the original adjoints F and G arise from the representations on the left and right of this het-bifunctor.

Heteromorphisms are formally treated using bifunctors of the form $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$. Such bifunctors and generalizations replacing $\mathbf{Set}$ by other categories have been studied by the Australian school under the name of *profunctors* [15], by the French school under the name of *distributors* [3], and by William Lawvere under the name of *bimodules* [19]. However, the guiding interpretation has been interestingly different. "Roughly speaking, a distributor is to a functor what a relation is to a mapping" [4, p. 308] (and hence the name "profunctor" in the Australian school). For instance, if $\mathbf{Set}$ was replaced by $\mathbf{2}$, then the bifunctor would just be the characteristic function of a relation from $\mathbf{X}$ to $\mathbf{A}$. Hence in the general context of enriched category theory, a "bimodule" $Y^{op} \otimes X \stackrel{\varphi}{\longrightarrow} \mathcal{V}$ would be interpreted as a "$\mathcal{V}$-valued relation" and an element of $\varphi(y, x)$ would be interpreted as the "truth-value of the φ-relatedness of y to x" [19, p. 158 (p. 28 of reprint)].

The subsequent development of profunctors-distributors-bimodules has been along the lines suggested by that guiding interpretation. For instance, composition is defined between distributors as "relational" generalizations of functors to define a category of distributors in analogy with composition defined between relations as generalizations of functions which allows the definition of a category of relations [4, Chapter 7].

The heteromorphic interpretation of the bifunctors $\mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ is rather different. Each such bifunctor is taken as defining how the chimeras in $\mathrm{Het}(x, a)$ compose with morphisms in $\mathbf{A}$ on one side and with morphisms in $\mathbf{X}$ on the other side to form other chimeras. This provides the formal treatment of the heteromorphisms that have always existed in mathematical practice. The principal novelty here is the use of the chimera morphism interpretation of these bifunctors to carry out a whole program of interpretation for adjunctions, i.e., a *theory* of adjoint functors. In the concrete examples, heteromorphisms have to be "found" as is done in the broad classes of examples treated here. However, in general, the adjunction representation theorem shows how abstract heteromorphisms (pairs of adjoint transposes in the product category $\mathbf{X} \times \mathbf{A}$) can always be found so that any adjunction arises (up to isomorphism) out of the representations on the left and right of the het-bifunctor of such heteromorphisms. Following this summary, we now turn to a slower development of the theory along with examples.

3 The Heteromorphic Theory of Adjoints

3.1 Definition and Directionality of Adjoints

There are many equivalent definitions of adjoint functors [20], but the most 'official' one seems to be the one using a natural isomorphism of hom-sets. Let $\mathbf{X}$ and $\mathbf{A}$ be categories and $F : \mathbf{X} \to \mathbf{A}$ and $G : \mathbf{A} \to \mathbf{X}$ functors between them. Then F and G are said to be a pair of *adjoint functors* or an *adjunction*, written $F \dashv G$, if for any x in $\mathbf{X}$ and a in $\mathbf{A}$, there is an isomorphism ϕ natural in x and in a:

$$\phi_{x,a} : \mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga).$$

With this standard way of writing the isomorphism of hom-sets, the functor F on the left is called the *left adjoint* and the functor G on the right is the *right adjoint*. Maps associated with each other by the adjunction isomorphism ("adjoint transposes" of one another) are indicated by an asterisk so if $g : Fx \to a$ then $g^* : x \to Ga$ is the associated map $\phi_{x,a}(g) = g^*$ and similarly if $f : x \to Ga$ then $\phi_{x,a}^{-1}(f) = f^* : Fx \to a$ is the associated map.

In much of the literature, adjoints are presented in a seemingly symmetrical fashion so that there appears to be no directionality of the adjoints between the categories $\mathbf{X}$ and $\mathbf{A}$. But there is a directionality and it is important in understanding adjoints. Both the maps that appear in the adjunction isomorphism,

$Fx \to a$ and $x \to Ga$, go from the "x-thing" (i.e., either x or the image Fx) to the "a-thing" (either the image Ga or a itself), so we see a direction emerging from $\mathbf{X}$ to $\mathbf{A}$. That direction of an adjunction is the direction of the left adjoint (which goes from $\mathbf{X}$ to $\mathbf{A}$). Then $\mathbf{X}$ might called the *sending* category and $\mathbf{A}$ the *receiving* category.[4]

In the theory of adjoints presented here, the directionality of adjoints results from being representations of heteromorphisms which have that directionality. Such morphisms can exhibited in concrete examples of adjoints (see the later examples). To abstractly define chimera morphisms or heteromorphisms that work for all adjunctions, we turn to the presentation of adjoints using adjunctive squares.

3.2 Embedding Adjunctions in a Product Category

Our approach to a theory of adjoints uses a certain "adjunctive square" diagram that is in the product category $\mathbf{X} \times \mathbf{A}$ associated with an adjunction $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$. With each object x in the category $\mathbf{X}$, we associate the element $\widehat{x} = (x, Fx)$ in the product category $\mathbf{X} \times \mathbf{A}$ so that Ga would have associated with it $\widehat{Ga} = (Ga, FGa)$. With each morphism in $\mathbf{X}$ with the form $h : x' \to x$, we associate the morphism $\widehat{h} = (h, Fh) : \widehat{x'} = (x', Fx') \to \widehat{x} = (x, Fx)$ in the product category $\mathbf{X} \times \mathbf{A}$ (maps compose and diagrams commute component-wise). Thus the mapping of x to (x, Fx) extends to an embedding $(1_{\mathbf{X}}, F) : \mathbf{X} \to \mathbf{X} \times \mathbf{A}$ whose image $\widehat{\mathbf{X}}$ (the graph of F) is isomorphic with $\mathbf{X}$.

With each object a in the category $\mathbf{A}$, we associate the element $\widehat{a} = (Ga, a)$ in the product category $\mathbf{X} \times \mathbf{A}$ so that Fx would have associated with it $\widehat{Fx} = (GFx, Fx)$. With each morphism in $\mathbf{A}$ with the form $k : a \to a'$,we associate the morphism $\widehat{k} = (Gk, k) : (Ga, a) \to (Ga', a')$ in the product category $\mathbf{X} \times \mathbf{A}$. The mapping of a to (Ga, a) extends to an embedding $(G, 1_{\mathbf{A}}) : \mathbf{A} \to \mathbf{X} \times \mathbf{A}$ whose image $\widehat{\mathbf{A}}$ (the graph of G) is isomorphic to $\mathbf{A}$.

The adjoint transpose of the identity map $1_{Fx} \in \mathrm{Hom}_A(Fx, Fx)$ is the unit morphism $\eta_x : x \to GFx \in \mathrm{Hom}_X(x, GFx)$. That pair $(\eta_x, 1_{Fx}) : (x, Fx) \to (GFx, Fx)$ of adjoint transposes is the left vertical 'heteromorphism' in the hom-pairs adjunctive square diagram. We use the raised-eyebrow quotes on 'heteromorphism' since it is a perfectly ordinary homomorphism in the product category $\mathbf{X} \times \mathbf{A}$ which plays the role of a heteromorphism from $\widehat{\mathbf{X}}$, the isomorphic copy of $\mathbf{X}$, to $\widehat{\mathbf{A}}$, the isomorphic copy of $\mathbf{A}$, both subcategories of $\mathbf{X} \times \mathbf{A}$. The adjoint transpose of the identity map $1_{Ga} \in \mathrm{Hom}_X(Ga, Ga)$ is the counit morphism $\varepsilon_a : FGa \to a \in \mathrm{Hom}_A(FGa, a)$. That pair $(1_G, \varepsilon_a) : (Ga, FGa) \to (Ga, a)$ of adjoint transposes is the right vertical 'heteromorphism' in the adjunctive square diagram.

[4] Sometimes adjunctions are written with this direction as in the notation $\langle F, G, \phi \rangle : \mathbf{X} \rightharpoonup \mathbf{A}$ (MacLane [20, p.78]). This also allows the composition of adjoints to be defined in a straightforward manner (MacLane [20, p.101]).

These various parts can then be collected together in the (hom-pair adjunctive square diagram of the representation theorem.

$$\begin{array}{ccccc} & (x,Fx) & \overset{(f,Ff)}{\longrightarrow} & (Ga,FGa) & \\ (\eta_x,1_{Fx}) & \downarrow & \searrow^{(f,g)} & \downarrow & (1_{Ga},\varepsilon_a) \\ & (GFx,Fx) & \overset{(Gg,g)}{\longrightarrow} & (Ga,a) & \end{array}$$

Hom-pair Adjunctive Square Diagram

The adjunctive square diagram conveniently represents the properties of an adjunction in the format of commutative squares. The map on the top is in $\widehat{\mathbf{X}}$ and the map on the bottom is in $\widehat{\mathbf{A}}$ and the vertical maps as well as the main diagonal (f,g) in a commutative adjunctive square are morphisms from $\widehat{\mathbf{X}}$-objects to $\widehat{\mathbf{A}}$-objects.

