The Objective Indefiniteness Interpretation of Quantum Mechanics

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Abstract

The common-sense view of reality is expressed logically in Boolean subset logic (each element is either definitely in or not in a subset, i.e., either definitely has or does not have a property). But quantum mechanics does not agree with this "properties all the way down" picture of micro-reality. Are there other coherent alternative views of reality? A logic of partitions, dual to the Boolean logic of subsets (partitions are dual to subsets), was recently developed along with a logical version of information theory. In view of the subset-partition duality, partition logic is *the* alternative to Boolean subset logic and thus it abstractly describes the alternative dual view of micro-reality. Perhaps QM is compatible with this dual view? Indeed, when the mathematics of partitions using sets is "lifted" from sets to vector spaces, then it yields the mathematics and relations of quantum mechanics. Thus the vision of micro-reality abstractly characterized by partition logic matches that described by quantum mechanics. The key concept explicated by partition logic is the old idea of "objective indefiniteness" (emphasized by Shimony). Thus partition logic, logical information theory, and the lifting program provide the back story so that the old idea then yields the *objective indefiniteness interpretation* of quantum mechanics.

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1 Introduction: the back story for objective indefiniteness

Classical physics is compatible with the common-sense view of reality that is expressed at the logical level in Boolean subset logic. Each element in the Boolean universe set is either definitely in or not in a subset, i.e., each element either definitely has or does not have a property. Each element is characterized by a full set of properties, a view that might be referred to as "properties all the way down."

It is now rather widely accepted that this common-sense view of reality is not compatible with quantum mechanics (QM). If we think in terms of only two positions, *here* and *there*, then in classical physics a particle is either definitely *here* or *there*, while in QM, the particle can be "neither definitely here nor there."[29, p. 144][1] This is not an epistemic or subjective indefiniteness of location; it is an ontological or objective indefiniteness. The notion of *objective indefiniteness* in QM has been most emphasized by Abner Shimony ([25],[26]).

> From these two basic ideas alone – indefiniteness and the superposition principle – it should be clear already that quantum mechanics conflicts sharply with common sense. If the quantum state of a system is a complete description of the system, then a quantity that has an indefinite value in that quantum state is objectively indefinite; its value is not merely unknown by the scientist who seeks to describe the system. ...Classical physics did not conflict with common sense in these fundamental ways.[25, p. 47]

Other quantum philosophers have used similar concepts. For instance, in his discussion of Heisenberg's uncertainty[2] principle, Paul Feyerabend asserted that "inherent indefiniteness is a universal and objective property of matter."[11, p. 202] Thus one path to arrive at the notion of "inherent indefiniteness" is to understand that Heisenberg's indefiniteness principle is *not* about the clumsiness of instruments in simultaneously measuring incompatible observables that always have definite values.

But there the development seems to have stalled. What was the logic that plays the role analogous to Boolean subset logic for the notion of objective indefiniteness? And given such a logic, how would one fill in the gap between the austere level of logic and the rich mathematical framework of quantum mechanics?

These questions can now be answered. The logic of objective indefiniteness that plays the role analogous to subset logic is the recently developed dual logic of partitions.[9] The dual relationship between subsets and partitions (explained below) shows that partition logic is not just an alternative but is *the* alternative to subset logic. Moreover, Boole developed a logical finite probability theory out of his logic of subsets [1], and the analogous theory developed out of the logic of partitions is a logical version of information theory.[8]

The concepts and operations of partition logic and logical information theory are developed in the rather austere set-theoretic context; they needed to be "lifted" to the richer environment of vector spaces. This lifting program from sets to vector spaces is part of the mathematical folklore (e.g., used intuitively by von Neumann). When applied to the concepts and operations of partition mathematics, the lifting program indeed yields the mathematics of quantum mechanics. This corroborates that the vision of micro-reality provided by *the* dual form of logic (i.e., partition logic rather than subset logic) is, in fact, the micro-reality described by QM. Thus the development of the logic of partitions, logical information theory, and the lifting program provides the back story to the notion of objective indefiniteness. The result is the objective indefiniteness interpretation of quantum mechanics.

[1] This is usually misrepresented in the popular literature as the particle being "both *here* and *there* at the same time." Weinberg also mentions a particle "spinning neither definitely clockwise nor counterclockwise" and then notes that for elementary particles, "it is possible to have a particle in a state in which it is neither definitely an electron nor definitely a neutrino until we measure some property that would distinguish the two, like the electric charge."[29, pp. 144-145 (thanks to Noson Yanofsky for this reference)]

[2] Heisenberg's German word was "Unbestimmtheit" which could well be translated as "indefiniteness" or "indeterminateness" rather than "uncertainty."

2 The logic of partitions

2.1 From "propositional" logic to subset logic

George Boole [1] originally developed his logic as the logic of subsets. As noted by Alonzo Church:

> The algebra of logic has its beginning in 1847, in the publications of Boole and De Morgan. This concerned itself at first with an algebra or calculus of classes,... a true propositional calculus perhaps first appeared... in 1877.[4, pp. 155-156]

In the logic of subsets, a *tautology* is defined as a formula such that no matter what subsets of the given universe U are substituted for the variables, when the set-theoretic operations are applied, then the whole formula evaluates to U. Boole noted that to determine these valid formulas, it suffices to take the special case of $U = 1$ which has only two subsets $0 = \emptyset$ and 1. Thus what was later called the "truth table" characterization of a tautology was a theorem, not a definition.[3]

But over the years, the whole became identified with the special case. The Boolean logic of subsets was reconceptualized as "propositional logic" and the truth-table characterization of a tautology became the definition of a tautology. This facilitated the further analysis of the propositional atoms into statements with quantifiers and the development of model theory. But the restricted notion of "propositional" logic also had a downside; it hid the idea of a dual logic since propositions don't have duals.

Subsets and partitions (or equivalence relations or quotient sets) are dual in the category-theoretic sense of the duality between monomorphisms and epimorphisms. This duality is familiar in abstract algebra in the interplay of subobjects (e.g., subgroups, subrings, etc.) and quotient objects. William Lawvere calls the general category-theoretic notion of a subobject a *part*, and then he notes: "The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition."[20, p. 85] The image of monomorphic or injective map between sets is a subset of the codomain, and dually the inverse-image of an epimorphic or surjective map between sets is a partition of the domain. But the development of the dual logic of partitions was delayed by the conceptualization of subset logic as "propositional" logic.