Given $f : x \to Ga$, the rest of the diagram is determined by the requirement that the square commutes. Commutativity in the second component uniquely determines that $g = g1_{Fx} = \varepsilon_a Ff$ so $g = f^* = \varepsilon_a Ff$ is the map associated with f in the adjunction isomorphism. Commutativity in the first component is the universal mapping property (UMP) factorization of any given $f : x \to Ga$ through the unit $x \xrightarrow{\eta_x} GFx \xrightarrow{Gf^*} Ga = x \xrightarrow{f} Ga$ which is often pictured as:

$$\begin{array}{cccc} & x & & \\ \eta_x & \downarrow & \searrow^{f} & \\ & GFx & \overset{Gf^*}{\longrightarrow} & Ga \\ & Fx & \overset{\exists! f^*}{\longrightarrow} & a \end{array}$$

Hom-pair adjunctive square south-west of the diagonal.

Similarly, if we were given $g : Fx \to a$, then commutativity in the first component implies that $f = 1_{Ga}f = Gg\eta_x = g^*$. And commutativity in the second component is the UMP factorization of any given $g : Fx \to a$ through the counit $Fx \xrightarrow{Fg^*} FGa \xrightarrow{\varepsilon_a} a = Fx \xrightarrow{g} a$ which is usually pictured as:

$$\begin{array}{cccc} x & \overset{\exists! g^*}{\longrightarrow} & Ga & \\ Fx & \overset{Fg^*}{\longrightarrow} & FGa & \\ & \searrow^{g} & \downarrow & \varepsilon_a \\ & & a & \end{array}$$

Hom-pair adjunctive square north-east of the diagonal.

Splicing together the two triangles along the diagonals so that the two diagonals form the hom-pair (f,g) (and supplying the identity maps 1_{Fx} and 1_{Ga} as required to form the left and right vertical hom-pairs), the hom-pair adjunctive square is put back together.

3.3 Heteromorphisms and Het-bifunctors

Heteromorphisms (in contrast to homomorphisms) are like mongrels or chimeras that do not fit into either of the two categories. Since the inter-category heteromorphisms are not morphisms in either of the categories, what can we say about them? The one thing we can reasonably say is that heteromorphisms can be precomposed or postcomposed with morphisms within the categories (i.e., intra-category morphisms) to obtain other heteromorphisms.[5] This is easily formalized using bifunctors similar to the hom-bifunctors $\mathrm{Hom}(x, y)$ in homomorphisms within a category. Using the sets-to-groups example to guide intuition, one might think of $\mathrm{Het}(x, a) = \{x \overset{c}{\Rightarrow} a\}$ as the set of set functions from a set x to a group a. For any $\mathbf{A}$-morphism $k : a \to a'$ and any chimera morphism $x \overset{c}{\Rightarrow} a$, intuitively there is a composite chimera morphism $x \overset{c}{\Rightarrow} a \xrightarrow{k} a' = x \overset{kc}{\Rightarrow} a'$, i.e., k induces a map $\mathrm{Het}(x, k) : \mathrm{Het}(x, a) \to \mathrm{Het}(x, a')$. For any $\mathbf{X}$-morphism $h : x' \to x$ and chimera morphism $x \overset{c}{\Rightarrow} a$, intuitively there is the composite chimera morphism $x' \xrightarrow{h} x \overset{c}{\Rightarrow} a = x' \overset{ch}{\Rightarrow} a$, i.e., h induces a map $\mathrm{Het}(h, a) : \mathrm{Het}(x, a) \to \mathrm{Het}(x', a)$ (note the reversal of direction). The induced maps would respect identity and composite morphisms in each category. Moreover, composition is associative in the sense that $(kc)h = k(ch)$. This means that the assignments of sets of chimera morphisms $\mathrm{Het}(x, a) = \{x \overset{c}{\Rightarrow} a\}$ and the induced maps between them constitute a *bifunctor* $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ (contravariant in the first variable and covariant in the second).

With this motivation, we may turn around and define *heteromorphisms* from $\mathbf{X}$-objects to $\mathbf{A}$-objects as the elements in the values of a bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$. This would be analogous to defining the homomorphisms in $\mathbf{X}$ as the elements in the values of a given hom-bifunctor $\mathrm{Hom}_{\mathbf{X}} : \mathbf{X}^{op} \times \mathbf{X} \to \mathbf{Set}$ and similarly for $\mathrm{Hom}_{\mathbf{A}} : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Set}$.

With heteromorphisms rigorously described using het-bifunctors, we can use Grothendieck's notion of a representable functor to show that an adjunction arises from a het-bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ that is "birepresentable" in the sense of being representable on both the left and right.

Given any bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$, it is *representable on the left* if for each $\mathbf{X}$-object x, there is an $\mathbf{A}$-object Fx that represents the functor $\mathrm{Het}(x, -)$, i.e., there is an isomorphism $\psi_{x,a} : \mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Het}(x, a)$ natural in a. For each x, let h_x be the image of the identity on Fx, i.e., $\psi_{x,Fx}(1_{Fx}) = h_x \in \mathrm{Het}(x, Fx)$. We first show that h_x is a universal element for the functor $\mathrm{Het}(x, -)$ and then use that to complete the construction of F as a functor. For any $c \in \mathrm{Het}(x, a)$, let $g(c) = \psi_{x,a}^{-1}(c) : Fx \to a$. Then naturality in a means that the following diagram commutes.

[5] The chimera genes are dominant in these mongrel matings. While mules cannot mate with mules, it is 'as if' mules could mate with either horses or donkeys to produce other mules.

$$\begin{array}{ccccc} & \mathrm{Hom}_{\mathbf{A}}(Fx,Fx) & \cong & \mathrm{Het}(x,Fx) & \\ {\scriptstyle \mathrm{Hom}(Fx,g(c))} & \downarrow & & \downarrow & {\scriptstyle \mathrm{Het}(x,g(c))} \\ & \mathrm{Hom}_{\mathbf{A}}(Fx,a) & \cong & \mathrm{Het}(x,a) & \end{array}$$

Het (x, a) representable on the left

Chasing 1_{Fx} around the diagram yields that $c = \mathrm{Het}(x, g(c))(h_x)$ which can be written as $c = g(c)h_x$. Since the horizontal maps are isomorphisms, $g(c)$ is the unique map $g : Fx \to a$ such that $c = gh_x$. Then (Fx, h_x) is a *universal element* (in MacLane's sense [20, p. 57]) for the functor $\mathrm{Het}(x, -)$ or equivalently $1 \xrightarrow{h_x} \mathrm{Het}(x, Fx)$ is a *universal arrow* [20, p. 58] from 1 (the one point set) to $\mathrm{Het}(x, -)$. Then for any $\mathbf{X}$-morphism $j : x \to x'$, $Fj : Fx \to Fx'$ is the unique $\mathbf{A}$-morphism such that $\mathrm{Het}(x, Fj)$ fills in the right vertical arrow in the following diagram.

$$\begin{array}{ccccc} & 1 & \xrightarrow{h_x} & \mathrm{Het}(x,Fx) & \\ {\scriptstyle h_{x'}} & \downarrow & & \downarrow & {\scriptstyle \mathrm{Het}(x,Fj)} \\ & \mathrm{Het}(x',Fx') & \xrightarrow{\mathrm{Het}(j,Fx')} & \mathrm{Het}(x,Fx') & \end{array}$$

It is easily checked that such a definition of $Fj : Fx \to Fx'$ preserves identities and composition using the functoriality of $\mathrm{Het}(x, -)$ so we have a functor $F : \mathbf{X} \to \mathbf{A}$. It is a further standard result that the isomorphism is also natural in x (e.g., [20, p. 81] or the "parameter theorem" [21, p. 525]).

Given a bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$, it is *representable on the right* if for each $\mathbf{A}$-object a, there is an $\mathbf{X}$-object Ga that represents the functor $\mathrm{Het}(-, a)$, i.e., there is an isomorphism $\varphi_{x,a} : \mathrm{Het}(x, a) \cong \mathrm{Hom}_X(x, Ga)$ natural in x. For each a, let e_a be the inverse image of the identity on Ga, i.e., $\varphi^{-1}_{Ga,a}(1_{Ga}) = e_a \in \mathrm{Het}(Ga, a)$. For any $c \in \mathrm{Het}(x, a)$, let $f(c) = \varphi_{x,a}(c) : x \to Ga$. Then naturality in x means that the following diagram commutes.

$$\begin{array}{ccccc} & \mathrm{Het}(Ga,a) & \cong & \mathrm{Hom}_{\mathbf{X}}(Ga,Ga) & \\ {\scriptstyle \mathrm{Het}(f(c),a)} & \downarrow & & \downarrow & {\scriptstyle \mathrm{Hom}(f(c),Ga)} \\ & \mathrm{Het}(x,a) & \cong & \mathrm{Hom}_{\mathbf{X}}(x,Ga) & \end{array}$$

Het (x, a) representable on the right

Chasing 1_{Ga} around the diagram yields that $c = \mathrm{Het}(f(c), a)(e_a) = e_a f(c)$ so (Ga, e_a) is a universal element for the functor $\mathrm{Het}(-, a)$ and that $1 \xrightarrow{e_a} \mathrm{Het}(Ga, a)$ is a universal arrow from 1 to $\mathrm{Het}(-, a)$. Then for any $\mathbf{A}$-morphism $k : a' \to a$, $Gk : Ga' \to Ga$ is the unique $\mathbf{X}$-morphism such that $\mathrm{Het}(Gk, a)$ fills in the right vertical arrow in the following diagram.

$$\begin{array}{ccccc} & 1 & \xrightarrow{e_a} & \mathrm{Het}(Ga,a) & \\ {\scriptstyle e_{a'}} & \downarrow & & \downarrow & {\scriptstyle \mathrm{Het}(Gk,a)} \\ & \mathrm{Het}(Ga',a') & \xrightarrow{\mathrm{Het}(Ga',k)} & \mathrm{Het}(Ga',a) & \end{array}$$

In a similar manner, it is easily checked that the functoriality of G follows from the functoriality of $\mathrm{Het}(-,a)$. Thus we have a functor $G : \mathbf{A} \to \mathbf{X}$ such that Ga represents the functor $\mathrm{Het}(-,a)$, i.e., there is a natural isomorphism $\varphi_{x,a} : \mathrm{Het}(x,a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga)$ natural in x. And in a similar manner, it can be shown that the isomorphism is natural in both variables.

Thus given a bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ representable on *both* sides, we have the adjunction natural isomorphisms:

$$\mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Het}(x,a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga).$$

Starting with $c \in \mathrm{Het}(x,a)$, the corresponding $f(c) \in \mathrm{Hom}_{\mathbf{X}}(x, Ga)$ and $g(c) \in \mathrm{Hom}_{\mathbf{A}}(Fx, a)$ are called *adjoint correlates* of one another. Starting with $1_{Fx} \in \mathrm{Hom}_{\mathbf{A}}(Fx, Fx)$, its adjoint correlates are the *het unit* $h_x \in \mathrm{Het}(x, Fx)$ and the ordinary unit $\eta_x \in \mathrm{Hom}_{\mathbf{X}}(x, GFx)$ where this usual unit η_x might also be called the "hom unit" to distinguish it from its het correlate. Starting with $1_{Ga} \in \mathrm{Hom}_{\mathbf{X}}(Ga, Ga)$, its adjoint correlates are the *het counit* $e_a \in \mathrm{Het}(Ga, a)$ and the usual (hom) counit $\varepsilon_a \in \mathrm{Hom}_{\mathbf{A}}(FGa, a)$. Starting with any $c \in \mathrm{Het}(x,a)$, the two factorizations $g(c)h_x = c = e_a f(c)$ combine to give what we will later call the "het adjunctive square" with c as the main diagonal [as opposed to the hom-pair adjunctive square previously constructed which had $(f(c), g(c))$ as the main diagonal].

There are cases (see below) where the het-bifunctor is only representable on the left $\mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Het}(x,a)$ or on the right $\mathrm{Het}(x,a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga)$, and in that case, it would make perfectly good sense to respectively take $F : \mathbf{X} \to \mathbf{A}$ as a *left half-adjunction* or $G : \mathbf{A} \to \mathbf{X}$ as a *right half-adjunction*. A half-adjunction is the simplest expression of a universal mapping property, and, of course, a left half-adjunction plus a right half-adjunction equals an adjunction.[6]

3.4 Adjunction Representation Theorem

Adjunctions may be and usually are presented without any thought to any underlying heteromorphisms. However, given any adjunction, there is always an "abstract" associated het-bifunctor given by the main diagonal maps in the commutative hom-pair adjunctive squares:

$$\mathrm{Het}(\widehat{x}, \widehat{a}) = \{\widehat{x} = (x, Fx) \xrightarrow{(f,f^*)} (Ga, a) = \widehat{a}\}$$

Het-bifunctor for any adjunction from hom-pair adjunctive squares.

The diagonal maps are closed under precomposition with maps from $\widehat{\mathbf{X}}$ and postcomposition with maps from $\widehat{\mathbf{A}}$. Associativity follows from the associativity in the ambient category $\mathbf{X} \times \mathbf{A}$.

[6] When there is a full adjunction, often only one half-adjunction is important while the other half-adjunction is a rather trivial piece of conceptual bookkeeping to round out the whole adjunction. In the later examples, for the free-group/underlying-set adjunction, the left half-adjunction carries the weight while in the Cartesian product adjunction, it is the right half-adjunction.

The representation is accomplished essentially by putting a $\widehat{hat}$ on objects and morphisms embedded in $\mathbf{X} \times \mathbf{A}$. The categories $\mathbf{X}$ and $\mathbf{A}$ are represented respectively by the subcategory $\widehat{\mathbf{X}}$ with objects $\widehat{x} = (x, Fx)$ and morphisms $\widehat{f} = (f, Ff)$ and by the subcategory $\widehat{\mathbf{A}}$ with objects $\widehat{a} = (Ga, a)$ and morphisms $\widehat{g} = (Gg, g)$. The *twist functor* $(F, G) : \mathbf{X} \times \mathbf{A} \to \mathbf{X} \times \mathbf{A}$ defined by $(F, G)(x, a) = (Ga, Fx)$ (and similarly for morphisms) restricted to $\widehat{\mathbf{X}} \cong \mathbf{X}$ is $\widehat{F}$ which has the action of F, i.e., $\widehat{F}\widehat{x} = (F, G)(x, Fx) = (GFx, Fx) = \widehat{Fx} \in \widehat{\mathbf{A}}$ and similarly for morphisms. The twist functor restricted to $\widehat{\mathbf{A}} \cong \mathbf{A}$ yields $\widehat{G}$ which has the action of G, i.e., $\widehat{G}\widehat{a} = (F, G)(Ga, a) = (Ga, FGa) = \widehat{Ga} \in \widehat{\mathbf{X}}$ and similarly for morphisms. These functors provide representations on the left and right of the abstract het-bifunctor $\mathrm{Het}(\widehat{x}, \widehat{a}) = \{\widehat{x} \xrightarrow{(f,f^*)} \widehat{a}\}$, i.e., the natural isomorphism

$$\mathrm{Hom}_{\widehat{\mathbf{A}}}(\widehat{F}\widehat{x}, \widehat{a}) \cong \mathrm{Het}(\widehat{x}, \widehat{a}) \cong \mathrm{Hom}_{\widehat{\mathbf{X}}}(\widehat{x}, \widehat{G}\widehat{a}).$$

This birepresentation of the abstract het-bifunctor gives an isomorphic copy of the original adjunction between the isomorphic copies $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{A}}$ of the original categories. This hom-pair representation is summarized in the following:

Adjunction Representation Theorem: Every adjunction $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$ can be represented (up to isomorphism) as arising from the left and right representing universals of a het-bifunctor $\mathrm{Het} : \widehat{\mathbf{X}}^{op} \times \widehat{\mathbf{A}} \to \mathbf{Set}$ giving the heteromorphisms from the objects in a category $\widehat{\mathbf{X}} \cong \mathbf{X}$ to the objects in a category $\widehat{\mathbf{A}} \cong \mathbf{A}$.[7]

3.5 Het Adjunctive Squares

We previously used the representations of $\mathrm{Het}(x, a)$ to pick out universal elements, the het unit $h_x \in \mathrm{Het}(x, Fx)$ and the het counit $e_a \in \mathrm{Het}(Ga, a)$, as the respective adjoint correlates of 1_{Fx} and 1_{Ga} under the isomorphisms. We showed that from the birepresentation of $\mathrm{Het}(x, a)$, any chimera morphism $x \overset{c}{\Rightarrow} a$ in $\mathrm{Het}(x, a)$ would have two factorizations: $g(c)h_x = c = e_a f(c)$. This two factorizations are spliced together along the main diagonal $c : x \Rightarrow a$ to form the het (commutative) adjunctive square.

[7] In a historical note [20, p. 103], MacLane noted that Bourbaki "missed" the notion of an adjunction because Bourbaki focused on the left representations of bifunctors $W : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Sets}$. MacLane remarks that given $G : \mathbf{A} \to \mathbf{X}$, they should have taken $W(x, a) = Hom_{\mathbf{X}}(x, Ga)$ and then focused on "the symmetry of the adjunction problem" to find Fx so that $Hom_{\mathbf{A}}(Fx, a) \cong Hom_{\mathbf{X}}(x, Ga)$. But MacLane thus missed the completely symmetrical adjunction problem which is: given $W(x, a)$, find both Ga and Fx such that $Hom_{\mathbf{A}}(Fx, a) \cong W(x, a) \cong Hom_{\mathbf{X}}(x, Ga)$.

$$\begin{array}{ccccc} & x & \xrightarrow{f(c)} & Ga & \\ h_x & \Downarrow & \searrow^{c} & \Downarrow & e_a \\ & Fx & \xrightarrow{g(c)} & a & \end{array}$$

Het Adjunctive Square[8]

Sometimes the two adjoint transposes are written vertically as in a Gentzen-style rule of inference:

$$\frac{x \to Ga}{Fx \to a}$$

Gentzen-style presentation of an adjunction

This can be seen as a proto-het-adjunctive square without the vertical morphisms—at least when the homomorphism involving the left adjoint is on the bottom.

Some of the rigmarole of the conventional treatment of adjoints (*sans* chimeras) is only necessary because of the restriction to morphisms within one category or the other. For instance, the UMP for the hom unit $\eta_x : x \to GFx$ is that given any morphism $f : x \to Ga$ in $\mathbf{X}$, there is a unique morphism $g = f^* : Fx \to a$ in the other category $\mathbf{A}$ such that G-functorial image back in the original category $\mathbf{X}$ gives the factorization of f through the unit: $x \xrightarrow{f} Ga = x \xrightarrow{\eta_x} GFx \xrightarrow{Gf^*} Ga$. The UMP has to go back and forth between homomorphisms in the two categories because it avoids mention of the heteromorphisms between the categories. The universal mapping property for the het unit $h_x : x \Rightarrow Fx$ is much simpler (i.e., no G and no over and back). Given any heteromorphism $c : x \Rightarrow a$, there is a unique homomorphism $g(c) : Fx \to a$ in the codomain category $\mathbf{A}$ such that $x \overset{c}{\Rightarrow} a = x \overset{h_x}{\Rightarrow} Fx \xrightarrow{g(c)} a$.