2.2 Basic concepts of partition logic

In the Boolean logic of subsets, the basic algebraic structure is the Boolean lattice $\wp(U)$ of subsets of a universe set U enriched by the implication $A \Rightarrow B = A^c \cup B$ to form the Boolean algebra of subsets of U. In a similar manner, we form the lattice of partitions on U enriched by the partition operation of implication and other partition operations.

Given a universe set U, a *partition* π on U is a set of non-empty subsets or blocks $\{B\}$ of U that are pairwise disjoint and whose union is U. Given two partitions $\pi = \{B\}$ and $\sigma = \{C\}$ on the same universe U, the partition σ is *refined* by π, written by $\sigma \preceq \pi$, if for every block $B \in \pi$, there is a block $C \in \sigma$ such that $B \subseteq C$. Given the two partitions on the same universe, their *join* $\pi \vee \sigma$ is the partition whose blocks are the non-empty intersections $B \cap C$. To define the meet $\pi \wedge \sigma$, consider an undirected graph on U where there is a link between any two elements $u, u' \in U$ if they are in the same block of π or the same block of σ. Then the blocks of $\pi \wedge \sigma$ are the connected components of that graph. The *top* of the lattice is the *discrete partition* $\mathbf{1} = \{\{u\} : u \in U\}$ whose

[3] Alfred Renyi [23] gave a generalization of the theorem to probability theory.

blocks are all the singletons, and the *bottom* is the *indiscrete partition* (nicknamed the "blob") $\mathbf{0} = \{\{U\}\}$ whose only block is all of U. This defines the *lattice of partitions* $\prod(U)$ on U.[4]

As late as 2001, it was noted that:

> the only operations on the family of equivalence relations fully studied, understood and deployed are the binary join $\vee$ and meet $\wedge$ operations.[2, p. 445]

For anything worthy to be called "partition logic," an operation of implication would be needed if not partition versions of all the sixteen binary subset operations. Given $\pi = \{B\}$ and $\sigma = \{C\}$, the *implication* $\sigma \Rightarrow \pi$ is the partition whose blocks are like the blocks of π except that whenever a block B is contained in some block $C \in \sigma$, then B is discretized, i.e., replaced by the singletons of its elements. If we think of a whole block B as a mini-$\mathbf{0}$ and a discretized B as a mini-$\mathbf{1}$, then the implication $\sigma \Rightarrow \pi$ is just the indicator function for the inclusion of the π-blocks in the σ-blocks. In the Boolean algebra $\wp(U)$, the implication is related to the partial order by the relation, $A \Rightarrow B = U$ iff $A \subseteq B$, and we immediately see that the corresponding relation holds in the partition lattice $\prod(U)$ enriched with implication, i.e., $\sigma \Rightarrow \pi = \mathbf{1}$ (discrete partition) iff $\sigma \preceq \pi$.

There are at least two algorithms to define partition operations in terms of the corresponding subset operations. We will use the representation of the partition lattice $\prod(U)$ as a lattice of subsets of $U \times U$.[5] Given a partition $\pi = \{B\}$ on U, the *distinctions* or *dits* of π are the ordered pairs (u, u') where u and u' are in distinct blocks of π, and $\operatorname{dit}(\pi)$ is the *set of distinctions* or *dit set* of π. Similarly, an *indistinction* or *indit* of π is an ordered pair (u, u') where u and u' are in the same block of π, and $\operatorname{indit}(\pi)$ is the *indit set* of π. Of course, $\operatorname{indit}(\pi)$ is just the equivalence relation determined by π, and it is the complement of $\operatorname{dit}(\pi)$ in $U \times U$.

The complement of an equivalence relation is properly called a *partition relation*. An equivalence relation is reflexive, symmetric, and transitive, so a partition relation is anti-reflexive [i.e., contains no diagonal pairs (u, u)], symmetric, and anti-transitive where a binary relation R is *anti-transitive* if for any $(u, u') \in R$, and for any chain of elements $u = u_1, u_2, ..., u_n = u'$ from u to u', then for at least one of the pairs, $(u_i, u_{i+1}) \in R$. Otherwise all the consecutive pairs in the chain would be in the complement R^c which is transitive so $(u, u') \in R^c$ contrary to the assumption.

Every subset $S \subseteq U \times U$ has a reflexive-symmetric-transitive *closure* $\overline{S}$ which is the smallest equivalence relation containing S. Hence we can define an *interior* operation as the complement of the closure of the complement, i.e., $\operatorname{int}(S) = \left(\overline{S^c}\right)^c$, which is the largest partition relation included in S. While some motivation might be supplied by thinking of the partition relations as "open" subsets and the equivalence relations as "closed" subsets, they do not form a topology. The closure operation is not a topological closure operation since the union of two closed subsets is not necessarily closed, and the intersection of two open subsets is not necessarily open.

Every partition π is represented by its dit set $\operatorname{dit}(\pi)$. The refinement relation between partitions, $\sigma \preceq \pi$ is represented by the inclusion relation between dit sets, i.e., $\sigma \preceq \pi$ iff $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$.[6] The join $\pi \vee \sigma$ is represented in $U \times U$ by the union of the dit sets, i.e., $\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$. But the intersection of two dit sets is not necessarily a dit set so to find the dit set of the meet $\pi \wedge \sigma$, we

[4] Unfortunately in much of the literature of combinatorial theory, the refinement partial ordering is written the other way around (so Gian-Carlo Rota sometimes called it "unrefinement"), and thus the "join" and "meet" are reversed, and the lattice of partitions is then "upside-down."

[5] The other method uses graph theory as in the above definition of the meet. See [9] for the details.

[6] The more customary upside-down representation of the "lattice of partitions" uses the indit sets so it is actually the lattice of equivalence relations rather than the lattice of partition relations.

have to take the interior of the intersection of their dit sets, i.e., $\operatorname{dit}(\pi \wedge \sigma) = \operatorname{int}(\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma))$. These equations for the dit sets of the join and meet are theorems, not definitions, since the join and meet were already defined above. The general algorithm to represent a partition operation is to apply the corresponding set operation to the dit sets and then apply the interior to the result (if it is not already a partition relation). Thus, for instance,

$$\operatorname{dit}(\sigma \Rightarrow \pi) = \operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi)).$$

It is a striking fact (see [9] for a proof) that $\operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi))$ is the dit set of $\sigma \Rightarrow \pi$ previously defined as the indicator function for the inclusion of π-blocks in σ-blocks. In this manner, the lattice of partitions $\prod(U)$ enriched by implication and other partition operations can be represented by the lattice of partition relations $\mathcal{O}(U \times U)$ on $U \times U$.