For instance, in the "old days" (before category theory), one might have stated the universal mapping property of the free group Fx on a set x by saying that for any map $c : x \Rightarrow a$ from x into a group a, there is a unique group homomorphism $g(c) : Fx \to a$ that preserves the action of c on the generators x, i.e., such that $x \overset{c}{\Rightarrow} a = x \hookrightarrow Fx \xrightarrow{g(c)} a$. That is just the left half-adjunction part of the free-group adjunction. There is nothing sloppy or 'wrong' in that old way of stating the universal mapping property.

Dually for the hom counit, given any morphism $g : Fx \to a$ in $\mathbf{A}$, there is a unique morphism $f = g^* : x \to Ga$ in the other category X, such that the F-functorial image back in the original category A gives the factorization of g though the counit: $Fx \xrightarrow{g} a = Fx \xrightarrow{Fg^*} FGa \xrightarrow{\varepsilon_a} a$. For the het counit, given any heteromorphism $c : x \Rightarrow a$, there is a unique homomorphism $f(c) : x \to Ga$ in the domain category $\mathbf{X}$ such that $x \overset{c}{\Rightarrow} a = x \xrightarrow{f(c)} Ga \overset{e_a}{\Rightarrow} a$. Putting these two

[8] For typographical reasons, the diagonal heteromorphism $c : x \Rightarrow a$ is represented as a single arrow rather than a double arrow.

het UMPs together yields the het adjunctive square diagram, just as previously putting the two hom UMPs together yielded the hom-pair adjunctive square diagram.

3.6 Het Natural Transformations

One of the main motivations for category theory was to mathematically characterize the intuitive notion of naturality for homomorphisms as in the standard example of the canonical linear homomorphism embedded a vector space into its double dual. Many heteromorphisms are rather arbitrary but certain ones are quite canonical so we should be able to mathematically characterize that canonicalness or naturality just as we do for homomorphisms. Indeed, the notion of a natural transformation immediately generalizes to functors with different codomains by taking the components to be heteromorphisms. Given functors $F : \mathbf{X} \to \mathbf{A}$ and $H : \mathbf{X} \to \mathbf{B}$ with a common domain *and* given a het-bifunctor $\mathrm{Het} : \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$, a *chimera* or *het natural transformation relative to* Het, $\varphi : F \Rightarrow H$, is given by a set of heteromorphisms $\{\varphi_x \in \mathrm{Het}(Fx, Hx)\}$ indexed by the objects of $\mathbf{X}$ such that for any $j : x \to x'$ the following diagram commutes.

$$\begin{array}{ccccc} & Fx & \overset{\varphi_x}{\Longrightarrow} & Hx & \\ Fj & \downarrow & & \downarrow & Hj \\ & Fx' & \overset{\varphi_{x'}}{\Longrightarrow} & Hx' & \end{array}$$

Het natural transformation

As with any commutative diagram involving heteromorphisms, composition and commutativity are defined using the het-bifunctor (similar remarks apply to any ordinary commutative hom diagram where it is the hom-bifunctor behind the scenes). For instance, the above commutative squares which define het natural transformations unpack as the following behind-the-scenes commutative squares in $\mathbf{Set}$ for the underlying het-bifunctor.

$$\begin{array}{ccccc} & 1 & \overset{\varphi_x}{\longrightarrow} & \mathrm{Het}(Fx, Hx) & \\ \varphi_{x'} & \downarrow & & \downarrow & \mathrm{Het}(Fx, Hj) \\ & \mathrm{Het}(Fx', Hx') & \underset{\mathrm{Het}(Fj, Hx')}{\longrightarrow} & \mathrm{Het}(Fx, Hx') & \end{array}$$

The composition $Fx \overset{\varphi_x}{\Longrightarrow} Hx \overset{Hj}{\longrightarrow} Hx'$ is $\mathrm{Het}(Fx, Hj)(\varphi_x) \in \mathrm{Het}(Fx, Hx')$, the composition $Fx \overset{Fj}{\longrightarrow} Fx' \overset{\varphi_{x'}}{\Longrightarrow} Hx'$ is $\mathrm{Het}(Fj, Hx')(\varphi_{x'}) \in \mathrm{Het}(Fx, Hx')$, and commutativity means they are the same element of $\mathrm{Het}(Fx, Hx')$. These het natural transformations do not compose like the morphisms in a functor category but they are acted upon by the natural transformations in the functor categories on each side to yield het natural transformations.

There are het natural transformations each way between any functor and the identity on its domain if the functor itself is used to define the appropriate het-bifunctor. That is, given *any* functor $F : \mathbf{X} \to \mathbf{A}$, there is a het natural transformation $1_{\mathbf{X}} \Rightarrow F$ relative to the bifunctor defined as $\mathrm{Het}(x, a) = \mathrm{Hom}_{\mathbf{A}}(Fx, a)$ as well as a het natural transformation $F \Rightarrow 1_{\mathbf{X}}$ relative to $\mathrm{Het}(a, x) = \mathrm{Hom}_{\mathbf{A}}(a, Fx)$.

Het natural transformations 'in effect' already occur with reflective (or coreflective) subcategories. A subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is a *reflective subcategory* if the inclusion functor $K : \mathbf{A} \hookrightarrow \mathbf{B}$ has a left adjoint. For any such reflective adjunctions, the heteromorphisms $\mathrm{Het}(b, a)$ are the $\mathbf{B}$-morphisms with their heads in the subcategory $\mathbf{A}$ so the representation on the right $\mathrm{Het}(b, a) \cong \mathrm{Hom}_{\mathbf{B}}(b, Ka)$ is trivial. The left adjoint $F : \mathbf{B} \to \mathbf{A}$ gives the representation on the left: $\mathrm{Hom}_{\mathbf{A}}(Fb, a) \cong \mathrm{Het}(b, a) \cong \mathrm{Hom}_{\mathbf{B}}(b, Ka)$. Then it is perfectly 'natural' to see the unit of the adjunction as defining a natural transformation $\eta : 1_{\mathbf{B}} \Rightarrow F$ but that is actually a het natural transformation (since the codomain of F is $\mathbf{A}$). Hence the conventional ("heterophobic"?) treatment (e.g., [20, p. 89]) is to define another functor R with the same domain and values on objects and morphisms as F except that its codomain is taken to be $\mathbf{B}$ so that we can then have a hom natural transformation $\eta : 1_{\mathbf{X}} \to R$ between two functors with the same codomain. Similar remarks hold for the dual coreflective case where the inclusion functor has a right adjoint and where the heteromorphisms are turned around, i.e., are $\mathbf{B}$-morphisms with their tail in the subcategory $\mathbf{A}$.

Given any adjunction isomorphism $\mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Het}(x, a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga)$, the adjoint correlates of the identities $1_{Fx} \in \mathrm{Hom}_{\mathbf{A}}(Fx, Fx)$ are the het units $h_x \in \mathrm{Het}(x, Fx)$ and the hom units $\eta_x \in \mathrm{Hom}_{\mathbf{X}}(x, GFx)$. The het units together give the het natural transformation $h : 1_{\mathbf{X}} \Rightarrow F$ while the hom units give the hom natural transformation $\eta : 1_{\mathbf{X}} \to GF$. The adjoint correlates of the identities $1_{Ga} \in \mathrm{Hom}_{\mathbf{X}}(Ga, Ga)$ are the het counits $e_a \in \mathrm{Het}(Ga, a)$ and the hom counits $\varepsilon_a \in \mathrm{Hom}_{\mathbf{A}}(FGa, a)$. The het counits together give the het natural transformation $e : G \Rightarrow 1_{\mathbf{A}}$ while the hom counits give the hom natural transformation $\varepsilon : FG \to 1_{\mathbf{A}}$.