Representation	$\prod(U)$	$\mathcal{O}(U \times U)$
Partition	π	$\operatorname{dit}(\pi)$
Refinement order	$\sigma \preceq \pi$	$\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$
Top	$\mathbf{1} = \{\{u\} : u \in U\}$	$\operatorname{dit}(\mathbf{1}) = U \times U - \Delta_U$ all dits
Bottom	$\mathbf{0} = \{\{U\}\}$	$\operatorname{dit}(\mathbf{0}) = \emptyset$ no dits
Join	$\pi \vee \sigma$	$\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$
Meet	$\pi \wedge \sigma$	$\operatorname{dit}(\pi \wedge \sigma) = \operatorname{int}(\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma))$
Implication	$\sigma \Rightarrow \pi$	$\operatorname{dit}(\sigma \Rightarrow \pi) = \operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi))$
Any logical op. #	$\sigma \# \pi$	Int. of subset op. # applied to dit sets

Lattice of partitions $\prod(U)$ represented as lattice of partition relations $\mathcal{O}(U \times U)$.

2.3 Analogies between subset logic and partition logic

The development of partition logic was guided by some basic analogies between the two dual forms of logic. The most basic analogy is that a distinction or dit of a partition is the analogue of an element of a subset–so saying that a partition distinguishes a pair (u, u') is the analogue of saying that a subset contains an element u. The top of the subset lattice is the universe set U of all possible elements and the top of the partition lattice is the partition $\mathbf{1}$ with all possible distinctions $\operatorname{dit}(\mathbf{1}) = U \times U - \Delta_U$ (all the ordered pairs minus the diagonal self-pairs which can never be distinctions). The bottoms of the lattices are the null subset $\emptyset$ of no elements and the indiscrete partition $\mathbf{0}$ of no distinctions. The partial order in the lattices is inclusion of elements and inclusion of distinctions.

Intuitively, a *property* on U is something that each element has or does not have (like a person being female or not), while intuitively an attribute on U is something that each element has but with various values (like the weight or height of a person). The subsets of U can be thought of as abstract versions of properties of the elements of U while the partitions on U are abstract versions of the attributes on U where the different blocks of a partition represent the different values of the attribute. Technically, an *attribute* is given by a function $f : U \to R$ (for some value set R) and the partition induced by the attribute is the inverse image partition $\{f^{-1}(r) \neq \emptyset : r \in R\}$.

We can use the same formulas for both subset logic and partition logic, and just interpret the variables as being either subsets of U or partitions π on U with the corresponding operations. In subset logic, we require $|U| \geq 1$ so the top and bottom of the subset lattice are not the same, and similarly in partition logic, we require $|U| \geq 2$ so the top and bottom of the partition lattice are not the same. A formula is a *subset tautology* if for any universe U ($|U| \geq 1$) and for any subsets of U substituted for the variables, the result of applying the subset operations is the top of the lattice U, i.e., the subset formula holds of all elements. A partition tautology is defined analogously. That is, a formula is a *partition tautology* if for any U ($|U| \geq 2$) and for any partitions on U substituted for the variables, the result of applying the partition operations is the top of the lattice, the discrete partition $\mathbf{1}$, i.e., the partition formula distinguishes all pairs (u, u') of distinct elements.

For the universe $U = 2 = \{0, 1\}$, the discrete partition $\mathbf{1} = \{\{0\}, \{1\}\}$ and the indiscrete partition $\mathbf{0} = \{\{0, 1\}\}$ are the only partitions. Moreover, the partition operations applied to these two partitions are isomorphic to the subset operations applied to the two subsets of a singleton set 1. For instance, we can describe the action of the partition implication by a "truth table."

σ	π	$\sigma \Rightarrow \pi$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

"Truth table" for partition implication in $\prod(2)$

Thus on the two partitions of 2, the partition implication is isomorphic to the subset implication in $\wp(1)$ and similarly for the other partition operations.

Proposition 1 *All partition tautologies are subset tautologies.*

Proof: If a formula is a partition tautology, then it will evaluate to $\mathbf{1}$ for all partitions on all universes $|U| \geq 2$ including $U = 2$. Then by the isomorphism $\prod(2) \cong \wp(1)$, the formula will evaluate to 1 for all subsets of 1 which suffices to make it a subset tautology. $\square$

The converse is not true. For instance, negation would be defined as $\neg\sigma = \sigma \Rightarrow \mathbf{0}$ which is always $\mathbf{0}$ except when $\sigma = \mathbf{0}$ in which case it is $\mathbf{0} \Rightarrow \mathbf{0} = \mathbf{1}$. Hence the excluded middle formula $\neg\sigma \vee \sigma$ would always evaluate to σ unless $\sigma = \mathbf{0}$ in which case, it evaluates to $\mathbf{1}$. Hence the formula does not evaluate to $\mathbf{1}$ for any $\sigma \neq \mathbf{0}, \mathbf{1}$. But the weak excluded middle formula $\neg\neg\sigma \vee \neg\sigma$ is a partition tautology even for the generalized relative π-negation $\overset{\pi}{\neg}\sigma = \sigma \Rightarrow \pi$, i.e., $((\sigma \Rightarrow \pi) \Rightarrow \pi) \vee (\sigma \Rightarrow \pi)$ is a partition tautology since each block $B \in \pi$ would be discretized by one of the disjuncts. The modus ponens formula $(\sigma \wedge (\sigma \Rightarrow \pi)) \Rightarrow \pi$ is also a partition tautology as can be readily checked.

There is no inclusion either way between the partition tautologies and the intuitionistic valid propositional formulas. The accumulation formula $\sigma \Rightarrow (\pi \Rightarrow (\pi \wedge \sigma))$ is valid in intuitionistic logic but not in partition logic, while the partition tautology of the weak excluded middle formula is not valid in intuitionistic logic.

	Subset Logic	Partition Logic
'Elements'	Elements $u \in U$	Distinctions $(u,u') \in (U\times U) - \Delta_U$
All 'elements'	Universe set U	Discrete partition **1** (all dits)
No 'elements'	Empty set $\emptyset$	Indiscrete partition **0** (no dits)
Variables in formulas	Subset $S \subseteq U$	Partition π on U
Interpretation	$f:S'\to U$ so $Im(S') = S$ defines *property* on U.	$f:U\to R$ so $f^{-1}(R) = \pi$ defines R-valued *attribute* on U.
Logical operations	Subset ops $\cup, \cap, \Rightarrow, \ldots$	Partition ops $\cong$ Interior of subset ops applied to dit-sets.
Formula $\Phi(\pi, \sigma,\ldots)$ holds of an 'element'	Element u is in $\Phi(\pi,\sigma,\ldots)$ as a subset.	A dit (u,u') is distinguished by $\Phi(\pi,\sigma,\ldots)$ as a partition.
Valid formula $\Phi(\pi,\sigma,\ldots)$	$\Phi(\pi,\sigma,\ldots) = U$ (top) for any subsets $\pi,\sigma,\ldots$ of any U ($1 \le \lvert U\rvert$).	$\Phi(\pi,\sigma,\ldots) = \mathbf{1}$ (top = discrete partition) for any partitions $\pi,\sigma,\ldots$ on any U ($2 \le \lvert U\rvert$).

Table of analogies between subset logic and partition logic

For a more complete treatment of the basics of partition logic, including the correctness and completeness theorems for a system of partition tableaus, see [9]. For our purposes in lifting the mathematics of partition logic to vector spaces, the most important operation is the join operation that "joins" together the distinctions of two partitions.

3 Logical information theory

We have so far made no assumptions about the finitude of the universe U. For a finite universe U, Boole developed the "logical" version of finite probability theory by assigning the quantitative measure of the relative cardinality $\Pr(S) = \frac{|S|}{|U|}$ to each subset which can be interpreted as a probability under the Laplacian assumption of equiprobable elements. Using the elements-distinctions analogy, we can assign the analogous quantitative measure of the relative cardinality of the dit set of a partition $h(\pi) = \frac{|\text{dit}(\pi)|}{|U\times U|}$ to each partition which can be interpreted as the *logical information content* or *logical entropy* of the partition. Under the assumption of equiprobable elements, the logical entropy of a partition can be interpreted as the probability that two drawings from U (with replacement) will give a distinction of the partition.

	Logical Finite Prob. Theory	Logical Information Theory
'Outcomes'	Elements $u \in U$ finite	Pairs $(u,u') \in U\times U$ finite
'Events'	Subsets $A \subseteq U$	Partitions π, i.e., $\text{dit}(\pi) \subseteq U\times U$
Normalized size	$\Pr(A) = \lvert A\rvert/\lvert U\rvert$ = number of elements (normalized).	$h(\pi) = \lvert\text{dit}(\pi)\rvert/\lvert U\times U\rvert$ = *Logical Entropy* of partition π = number of distinctions (normalized).
Equiprobable outcomes	$\Pr(A)$ = probability randomly drawn element is in subset A	$h(\pi)$ = probability randomly drawn pair (w/replacement) is distinguished by partition π

Logical probability theory is to subset logic
as logical information theory is to partition logic

The probability of drawing an element from a block $B \in \pi$ is $p_B = \frac{|B|}{|U|}$ so the logical entropy of a partition can be written in terms of these block probabilities since $|\text{dit}(\pi)| = \sum_{B \neq B' \in \pi} |B \times B'| = |U|^2 - \sum_{B \in \pi} |B|^2$. Hence:

$$h(\pi) = \frac{|\text{dit}(\pi)|}{|U \times U|} = \frac{|U|^2 - \sum_{B \in \pi} |B|^2}{|U|^2} = 1 - \sum_{B \in \pi} p_B^2.$$

This formula has a long history (see [8]) and is usually called the *Gini-Simpson diversity index* in the biological literature [22]. For instance, if we partition animals by species, then it is the probability in two independent samples that we will find animals of different species.

This version of the logical entropy formula also makes clear the generalization path to define the logical entropy of any finite probability distribution $p = (p_1, ..., p_n)$:

$$h(p) = 1 - \sum_i p_i^2.$$[7]

C. R. Rao [22] has defined a general notion of quadratic entropy in terms of a distance function $d(u, u')$ between the elements of U. In the most general "logical" case, the natural logical distance function is:

$$d(u, u') = 1 - \delta(u, u') = \begin{cases} 1 \text{ if } u \neq u' \\ 0 \text{ if } u = u' \end{cases}$$

and, in that case, the quadratic entropy is just the logical entropy.

The Shannon entropy for a partition:

$$H(\pi) = \sum_{B \in \pi} p_B \log_2 \left(\frac{1}{p_B}\right)$$

can also be interpreted in terms of distinctions. In the special case where π has 2^n equal-sized blocks, then $H(\pi) = 2^n \frac{1}{2^n} \log_2 \left(\frac{1}{1/2^n}\right) = \log_2(2^n) = n$. Think of the 2^n blocks as being enumerated by an n-digit binary number. Then the n questions, "Is the i^{th} digit a 1?" will partition the blocks into two equal groups and thus will partition U into two equal blocks. Thus each of the n questions gives a binary partition of U into two equal parts, and the join of those n binary partitions is the original partition π. Thus the Shannon entropy $H(\pi) = \log_2 \left(\frac{1}{p_B}\right) = \log_2 \left(\frac{1}{1/2^n}\right) = n$ in this case is the number of equal binary partitions ("bits") necessary to make all the distinctions of π. The general formula $H(\pi) = \sum_{B \in \pi} p_B \log_2 \left(\frac{1}{p_B}\right)$ can then be seen as the *average* number of equal binary partitions or bits necessary to make all the distinctions of π. In contrast, the logical entropy $h(\pi)$ involves no averaging in its interpretation as the normalized number of distinctions in π.

A number of compound entropy concepts (e.g., mutual entropy, cross-entropy, divergence) are defined for Shannon entropy, and corresponding concepts are easily defined for logical entropy. The following table summarizes the relationships where two partitions $\pi = \{B\}$ and $\sigma = \{C\}$ are *stochastically independent* if for any $B \cap C \neq \emptyset$, $p_{B \cap C} = p_B p_C$, and where $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ are two finite probability distributions.

[7] In the general case, the p_i becomes a probability density function and the summation an integral.