4 Examples

4.1 The Product Adjunction

Let $\mathbf{X}$ be the category $\mathbf{Set}$ of sets and let $\mathbf{A}$ be the category $\mathbf{Set}^2 = \mathbf{Set} \times \mathbf{Set}$ of ordered pairs of sets. A heteromorphism from a set to a pair of sets is a pair of set maps with a common domain $(f_1, f_2) : W \Rightarrow (X, Y)$ which is called a *cone*. The het-bifunctor is given by $\mathrm{Het}(W, (X, Y)) = \{W \Rightarrow (X, Y)\}$, the set of all cones from W to (X, Y). To construct a representation on the right, suppose we are given a pair of sets $(X, Y) \in \mathbf{Set}^2$. How could one construct a set, to be denoted $X \times Y$, such that all cones $W \Rightarrow (X, Y)$ from any set W could be represented by set functions (morphisms within $\mathbf{Set}$) $W \to X \times Y$? In the "atomic" case of $W = 1$ (the one element set), a 1-cone $1 \Rightarrow (X,Y)$ would just pick out an ordered pair (x,y) of elements, the first from X and the second from Y. Any cone $W \Rightarrow (X,Y)$ would just pick out a set of pairs of elements. Hence the universal object would have to be the set $\{(x,y) : x \in X, y \in Y\}$ of *all* such pairs which yields the Cartesian product of sets $X \times Y$. The assignment of that set to each pair of sets gives the right adjoint $G : \mathbf{Set}^2 \to \mathbf{Set}$ where $G((X,Y)) = X \times Y$ (and similarly for morphisms). The het counit $e_{(X,Y)} : X \times Y \Rightarrow (X,Y)$ canonically takes each ordered pair (x,y) as a single element in $X \times Y$ to that pair of elements in (X,Y). The universal mapping property of the Cartesian product $X \times Y$ then holds; given any set W and a cone $(f_1, f_2) : W \Rightarrow (X,Y)$, there is a unique set function $\langle f_1, f_2 \rangle : W \to X \times Y$ defined by $\langle f_1, f_2 \rangle (w) = (f_1(w), f_2(w))$ that factors the cone through het counit:

$$\begin{array}{ccccc} & W & \xrightarrow{\langle f_1,f_2\rangle} & X \times Y & \\ & (f_1,f_2) & \searrow & \Downarrow & e_{(X,Y)} \\ & & & (X,Y) & \end{array}$$

Right half-adjunction of the product adjunction.

Fixing W in the domain category, how could we find a universal object in $\mathbf{Set}^2$ so that all heteromorphisms $(f_1, f_2) : W \Rightarrow (X,Y)$ could be uniquely factored through it. The obvious suggestion is the pair (W,W) which defines a functor $F : \mathbf{Set} \to \mathbf{Set}^2$ and where the het unit $h_W : W \Rightarrow (W,W)$ is just the pair of identity maps $h_W = (1_W, 1_W)$. Then for each cone $(f_1, f_2) : W \Rightarrow (X,Y)$, there is a unique pair of maps, also denoted $(f_1, f_2) : (W,W) \to (X,Y)$, which are a morphism in $\mathbf{Set}^2$ and which factors the cone through the het unit:

$$\begin{array}{cccc} & W & & \\ h_W & \Downarrow & \searrow & (f_1,f_2) \\ & (W,W) & \xrightarrow{(f_1,f_2)} & (X,Y) \end{array}$$

Left half-adjunction of the product adjunction.

Splicing the two half-adjunctions along the diagonal gives the:

$$\begin{array}{ccccc} & W & \xrightarrow{\langle f_1,f_2\rangle} & X \times Y & \\ h_W & \Downarrow & \searrow^{(f_1,f_2)} & \Downarrow & e_{(X,Y)} \\ & (W,W) & \xrightarrow{(f_1,f_2)} & (X,Y) & \end{array}$$

Het adjunctive square for the product adjunction.

The two factor maps on the top and bottom are uniquely associated with the diagonal cones, and the isomorphism is natural so that we have natural isomorphisms between the hom-bifunctors and the het-bifunctor:

$$\mathrm{Hom}_{\mathbf{Set}^2}((W,W),(X,Y)) \cong \mathrm{Het}(W,(X,Y)) \cong \mathrm{Hom}_{\mathbf{Set}}(W, X \times Y).$$

4.2 Limits in Sets

Let $\mathbf{D}$ be a small (diagram) category and $D : \mathbf{D} \rightarrow \mathbf{Set}$ a functor considered as a diagram in the category of $\mathbf{Set}$. Limits in sets generalize the previous example where $\mathbf{D} = 2$. The diagram D is in the functor category $\mathbf{Set}^{\mathbf{D}}$ where the morphisms are natural transformation between the functors. A heteromorphism from a set W to a diagram functor D is concretely given by a *cone* $c : W \Rightarrow D$ which is defined as a set of maps $\{W \xrightarrow{c_i} D_i\}_{i \in Ob(\mathbf{D})}$ (where $Ob(\mathbf{D})$ is the set of objects of $\mathbf{D}$) such that for any morphism $\alpha : i \rightarrow j$ in $\mathbf{D}$, $W \xrightarrow{c_i} D_i \xrightarrow{D_\alpha} D_j = W \xrightarrow{c_j} D_j$. The adjunction is then given by the birepresentation of the het-bifunctor where $\mathrm{Het}(W, D) = \{W \Rightarrow D\}$ is the set of cones from the set W to the diagram functor D.

To construct the limit functor (right adjoint), take the product $\prod_{i \in Ob(\mathbf{D})} D_i$ and then take $LimD$ as the set of elements $(..., x_i, ...)$ of the product such that for any morphism $\alpha : i \rightarrow j$ in $\mathbf{D}$, $D_\alpha(x_i) = x_j$.

Since any cone $c : W \Rightarrow D = \left\{W \xrightarrow{c_i} D_i\right\}_{i \in Ob(\mathbf{D})}$ would carry each element $w \in W$ to a compatible set of elements from the $\{D_i\}$, the adjoint correlate map $f(c) : W \rightarrow LimD$ would just carry w to element $(..., c_i(w), ...)$ in the set $LimD$. The het counit $e_D : LimD \Rightarrow D$ would be the cone $\left\{LimD \xrightarrow{(e_D)_i} D_i\right\}$ of maps such that $(e_D)_i : LimD \rightarrow D_i$ carries each element $(..., x_i, ...)$ of the set $LimD$ to the element $x_i \in D_i$ picked out in the i^{th} component of $(..., x_i, ...)$. The universal mapping property for the het counit e_D is that given any cone $c : W \Rightarrow D$ (for any set W), there is a unique set morphism $f(c) : W \rightarrow LimD$ (constructed above) such that $W \xrightarrow{f(c)} LimD \overset{e_D}{\Rightarrow} D = W \overset{c}{\Rightarrow} D$ which might be diagrammed as:

$$\begin{array}{ccccc} W & \overset{f(c)}{\longrightarrow} & LimD & & \\ c & \searrow & \Downarrow & & e_D \\ & & D & & \end{array}$$

Right half-adjunction of the limits adjunction.

Fixing W, how might we construct a functor $\mathbf{D} \rightarrow \mathbf{Set}$ such that all cones $c : W \Rightarrow D$ might be factored through it? Since the only given data is W, the obvious thing to try is the constant or diagonal functor $\Delta W : \mathbf{D} \rightarrow \mathbf{Set}$ which takes each object i in $\mathbf{D}$ to W and each morphism $\alpha : i \rightarrow j$ to the identity map 1_W. Given a cone $c : W \Rightarrow D$, the obvious natural transformation $g(c)$ from the diagonal functor ΔW to D is given by the same set of maps $g(c) = \left\{W \xrightarrow{c_i} D_i\right\}_{i \in Ob(\mathbf{D})}$ which constitutes a morphism in $\mathbf{Set}^{\mathbf{D}}$.[9] The het unit $h_W : W \Rightarrow \Delta W$ is the set of identity maps $\left\{W \xrightarrow{(h_W)_i = 1_W} (\Delta W)_i = W\right\}$. The

[9] It should be carefully noted that much of the literature refers to this natural transformation morphism within the category $\mathbf{Set}^{\mathbf{D}}$ interchangably with the cone $W \Rightarrow D$ which is a heteromorphism from a set to a functor.

(rather trivial) universal mapping property of the het unit h_W is that given any cone $c : W \Rightarrow D$, there exists a unique natural transformation $g(c) : \Delta W \to D$ which factors the cone through the het unit:

$$\begin{array}{ccccc} & W & & \\ h_W & \Downarrow & \searrow & c \\ & \Delta W & \xrightarrow{g(c)} & D \end{array}$$

Left half-adjunction of the limit adjunction.

Splicing the two half-adjunctions together along the main diagonal gives the:

$$\begin{array}{ccccc} & W & \xrightarrow{f(c)} & LimD & \\ h_W & \Downarrow & \searrow^{c} & \Downarrow & e_D \\ & \Delta W & \xrightarrow{g(c)} & D & \end{array}$$

Het adjunctive square for the limit adjunction.

The two factor maps on the top and bottom are uniquely associated with the diagonal cones, and the isomorphism is natural so that we have natural isomorphisms between the hom-bifunctors and the het-bifunctor:

$$\mathrm{Hom}_{\mathbf{Set}^D}(\Delta W, D) \cong \mathrm{Het}(W, D) \cong \mathrm{Hom}_{\mathbf{Set}}(W, LimD).$$

4.3 Colimits in Sets

The duals to products are coproducts which in *Set* is the disjoint union of sets. Instead of developing that special case, we move directly to the adjunction for colimits in **Set** which is dual to the limits adjunction. The argument here (as with limits) would work for any other cocomplete (or complete) category of algebras replacing the category of sets. Given the same data as in the previous section, the diagonal functor $\Delta : \mathbf{Set} \to \mathbf{Set}^{\mathbf{D}}$ is defined as before and it has a *left* adjoint $\mathrm{Colim} : \mathbf{Set}^{\mathbf{D}} \to \mathbf{Set}$.