	Shannon Entropy	Logical Entropy
Entropy	$H(\pi) = \Sigma_B p_B \log(1/p_B)$	$h(\pi) = \Sigma_B p_B (1-p_B)$
Mutual Information	$I(\pi;\sigma) =$ $H(\pi)+H(\sigma)-H(\pi\vee\sigma)$	$\mathrm{mut}(\pi;\sigma) =$ $h(\pi)+h(\sigma)-h(\pi\vee\sigma)$
Independence	$I(\pi;\sigma) = 0$	$\mathrm{mut}(\pi;\sigma) = h(\pi)h(\sigma)$
Cross entropy	$H(p\|\|q) = \Sigma p_i \log(1/q_i)$	$h(p\|\|q) = \Sigma p_i(1-q_i)$
Divergence	$D(p\|\|q) = H(p\|\|q)-H(p)$	$d(p\|\|q) = 2h(p\|\|q) - h(p) - h(q)$
Information Inequality	$D(p\|\|q) \geq 0$ with $= 0$ iff $p_i = q_i$ for all i.	$d(p\|\|q) \geq 0$ with $= 0$ iff $p_i = q_i$ for all i.

Corresponding concepts for Shannon entropy and logical entropy

Further details about logical information theory can be found in [8]. For our purposes here, the important thing is the lifting of logical entropy to the context of vector spaces and quantum mathematics where for any density matrix ρ, the logical entropy $h(\rho) = 1 - \mathrm{tr}\left[\rho^2\right]$ allows us to directly measure and interpret the changes made in a measurement.

4 Partitions and objective indefiniteness

4.1 Representing objective indistinctness

It has already been emphasized how Boolean subset logic captures at the logical level the common sense vision of reality where an entity definitely has or does not have any property. We can now describe how the dual logic of partitions captures at the logical level a vision of reality with objectively indefinite (or indistinct)[8] entities. The key step is to interpret a subset such as a block B in a partition, not as a subset of the distinct elements $u \in B$, but as a *single objectively indistinct element* that, with further distinctions, could become any of the fully distinct elements $u \in B$. To anticipate the lifted concepts in vector spaces, the fully distinct elements $u \in U$ might be called "eigen-elements" and the single indistinct element B is a "superposition" of the eigen-elements $u \in B$ (thinking of the collecting together $\{u, u', ...\} = B$ of the elements of B as their "superposition"). With distinctions, the indistinct element B might be refined into one of the singletons $\{u\}$ for $u \in B$ [where $\{u\}$ is the "superposition" consisting of a single eigen-element so it just denotes that element u].

Abner Shimony ([25] and [26]), in his description of a superposition state as being objectively indefinite, adopted Heisenberg's [15] language of "potentiality" and "actuality" to describe the relationship of the eigenstates that are superposed to give an objectively indefinite superposition. This terminology could be adapted to the case of the sets. The elements $u \in B$ are "potential" in the objectively indefinite "superposition" B, and, with further distinctions, the indefinite element B might "actualize" to $\{u\}$ for one of the "potential" $u \in B$. Starting with B, the other $u \notin B$ are not "potentialities" that could be "actualized" with further distinctions.

[8] The adjectives "indefinite" and "indistinct" will be used interchangeably as synonyms. The word "indefiniteness" is more common in the QM literature, but "indistinctness" has a better noun form as "indistinctions" (with the opposite as "distinctions").

This terminology is, however, somewhat misleading since the indefinite element B is perfectly actual; it is only the multiple eigen-elements $u \in B$ that are "potential" until "actualized" by some further distinctions. In a "measurement," a single actual indefinite element becomes a single actual definite element. Since the "measurement" goes from actual indefinite to actual definite, the potential-to-actual language of Heisenberg should only be used with proper care–if at all.

Consider a three-element universe $U = \{a, b, c\}$ and a partition $\pi = \{\{a\}, \{b, c\}\}$. The block $\{b, c\}$ is objectively indefinite between $\{b\}$ and $\{c\}$ so those singletons are its "potentialities" in the sense that a distinction could result in either $\{b\}$ or $\{c\}$ being "actualized." However $\{a\}$ is not a "potentiality" when one is starting with the indefinite element $\{b, c\}$.

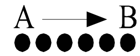
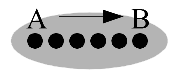
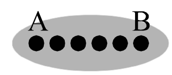
Note that this objective indefiniteness is not well-described as saying that indefinite pre-distinction element is "simultaneously both b and c"; it is indefinite between b and c. That is, a "superposition" should *not* be thought of like a double exposure photograph which has two fully definite images. That imagery is a holdover from classical wave imagery (e.g., in Fourier analysis) where definite eigen-waveforms are superposed to give a superposition waveform. Instead, the objectively indistinct element is like an out-of-focus photograph that with some sharpening could be resolved into one of two or more definite images. Yet one needs some way to indicate what are the definite eigen-elements that could be "actualized" from a single indefinite element B, and that is the role in the set case of conceptualizing B as a collecting together or a "superposition" of certain "potential" eigen-elements u.

The following is another attempt to clarify the imagery.

Eigenstate 1: Guy Fawkes with goatee	
Eigenstate 2: Guy Fawkes with mustache	
Objectively indistinct state before (facial hair) distinctions were made is pre-distinction state.	
But—objectively indistinct state may be *represented* by math superposition of the possible distinct alternatives: $[\vert goatee\rangle + \vert mustache\rangle]/\sqrt{2}$.	

Indistinct pre-distinction state *represented* as superposition

The following table gives yet another attempt at visualization by contrasting a classical picture and an objectively indefinite (or "quantum") picture of a "particle" getting from A to B.

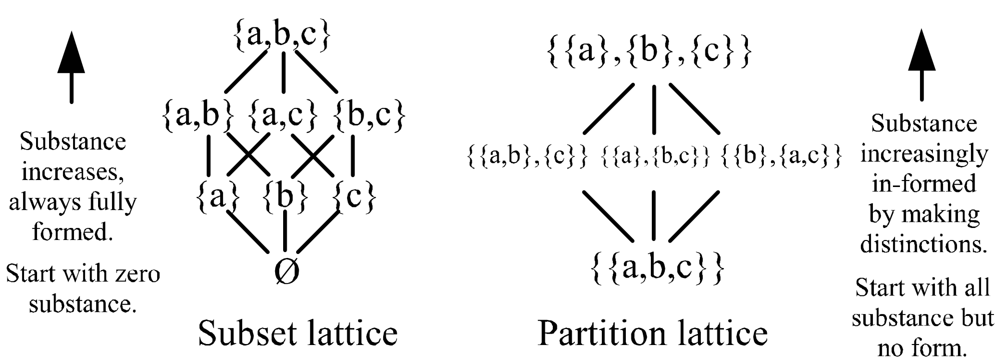
Classical trajectory from A to B	
Subjective indefiniteness of classical position ("cloud of ignorance")	
"Quantum trajectory": like def. focus at A, going out of focus, & new def. focus at B	
Particle with objectively indefinite location…	
may be *represented* as superposition of possible eigen-positions.	

Getting from A to B in classical and quantum ways

The classical trajectory is a sequence of definite positions. A state of subjective indefiniteness is compatible with a classical trajectory when we have a "cloud of ignorance" about the actual definite location of the particle. The "quantum trajectory" might be envisaged as starting with a definite focus or location at A, then evolving to an objectively indefinite state (with the various positions as potentialities), and then finally another "look" or measurement that achieves a definite focus at location B. The particle in its objectively indefinite position state is represented as the superposition of the possible definite position states.

4.2 The conceptual duality between the two lattices

The conceptual duality between the lattice of subsets and the lattice of partitions could be described using the rather meta-physical notions of substance and form. Consider what happens when one starts at the bottom of each lattice and moves towards the top.



Conceptual duality between the two lattices

At the bottom of the Boolean lattice is the empty set $\emptyset$ which represents no substance. As one moves up the lattice, new fully propertied elements of substance appear until finally one reaches the top, the universe U. Thus new substance is created but each element is fully formed and distinguished in terms of its properties.

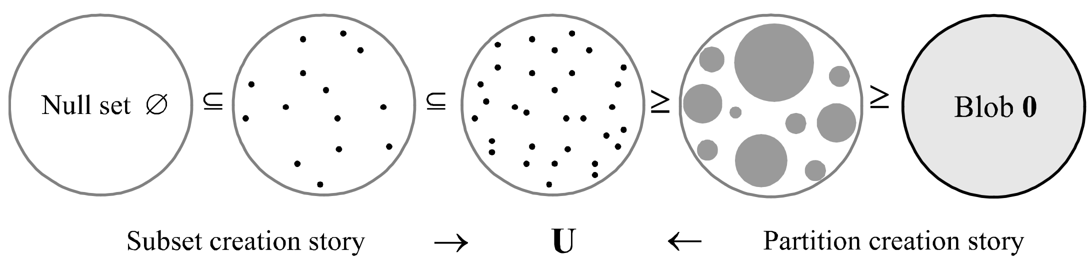
At the bottom of the partition lattice is the blob $\mathbf{0}$ which represents all the substance but with no distinctions to in-form the substance. As one moves up the lattice, no new substance appears but distinctions objectively in-form the indistinct elements as they become more and more distinct, until one finally reaches the top, the discrete partition $\mathbf{1}$, where all the eigen-elements of U have been fully distinguished from each other. Thus one ends up at the same place either way, but by two totally different but dual ways.

The notion of logical entropy expresses this idea of objective in-formation as the normalized count of the informing distinctions. For instance, in the partition lattice on a three element set pictured above, the logical entropy of the blob is always $h(\mathbf{0}) = 0$ since there are no distinctions. For a middle partition such as $\pi = \{\{a\}, \{b, c\}\}$, the distinctions are (a, b), (b, a), (a, c), and (c, a) for a total of 4 where $|U|^2 = 3^2 = 9$ so the logical entropy is $h(\pi) = \frac{|\operatorname{dit}(\pi)|}{|U \times U|} = \frac{4}{9}$. For the discrete partition, there are all possible distinctions for a total of $|U|^2 - |\Delta_U| = 9 - 3 = 6$ so the logical entropy is $h(\mathbf{1}) = 1 - \frac{1}{|U|} = \frac{6}{9}$. In each case, the logical entropy of a partition is the probability that two independent draws from U will yield a distinction of the partition.

The progress from bottom to top of the two lattices could also be described as two creation stories.

- *Subset creation story*: "In the Beginning was the Void", and then elements are created, fully propertied and distinguished from one another, until finally reaching all the elements of the universe set U.
- *Partition creation story*: "In the Beginning was the Blob", which is an undifferentiated "substance," and then there is a "Big Bang" where elements ("its") are created by being objectively in-formed (objective "dits") by the making of distinctions (e.g., breaking symmetries) until the result is finally the singletons which designate the elements of the universe U.[9]

These two creation stories might also be illustrated as follows.



[9]Heisenberg identifies the "substance" with energy.

> Energy is in fact the substance from which all elementary particles, all atoms and therefore all things are made, and energy is that which moves. Energy is a substance, since its total amount does not change, and the elementary particles can actually be made from this substance as is seen in many experiments on the creation of elementary particles.[15, p. 63]

In his sympathetic interpretation of Aristotle's treatment of substance and form, Heisenberg refers to the substance as: "a kind of indefinite corporeal substratum, embodying the possibility of passing over into actuality by means of the form."[15, p. 148] It was previously noted that Heisenberg's "potentiality" "passing over into actuality by means of the form" should be seen as the actual indefinite "passing over into" the actual definite by being objectively in-formed through the making of distinctions.

Two ways to create a universe U

One might think of the universe U (in the middle of the above picture) as the macroscopic world of fully definite entities that we ordinarily experience. Common sense and classical physics assumes, as it were, the subset creation story on the left. But *a priori,* it could just as well have been the dual story, the partition creation story pictured on the right, that leads to the *same* macro-picture U. And, as we will see, that is indeed the message of quantum mechanics.

5 The Lifting Program

5.1 From sets to vector spaces

We have so far outlined the mathematics of set partitions such as the representation of an indefinite element as a (non-singleton) block in a partition and carving out the fully distinct eigen-elements by making more distinctions, e.g., joining together the distinctions of different partitions (on the same universe). The lifting program lifts these set-based concepts to the much richer environment of vector spaces.

Why vector spaces? Dirac [7] noted that the notion of superposition was basic to and characteristic of quantum mechanics. At the level of sets, there is only a very simple and austere notion of "superposition," namely collecting together definite eigen-elements into one subset interpreted as one indefinite element (indistinct between the "superposed" eigen-elements). In a vector space, superposition is represented by a weighted vector sum with weights drawn from the base field.[10] Thus the lifting of set concepts to vector spaces (Hilbert spaces in particular) gives a much richer version of partition mathematics, and, as we will see, the lifting gives the mathematics of quantum mechanics.