For this adjunction, a heteromorphism from a diagram functor D to a set Z is a *cocone* $c : D \Rightarrow Z$ which is defined as a set of maps $\{D_i \xrightarrow{c_i} Z\}_{i \in Ob(\mathbf{D})}$ such that for any morphism $\alpha : i \to j$ in $\mathbf{D}$, $D_i \xrightarrow{D_\alpha} D_j \xrightarrow{c_j} Z = D_i \xrightarrow{c_i} Z$. The adjunction is then given by the birepresentations of the het-bifunctor where $\mathrm{Het}(D, Z) = \{D \Rightarrow Z\}$ is the set of cocones from the diagram functor D to the set Z.

Fixing a diagram functor $D : \mathbf{D} \to Set$ in $\mathbf{Set}^{\mathbf{D}}$, how can we construct a universal object $\mathrm{Colim}\, D$ so that any cocone $c : D \Rightarrow Z$ can be factored through it? Instead of taking the product $\prod_{i \in Ob(\mathbf{D})} D_i$, take the dual construction of the coproduct which is the disjoint union $\coprod_{i \in Ob(\mathbf{D})} D_i$. Then instead of the subset of compatible elements, take $\mathrm{Colim}\, D$ as the quotient set by the compatibility equivalence relation $x_i \sim x_j$ if $D_\alpha(x_i) = x_j$ for any morphism D_α between the D_is.

To construct the het unit cocone $h_D : D \Rightarrow \operatorname{Colim} D$, define each map $(h_D)_i : D_i \to \operatorname{Colim} D$ by taking each element $x \in D_i$ to its equivalence class in $\operatorname{Colim} D$. All the maps in $h_D = \left\{ D_i \xrightarrow{(h_D)_i} \operatorname{Colim} D \right\}_{i \in Ob(\mathbf{D})}$ will commute with the maps $D\alpha : D_i \to D_j$ for $\alpha : i \to j$ in $\mathbf{D}$ by the construction of the coelements so that h_D is a cocone $D \Rightarrow \operatorname{Colim} D$. Given a cocone $c : D \Rightarrow Z = \left\{ D_i \xrightarrow{c_i} Z \right\}_{i \in Ob(\mathbf{D})}$, an $x \in D_i$ must be carried by c_i to the same element z of Z as all the other elements $x' \in D_j$ in the equivalence class of x are carried by their maps $c_j : D_j \to Z$, so the factor set map $g(c) : \operatorname{Colim} D \to Z$ would just carry that equivalence class of x as an element of $\operatorname{Colim} D$ to that $z \in Z$. This factor map uniquely completes the following diagram:

$$\begin{array}{ccccc} & D & & & \\ h_D & \Downarrow & & \searrow & c \\ & \operatorname{Colim} D & \xrightarrow{g(c)} & Z & \end{array}$$

Left half-adjunction for colimit adjunction.

Fixing Z, the same diagonal functor $\Delta : \mathbf{Set} \to \mathbf{Set}^{\mathbf{D}}$ gives a universal object ΔZ in $\mathbf{Set}^{\mathbf{D}}$. The het counit e_Z is the cocone $e_Z = \left\{ (\Delta Z)_i = Z \overset{1_Z}{\to} Z \right\}_{i \in Ob(\mathbf{D})}$ of identity maps. Given any cocone $c : D \Rightarrow Z = \left\{ D_i \overset{c_i}{\to} Z \right\}_{i \in Ob(\mathbf{D})}$, the adjoint correlate natural transformation $f(c) : D \to \Delta Z$ would be defined by the same set of maps $\left\{ D_i \overset{c_i}{\to} Z \right\}$. That is clearly the unique natural transformation $D \to \Delta Z$ to make the following diagram commute:

$$\begin{array}{cccc} D & \xrightarrow{f(c)} & \Delta Z & \\ c & \searrow & \Downarrow & e_Z \\ & & Z & \end{array}$$

Right half-adjunction for colimit adjunction.

Splicing the two half-adjunctions together along the main diagonal yields the:

$$\begin{array}{ccccc} & D & \xrightarrow{f(c)} & \Delta Z & \\ h_D & \Downarrow & \searrow^{c} & \Downarrow & e_Z \\ & \operatorname{Colim} D & \xrightarrow{g(c)} & Z & \end{array}$$

Het adjunctive square for the colimit adjunction.

The two factor maps on the top and bottom are uniquely associated with the diagonal cocone, and the isomorphism is natural so that we have natural isomorphisms between the hom-bifunctors and the het-bifunctor:

$$\operatorname{Hom}_{\mathbf{Set}}(\operatorname{Colim} D, Z) \cong \operatorname{Het}(D, Z) \cong \operatorname{Hom}_{\mathbf{Set}^{\mathbf{D}}}(D, \Delta Z).$$

4.4 Adjoints to Forgetful Functors

Perhaps the most accessible adjunctions are the free-forgetful adjunctions between $\mathbf{X} = \mathbf{Set}$ and a category of algebras such as the category of groups $\mathbf{A} = \mathbf{Grps}$. The right adjoint $G : \mathbf{A} \to \mathbf{X}$ forgets the group structure to give the underlying set GA of a group A. The left adjoint $F : \mathbf{X} \to \mathbf{A}$ gives the free group FX generated by a set X.

For this adjunction, the heteromorphisms are any set functions $X \overset{c}{\Rightarrow} A$ (with the codomain being a group A) and the het-bifunctor is given by such functions: $\mathrm{Het}(X, A) = \{X \Rightarrow A\}$ (with the obvious morphisms). A heteromorphism $c : X \Rightarrow A$ determines a set map $f(c) : X \to GA$ trivially and it determines a group homomorphism $g(c) : FX \to A$ by mapping the generators $x \in X$ to their images $c(x) \in A$ and then mapping the other elements of FX as they must be mapped in order for $g(c)$ to be a group homomorphism. The het unit $h_X : X \Rightarrow FX$ is insertion of the generators into the free group and the het counit $e_A : GA \Rightarrow A$ is just the retracting of the elements of the underlying set back to the group. These factor maps $f(c)$ and $g(c)$ uniquely complete the usual two half-adjunction triangles which together give the:

$$\begin{array}{ccccc} & X & \xrightarrow{f(c)} & GA & \\ h_X & \Downarrow & \searrow^{c} & \Downarrow & e_A \\ & FX & \xrightarrow{g(c)} & A & \end{array}$$

Het adjunctive square for the free group adjunction.

These associations also give us the two representations:

$$\mathrm{Hom}(FX, A) \cong \mathrm{Het}(X, A) \cong \mathrm{Hom}(X, GA).$$

In general, the existence of a left adjoint to $U : \mathbf{A} \to \mathbf{Set}$ (i.e., a left representation of $\mathrm{Het}(X, A) = \{X \Rightarrow A\}$) will depend on whether or not there is an $\mathbf{A}$-object FX with the least or minimal structure so that every chimera $X \overset{c}{\Rightarrow} A$ will determine a unique representing $\mathbf{A}$-morphism $g(c) : FX \to A$.

The existence of a *right* adjoint to U will depend on whether or not for any set X there is an $\mathbf{A}$-object IX with the greatest or maximum structure so that any chimera $A \Rightarrow X$ can be represented by an $\mathbf{A}$-morphism $A \to IX$.

Consider the underlying set functor $U : \mathbf{Pos} \to \mathbf{Set}$ from the category of partially ordered sets (an ordering that is reflexive, transitive, and anti-symmetric) with order-preserving maps to the category of sets. It has a left adjoint since each set has a least partial order on it, namely the discrete ordering. Hence any chimera function $X \overset{c}{\Rightarrow} A$ from a set X to a partially ordered set or poset A could be represented as a set function $X \xrightarrow{f(c)} UA$ or as an order-preserving function $DX \xrightarrow{g(c)} A$ where DX gives the discrete ordering on X. The functor giving the discrete partial ordering on a set is left adjoint to the underlying set function.

In the other direction, one could take as a chimera any function $A \overset{c}{\Rightarrow} X$ (from a poset A to a set X) and it is represented on the left by the ordinary set function $UA \xrightarrow{g(c)} X$ so the left half-adjunction trivially exists:

$$\begin{array}{cccc} & A & \xrightarrow{?} & IX? \\ h_A & \Downarrow & \searrow^{c} & \downarrow ? \\ & UA & \xrightarrow{g(c)} & X \end{array}$$

Left half-adjunction (with no right half-adjunction).

But the underlying set functor U does not have a right adjoint since there is no maximal partial order IX on X so that any chimera $A \overset{c}{\Rightarrow} X$ could be represented as an order-preserving function $f(c) : A \to IX$. To receive all the possible orderings, the ordering relation would have to go both ways between any two points which would then be identified by the anti-symmetry condition so that IX would collapse to a single point and the factorization of c through IX would fail.[10] Thus poset-to-set chimera $A \Rightarrow X$ can only be represented on the left.

Relaxing the anti-symmetry condition, let $U : \mathbf{Ord} \to \mathbf{Set}$ be the underlying set functor from the category of preordered sets (reflexive and transitive orderings) to the category of sets. The discrete ordering again gives a left adjoint. But now there is also a maximal ordering on a set X, namely the 'indiscrete' ordering IX on X (the 'indiscriminate' or 'chaotic' preorder on X) which has the ordering relation both ways between any two points. Then a preorder-to-set chimera morphism $A \Rightarrow X$ (just a set function ignoring the ordering) can be represented on the left as a set function $UA \xrightarrow{g(c)} X$ and on the right as an order-preserving function $A \xrightarrow{f(c)} IX$ so that U also has a right adjoint I and we have the following:

$$\begin{array}{ccccc} & A & \xrightarrow{f(c)} & IX & \\ h_A & \Downarrow & \searrow^{c} & \Downarrow & e_X \\ & UA & \xrightarrow{g(c)} & X & \end{array}$$

Het adjunctive square for the indiscrete-underlying adjunction on preorders.