The lifting program is not an algorithm but there is a guiding:

Basis Principle: *Apply the set concept to a basis set and then generate the lifted vector space concept.*

For instance, what is the vector space lift of the set concept of cardinality? We apply the set concept of cardinality to a basis set of a vector space where it yields the notion of *dimension* of the vector space (after checking that all bases have equal cardinality). Thus the lift of set-cardinality is not the cardinality of a vector space but its dimension.[11] Thus the null set $\emptyset$ with cardinality 0 lifts to the trivial zero vector space with dimension 0.

It is often convenient to refer to a set concept in terms of its lifted vector space concept. This will be done by using the name of the vector space concept enclosed in scare quotes, e.g., the cardinality of a set is its "dimension."

5.2 Lifting set partitions

To lift set partition mathematics to vector spaces, the first question is the lift of a set partition. The answer is immediately obtained by applying the set concept of a partition to a basis set and

[10] A vector expressed in a certain basis can be thought as a "multi-set" with a field element assigned as a weight to each element of the basis set.

[11] In QM, the extension of concepts on finite dimensional Hilbert space to infinite dimensional ones is well-known. Since our expository purpose is conceptual rather than mathematical, we will stick to finite dimensional spaces.

then seeing what it generates. Each block B of the set partition of a basis set generates a subspace $W_B \subseteq V$, and the subspaces together form a *direct sum decomposition*: $V = \sum_B \oplus W_B$. Thus the proper lifted notion of a partition for a vector space is *not* a set partition of the space, e.g., defined by a subspace $W \subseteq V$ where $v \sim v'$ if $v - v' \in W$, but is a direct sum decomposition of the vector space.[12] Or put the other way around (i.e., delifted), a set partition is a "direct sum decomposition" of a set.

5.3 Lifting partition joins

The main partition operation that we need to lift to vector spaces is the join operation. Two set partitions cannot be joined unless they are *compatible* in the sense of being defined on the same universe set. This notion of compatibility lifts to vector spaces by defining two vector space partitions $\omega = \{W_\lambda\}$ and $\xi = \{X_\mu\}$ on V as being *compatible* if there is a basis set for V so that the two vector space partitions arise from two set partitions of that common basis set.

If two set partitions $\pi = \{B\}$ and $\sigma = \{C\}$ are compatible, then their *join* $\pi \vee \sigma$ is defined as the set partition whose blocks are the non-empty intersections $B \cap C$. Similar the lifted concept is that if two vector space partitions $\omega = \{W_\lambda\}$ and $\xi = \{X_\mu\}$ are compatible, then their *join* $\omega \vee \xi$ is defined as the vector space partition whose subspaces are the non-zero intersections $W_\lambda \cap X_\mu$. And by the definition of compatibility, we could generate the subspaces of the join $\omega \vee \xi$ by the blocks in the join of the two set partitions of the common basis set.

5.4 Lifting attributes

A set partition might be seen as an abstract rendition of the inverse image partition $\{f^{-1}(r)\}$ defined by some concrete attribute $f : U \to \mathbb{R}$ on U (where we take the value set as the reals since that is also the relevant value set for QM). What is the lift of an attribute? At first glance, the basis principle would seem to imply: define a set attribute on a basis set (with values in the base field) and then linearly generate a functional from the vector space to the base field. But a functional does not define a vector space partition; it only defines the set partition of the vector space determined by the kernel of the functional. Hence we need to try a more careful application of the basis principle.

It is helpful to first give a suggestive reformulation of a set attribute $f : U \to \mathbb{R}$. If f is constant on a subset $S \subseteq U$ with a value r, then we might symbolize this as:

$$f \upharpoonright S = rS$$

and suggestively call S an "eigenvector" and r an "eigenvalue." For any "eigenvalue" r, define $f^{-1}(r)$ = "eigenspace of r" as the union of all the "eigenvectors" with that "eigenvalue." Since the "eigenspaces" span the set U, the attribute $f : U \to \mathbb{R}$ can be represented by:

$$f = \textstyle\sum_r r\chi_{f^{-1}(r)}$$

"Spectral decomposition" of set attribute $f : U \to \mathbb{R}$

[12] The usual quantum logic approach to define a 'propositional' logic for QM focused on the question of whether or not a vector was in a subspace, which in turn lead to a misplaced focus on the set equivalence relations defined by the subspaces, equivalence relations that have a special property of being *commuting* [14]. If "quantum logic" is to be the logic that is to QM as Boolean subset logic is to classical mechanics, then that is partition logic.

[where $\chi_{f^{-1}(r)}$ is the characteristic function for the "eigenspace" $f^{-1}(r)$]. Thus a set attribute determines a set partition and has a constant value on the blocks of the set partition, so by the basis principle, that lifts to a vector space concept that determines a vector space partition and has a constant value on the blocks of a vector space partition.

The suggestive terminology gives the proper lift. The lift of $f \upharpoonright S = rS$ is the eigenvector equation $Lv = \lambda v$ where L is a linear operator on V. In particular, if $Lv_1 = \lambda v_1$ and $Lv_2 = \lambda v_2$ for two basis vectors v_1 and v_2, then $Lv = \lambda v$ for all $v \in [v_1, v_2]$ (the subspace generated by v_1 and v_2). The lift of an "eigenspace" $f^{-1}(r)$ is the eigenspace W_λ of an eigenvalue λ. The lift of the simplest attributes, which are the characteristic functions $\chi_{f^{-1}(r)}$, are the projection operators P_λ that project to the eigenspaces W_λ. The characteristic property of the characteristic functions $\chi : U \to \mathbb{R}$ is that they are idempotent in the sense that $\chi(u)\chi(u) = \chi(u)$ for all $u \in U$, and the lifted characteristic property of the projection operators $P : V \to V$ is that they are idempotent in the sense that $P^2 : V \to V \to V = P : V \to V$. Finally, the "spectral decomposition" of a set attribute lifts to the spectral decomposition of a *vector space attribute*:

$$f = \sum_r r\chi_{f^{-1}(r)} \text{ lifts to } L = \sum_\lambda \lambda P_\lambda.$$

Lift of a set attribute to a vector space attribute

Thus a vector space attribute is just a linear operator whose eigenspaces span the whole space which is called a *diagonalizable linear operator* [17]. Then we see that the proper lift of a set attribute using the basis principle does indeed define a vector space partition, namely that of the eigenspaces of a diagonalizable linear operator, and that the values of the attribute are constant on the blocks of the vector space partition–as desired. To keep the eigenvalues of the linear operator real, quantum mechanics restricts the vector space attributes to *Hermitian* (or *self-adjoint*) linear operators, which represent *observables*, on a Hilbert space.