4.5 Reflective Subcategories

Suppose that $\mathbf{A}$ is a subcategory of $\mathbf{X}$ with $G : \mathbf{A} \hookrightarrow \mathbf{X}$ the inclusion functor and suppose that it has a left adjoint $F : \mathbf{X} \to \mathbf{A}$. Then $\mathbf{A}$ is said to be a *reflective subcategory* of $\mathbf{X}$, the left adjoint F is the **reflector**, and the adjunction is called a *reflection*: $\mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga)$. For all reflections, the chimera morphisms are the morphisms $x \Rightarrow a$ in the ambient category $\mathbf{X}$ with their heads in the reflective subcategory $\mathbf{A}$. Hence the het-bifunctor would be:

[10] Thanks to Vaughn Pratt for the example.

$$\mathrm{Het}(x,a) = \mathrm{Hom}_{\mathbf{XA}}(x,a)$$

where the **XA** subscript indicates that x can be any object in **X** but that a is any element of the subcategory **A**. Note the two ways of seeing any $c \in \mathrm{Het}(x,a) = \mathrm{Hom}_{\mathbf{XA}}(x,a)$. From one viewpoint, $c \in \mathrm{Hom}_{\mathbf{XA}}(x,a) \subseteq \mathrm{Hom}_{\mathbf{X}}(x,a)$ so that c is just a morphism inside the category **X**, but we also view it as a chimera with its tail in **X** and head in **A**. Since G is the inclusion functor, it just takes a as an element of **A** to itself as an element of **X** and similarly for morphisms. Thus we insert $\mathrm{Het}(x,a)$ in the middle to get the two representation isomorphisms:

$$\mathrm{Hom}_{\mathbf{A}}(Fx,a) \cong \mathrm{Het}(x,a) \cong \mathrm{Hom}_{\mathbf{X}}(x,Ga).$$

There is also the dual case of a *coreflective subcategory* where the inclusion functor has a right adjoint and where the chimera morphisms are turned around (tail in subcategory and head in ambient category). This case will be used in the next section but here I will focus on reflective subcategories.

For an interesting example of a reflector dating back five centuries, we use the modern mathematical formulation of double-entry bookkeeping [7] [8]. Let **Ab** be the category of abelian (i.e., commutative) groups where the operation is written as addition. Thus 0 is the identity element, $a + a' = a' + a$, and for each element a, there is an element $-a$ such that $a + (-a) = 0$. Let **CMon** be the category of commutative monoids so the addition operation has the identity 0 but does not necessarily have an inverse. Let $G : \mathbf{Ab} \hookrightarrow \mathbf{CMon}$ be the inclusion functor.

In 1494, the mathematician Luca Pacioli published an accounting technique that had been developed in practice during the 1400s and which became known as *double-entry bookkeeping* [22]. In essence, the idea was to do additive arithmetic with additive inverses using ordered pairs $[x \mathbin{//} x']$ of non-negative numbers called *T-accounts*.[11] The number on the left side was called the *debit entry* and the number on the right the *credit entry*. T-accounts added by adding the corresponding entries: $[x \mathbin{//} x'] + [y \mathbin{//} y'] = [x + y \mathbin{//} x' + y']$. Two T-accounts were deemed equal if their cross-sums were equal (the additive version of the equal cross-multiples used to define equality of multiplicative ordered pairs or fractions). Thus

$$[x \mathbin{//} x'] = [y \mathbin{//} y'] \text{ if } x + y' = x' + y.$$

Hence the additive inverse was obtained by "reversing the entries" (as accountants say):

$$[x \mathbin{//} x'] + [x' \mathbin{//} x] = [x + x' \mathbin{//} x' + x] = [0 \mathbin{//} 0].$$

[11] The double-slash separator was suggested by Pacioli. "At the beginning of each entry, we always provide 'per', because, first, the debtor must be given, and immediately after the creditor, the one separated from the other by two little slanting parallels (virgolette), thus, //,. . . ." [22, p. 43]

To obtain the reflector or left adjoint $F : \mathbf{CMon} \to \mathbf{Ab}$ to G, we need only note that Pacioli was implicitly using the fact that the normal addition of numbers is *cancellative* in the sense that $x + z = y + z$ implies $x = y$. Since commutative monoids do not in general have that property we need only to tweak the definition of equality of T-accounts [5, p. 17]:

$$[x \mathbin{//} x'] = [y \mathbin{//} y'] \text{ if there is a } z \text{ such that } x + y' + z = x' + y + z.$$

This construction with the induced maps then yields a functor $F : \mathbf{CMon} \to \mathbf{Ab}$ that takes a commutative monoid m to a commutative group $Fm = P(m)$. The group $P(m)$ is usually called the "group of differences" or "inverse-completion" and, in algebraic geometry, its generalization is called the "Grothendieck group." However, due to about a half-millennium of priority, we will call the additive group of differences the *Pacioli group* of the commutative monoid m. For any such m, the het unit $h_m : m \Rightarrow Fm = P(m)$ which takes an element x to the T-account $[0 \mathbin{//} x]$ with that credit balance (the debit balance mapping would do just as well).

For this adjunction, a heteromorphism $c : m \Rightarrow a$, is any *monoid homomorphism* from a commutative monoid m to any abelian group a (being only a monoid homomorphism, it does not need to preserve any inverses that might exist in m). The Pacioli group has the following universality property: for any heteromorphism $c : m \Rightarrow a$, there is a unique group homomorphism $g(c) : Fm \to a$ such that $m \overset{h_m}{\Rightarrow} Fm \overset{g(c)}{\to} a = m \overset{c}{\Rightarrow} a$. The group homomorphism factor map is: $g(c)([x' \mathbin{//} x]) = c(x) + (-c(x'))$. This establishes the other representation isomorphism of the adjunction:

$$\mathrm{Hom}_{\mathbf{Ab}}(Fm, a) \cong \mathrm{Het}(m, a) \cong \mathrm{Hom}_{\mathbf{CMon}}(m, Ga).$$

4.6 The Special Case of Endo-Adjunctions

Since the heteromorphic theory of adjoints is based on representing the heteromorphisms between the objects of two different categories with homomorphisms within each category, the case of an endo-adjunction all within one category is clearly going to require some special attention. The product-exponential adjunction in **Set** is an important example of adjoint endo-functors $F : \mathbf{Set} \rightleftarrows \mathbf{Set} : G$. For any fixed (non-empty) "index" set A, the product functor $F(-) = - \times A : \mathbf{Set} \to \mathbf{Set}$ has a right adjoint $G(-) = (-)^A : \mathbf{Set} \to \mathbf{Set}$ which makes **Set** a Cartesian-closed category. For any sets X and Y, the adjunction has the form: $\mathrm{Hom}(X \times A, Y) \cong \mathrm{Hom}(X, Y^A)$.

Since both functors are endo-functors on **Set**, we don't have the two categories between which to have heteromorphisms. Moreover, we don't have the expected canonical maps as the het unit or counit. For instance, the het unit should be a canonical morphism $h_X : X \Rightarrow FX$ but if $FX = X \times A$, there is no *canonical* (het or otherwise) map $X \to X \times A$ (except in the special case where A

is a singleton). Similarly, the het counit should be a canonical map $e_Y : GY \Rightarrow Y$ but if $GY = Y^A$ then there is no canonical (het or otherwise) map $Y^A \to Y$ (unless A is a singleton). Hence a special treatment is required. It consists of showing that the endo-adjunction can be parsed in two ways as adjunctions each of which is between different categories and then the heteromorphic theory applies.

The key is the following special case of a result by Freyd [12, p. 83]. Consider any endo-adjunction $F : \mathbf{C} \rightleftarrows \mathbf{C} : G$ on a category $\mathbf{C}$ where the functors are assumed one-one on objects. Then the image of G is a subcategory of $\mathbf{C}$, i.e., $\mathrm{Im}\,(G) \hookrightarrow \mathbf{C}$, and similarly the image of F is also a subcategory of $\mathbf{C}$, i.e., $\mathrm{Im}\,(F) \hookrightarrow \mathbf{C}$. The operation of taking the G-image of the hom-set $\mathrm{Hom}_{\mathbf{C}}(Fx, a)$ to obtain $\mathrm{Hom}_{\mathrm{Im}(G)}(GFx, Ga)$ is onto by construction. It is one-one on objects by assumption and one-one on maps since if for $g, g' : Fx \to a$ and $Gg = Gg'$, then $g = g'$ by the uniqueness of the factor map $f^* : Fx \to a$ to factor $x \xrightarrow{f} Ga := x \xrightarrow{\eta_x} GFx \xrightarrow{Gg=Gg'} Ga$ through the hom unit η_x. The isomorphism is also natural in x and a so we have:

$$\mathrm{Hom}_{\mathrm{Im}(G)}(GFx, Ga) \cong \mathrm{Hom}_{\mathbf{C}}(Fx, a).$$

If $x' = Ga$, then using that isomorphism and the adjunction, we have:

$$\mathrm{Hom}_{\mathrm{Im}(G)}(GFx, x') \cong \mathrm{Hom}_{\mathrm{Im}(G)}(GFx, Ga) \cong \mathrm{Hom}_{\mathbf{C}}(Fx, a) \cong \mathrm{Hom}_{\mathbf{C}}(x, Ga) \cong \mathrm{Hom}_{\mathbf{C}}(x, x').$$

Thus $\mathrm{Hom}_{\mathrm{Im}(G)}(GFx, x') \cong \mathrm{Hom}_{\mathbf{C}}(x, x')$ so that $GF : \mathbf{C} \to \mathrm{Im}(G)$ (where G is construed as having the codomain $\mathrm{Im}\,(G)$) is left adjoint to the inclusion $\mathrm{Im}\,(G) \hookrightarrow \mathbf{C}$ and thus $\mathrm{Im}\,(G)$ is a reflective subcategory of $\mathbf{C}$.

Dually, we also have the natural isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(x, Ga) \cong \mathrm{Hom}_{\mathrm{Im}(F)}(Fx, FGa)$$

by taking the F-image of $\mathrm{Hom}_{\mathbf{C}}(x, Ga)$ and using the universal mapping property of the hom counit. If $a' = Fx$ then using that isomorphism and the adjunction, we have:

$$\mathrm{Hom}_{\mathbf{C}}(a', a) \cong \mathrm{Hom}_{\mathbf{C}}(Fx, a) \cong \mathrm{Hom}_{\mathbf{C}}(x, Ga) \cong \mathrm{Hom}_{\mathrm{Im}(F)}(Fx, FGa) \cong \mathrm{Hom}_{\mathrm{Im}(F)}(a', FGa).$$

Thus $\mathrm{Hom}_{\mathbf{C}}(a', a) \cong \mathrm{Hom}_{\mathrm{Im}(F)}(a', FGa)$ so that $FG : \mathbf{C} \to \mathrm{Im}\,(F)$ (where F is construed as having the codomain $\mathrm{Im}\,(F)$) is right adjoint to the inclusion $\mathrm{Im}\,(F) \hookrightarrow \mathbf{C}$ and thus $\mathrm{Im}(F)$ is a coreflective subcategory of $\mathbf{C}$.

Therefore the endo-adjunction can be analyzed or parsed into two adjunctions between *different* categories, a reflection and a coreflection. It was previously noted that in the case of a reflection, i.e., a left adjoint to the inclusion functor, heteromorphisms can be found as the morphisms with their tails in the ambient category and their heads in the subcategory. For a coreflection (right adjoint to the inclusion functor), the heteromorphisms would be turned around, i.e., would have their tails in the subcategory and their heads in the ambient category. The heteromorphic theory applies to each of these adjunctions.

In the case of the exponential endo-adjunction on **Set**, $\mathbf{X} = \mathbf{Set} = \mathbf{A}$. To parse the adjunction as a reflection, let **APower** be the subcategory of $G(-) = (-)^A$ images (G is one-one since A is non-empty) so that $\mathbf{APower} \hookrightarrow \mathbf{Set}$, and that inclusion functor has a left adjoint $GF(-) = (- \times A)^A : \mathbf{Set} \to \mathbf{APower}$. Then the heteromorphisms are those with their tail in **Set** and head in **APower**, i.e., the morphisms of the form $X \to Y^A$. But now we have the het unit and counit in accordance with the heteromorphic theory. The het unit $h_X : X \Rightarrow GFX = (X \times A)^A$ is the canonical map that takes an x in X to the function $(x,-) : A \to X \times A$ which takes a in A to $(x,a) \in X \times A$. This is the 'same' as the ordinary unit $\eta_X : X \to (X \times A)^A$ in the original product-exponential adjunction, i.e., it is the 'same' modulo the fact that h_x is viewed as a heteromorphism with its tail in **Set** and its head in **APower** while the same set-map η_X is viewed as a homomorphism in **Set**. Since the right adjoint in the reflective case is the inclusion, the het counit $e_{Y^A} : Y^A \to Y^A$ is the identity but seen as a heteromorphism from an object in **Set** to the same object in **APower**. The het adjunctive square then is the following commutative diagram.

$$\begin{array}{ccccc} & X & \xrightarrow{f} & Y^A & \\ h_X & \Downarrow & \searrow^{f} & \Downarrow & e_{Y^A} \\ & (X \times A)^A & \xrightarrow{(f^*)^A} & Y^A & \end{array}$$

Het adjunctive square for exponential endo-adjunction parsed as a reflection

As a coreflection, let **AProd** be the subcategory of $F(-) = - \times A$ images (F is one-one since A is non-empty) so that $\mathbf{AProd} \hookrightarrow \mathbf{Set}$, and that inclusion functor has a right adjoint $FG(-) = (-)^A \times A : \mathbf{Set} \to \mathbf{AProd}$. Then the heteromorphisms are those with their tail in **AProd** and their head in **Set**, i.e., the morphisms of the form $X \times A \to Y$. And now we again have the het unit and counit. The het counit $e_Y : Y^A \times A = FGY \Rightarrow Y$ is the evaluation map which is the 'same' as the ordinary counit $\varepsilon_Y : Y^A \times A \to Y$ in the original product-exponential adjunction. Since the left adjoint in the coreflective case is the inclusion, the het unit $h_{X \times A} : X \times A \Rightarrow X \times A$ is the identity (again seen as a heteromorphism). The het adjunctive square is then the following commutative diagram.

$$\begin{array}{ccccc} & X \times A & \xrightarrow{g^* \times A} & Y^A \times A & \\ h_{X \times A} & \Downarrow & \searrow^{g} & \Downarrow & e_Y \\ & X \times A & \xrightarrow{g} & Y & \end{array}$$

Het adjunctive square for exponential endo-adjunction parsed as a coreflection

One might well ask: "Why the special treatment since the heteromorphisms are supposed to given (up to isomorphism) by the hom-pair adjunctive square diagram of the adjunction representation theorem?" The answer is that this is exactly what has been derived. For the exponential adjunction, the hom-pair adjunctive square diagram is as follows:

$$\begin{array}{ccccc} & (X, X \times A) & \stackrel{(f, g^* \times A)}{\longrightarrow} & \left(Y^A, Y^A \times A\right) & \\ (\eta_X, 1_{X \times A}) & \downarrow & \searrow^{(f,g)} & \downarrow & (1_{Y^A}, \varepsilon_Y) \\ & \left((X \times A)^A, X \times A\right) & \stackrel{((f^*)^A, g)}{\longrightarrow} & \left(Y^A, Y\right) & \end{array}$$

Hom-pair adjunctive square for exponential adjunction

The left component of each pair is the 'same' as the het adjunctive square for the adjunction parsed as a reflection (see the previous diagram for the reflection) modulo the point that the codomains of the vertical maps are taken as **APower** so they become heteromorphisms (e.g., η_X becomes h_X). Dually, the right component of each pair is the 'same' as the het adjunctive square for the adjunction parsed as a coreflection (see the previous diagram) modulo the point that the domains of the vertical maps are taken as **AProd** so they become heteromorphisms (e.g., ε_Y becomes e_Y).

5 Concluding Remarks

This paper inevitably has two themes. The main theme is showing how adjoint functors arise from the representations within two categories of the heteromorphisms between the categories. But the logically prior theme is showing that heteromorphisms can be rigorously treated as part of category theory—rather than just as stray chimeras roaming in the wilds of mathematical practice.

Taking the logically prior theme first, category theory has always been presented as embodying the idea of grouping mathematical objects of a certain sort together with their appropriate morphisms in a "category." In some respects, this homomorphic theme became the leading theme just as in Felix Klein's Erlanger Program where geometries were characterized by the invariants of a specified class of transformations. Indeed, in their founding paper, Eilenberg and MacLane noted that category theory "may be regarded as a continuation of the Klein Erlanger Program, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings." [6, p. 237] Hence the whole concept of a "heteromorphism" between objects of different categories has seemed like a cross-species hybrid that is out-of-place and running against the spirit of the enterprise. Functors were defined to handle all the external relations between categories so object-to-object inter-category morphisms had no 'official' role.

At the outset of this paper, a number of testimonials were quoted about the centrality of adjunctions in category theory. Once heteromorphisms were rigorously treated using het-bifunctor (in analogy to treating homomorphisms with hom-bifunctors), it quickly became clear that an adjunction between two categories was closely related to the heteromorphisms between the objects of the two categories. Our main theme is that left and right adjoints arise as the left and right representations within the categories of the heteromorphisms between the categories. Given the importance of adjoints, this made an argument for taking heteromorphisms "out of the closet" and recognizing them as part of the conceptual family of category theory.[12]

When heteromorphisms (and the related concepts such as het natural transformations) are accepted as a part of category theory—as is argued here, then it does imply some "broadening" of that Erlanger-style theme. Mathematical objects in different categories may nevertheless have *partial* similarities of structure which would be expressed by heteromorphisms between the categories. Adjoints arise from universals that represent and internalize those structural external relationships within each of the categories. And that may help explain why adjoints have emerged as the principal lens to focus on what is important in mathematics. Structures are important [11] that have within them universal models representing the relationships with external entities of a different kind.

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[12] Indeed, one might ask why it has taken so long for heteromorphisms to be formally recognized? The reticence seems to come from heteromorphisms not fitting into the Erlanger-style theme of emphasizing homomorphisms between mathematical objects of the same category.

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