Lift program:	Set concept	Vector space concept
Eigenvalues	r s.t. $f \upharpoonright S = rS$ for some S	λ s.t. $Lv = \lambda v$ for some v
Eigenvectors	S s.t. $f \upharpoonright S = rS$ for some r	v s.t. $Lv = \lambda v$ for some λ
Eigenspaces	$\cup\{S: f \upharpoonright S = rS\} = f^{-1}(r)$ for an "eigenvalue" r	$\{v: Lv = \lambda v\} = W_\lambda$ for an eigenvalue λ
Partition	Set partition of "Eigenspaces" $f^{-1}(r)$	Vector space partition of Eigenspaces W_λ
Characteristic functions	$\chi_S: U \to \{0,1\}$ for subsets S like $f^{-1}(r)$	Projection operators for subspaces like $W_\lambda = P_\lambda(V)$
Spectral decomposition	Set attribute $f: U \to \mathbb{R}$ $f = \Sigma_r r\chi_{f^{-1}(r)}$	Hermitian linear operator $L = \Sigma_\lambda \lambda P_\lambda$

Set attributes lift to linear operators

One of the mysteries of quantum mechanics is that the set attributes such as position or momentum on the phase spaces of classical physics become linear operators on the state spaces of QM. The lifting program should take away some of that mystery.

5.5 Lifting compatible attributes

Since two set attributes $f: U \to \mathbb{R}$ and $g: U' \to \mathbb{R}$ define two inverse image partitions $\left\{f^{-1}(r)\right\}$ and $\left\{g^{-1}(s)\right\}$ on their domains, we need to extend the concept of compatible partitions to the attributes that define the partitions. That is, two attributes $f: U \to \mathbb{R}$ and $g: U' \to \mathbb{R}$ are *compatible* if they have the same domain $U = U'$.[13] We have previously lifted the notion of compatible set partitions to compatible vector space partitions. Since real-valued set attributes lift to Hermitian linear operators, the notion of compatible set attributes just defined would lift to two linear operators being *compatible* if their eigenspace partitions are compatible. It is a standard fact of the QM literature (e.g., [18, pp. 102-3] or [17, p. 177]) that two (Hermitian) linear operators $L, M: V \to V$ are compatible if and only if they commute, $LM = ML$. Hence the *commutativity* of linear operators is the lift of the compatibility (i.e., defined on the same set) of set attributes.

Given two compatible set attributes $f: U \to \mathbb{R}$ and $g: U \to \mathbb{R}$, the join of their "eigenspace" partitions has as blocks the non-empty intersections $f^{-1}(r) \cap g^{-1}(s)$. Each block in the join of the "eigenspace" partitions could be characterized by the ordered pair of "eigenvalues" (r, s). An "eigenvector" $S \subseteq f^{-1}(r)$ and $S \subseteq g^{-1}(s)$ would be a "simultaneous eigenvector": $S \subseteq f^{-1}(r) \cap g^{-1}(s)$.

In the lifted case, two commuting Hermitian linear operator L and M have compatible eigenspace partitions $W_L = \{W_\lambda\}$ (for the eigenvalues λ of L) and $W_M = \{W_\mu\}$ (for the eigenvalues μ of M). The blocks in the join $W_L \vee W_M$ of the two compatible eigenspace partitions are the non-zero subspaces $\{W_\lambda \cap W_\mu\}$ which can be characterized by the ordered pairs of eigenvalues (λ, μ). The nonzero vectors $v \in W_\lambda \cap W_\mu$ are *simultaneous eigenvectors* for the two commuting operators, and there is a basis for the space consisting of simultaneous eigenvectors.[14]

A set of compatible set attributes is said to be *complete* if the join of their partitions is discrete, i.e., the blocks have cardinality 1. Each element of U is then characterized by the ordered n-tuple $(r, ..., s)$ of attribute values.

In the lifted case, a set of commuting linear operators is said to be *complete* if the join of their eigenspace partitions is nondegenerate, i.e., the blocks have dimension 1. The eigenvectors that generate those one-dimensional blocks of the join are characterized by the ordered n-tuples $(\lambda, ..., \mu)$ of eigenvalues so the eigenvectors are usually denoted as the eigenkets $|\lambda, ..., \mu\rangle$ in the Dirac notation. These *Complete Sets of Commuting Operators* are Dirac's CSCOs [7].

5.6 Summary of lifting program

The lifting program so far is summarized in the following table.

[13] This simplified definition is justified by the later treatment of compatible attributes in the context of "quantum mechanics" on sets.

[14] One must be careful not to assume that the simultaneous eigenvectors are the eigenvectors for the operator $LM = ML$ due to the problem of degeneracy.

Summary of Lifting Program	Set concept	Vector space concept appropriate for QM
Partition	Set partition $\pi = \{B\}$ of set $U = \cup B$	Direct sum decomposition $\{W_i\}$ of space $V = \Sigma \oplus W_i$
Attribute/observable	Function $f : U \rightarrow \mathbb{R}$	Hermitian linear operator $L : V \rightarrow V$
Compatible partitions	Partitions π, σ on same set U	Vector space partitions $\{W_i\}$, $\{X_j\}$ with common basis
Compatible attributes	Functions $f, g : U \rightarrow \mathbb{R}$ on same domain U	Commuting linear operators $LM = ML$
Partition of attribute	Inverse-image partition $\{f^{-1}(r)\}$ for $f : U \rightarrow \mathbb{R}$	Eigenspace partition $\{W_\lambda\}$ for $L : V \rightarrow V$
Join compatible attributes	$f^{-1} \vee g^{-1} = \{f^{-1}(r) \cap g^{-1}(s)\}$ for $f, g : U \rightarrow \mathbb{R}$	$W_L \vee W_M = \{W_\lambda \cap W_\mu\}$ for $LM = ML$
Complete set of commuting operators	$\vee f_i^{-1}$ is discrete on U for complete set $\{f_i : U \rightarrow \mathbb{R}\}$.	$\vee W_{Li}$ is nondegenerate for CSCO $\{L_i : V \rightarrow V\}$.

Summary of Lifts

5.7 Some subtleties of the lifting program

The relation between set concepts and the lifted vector space concepts is not a one-to-one mapping.[15] For instance, the same subset $S = f^{-1}(r)$ appears both as an "eigenvector" S such that $f \upharpoonright S = rS$ and as an "eigenspace"–which are two very different vector space concepts. The two-dimensional space $[a, b]$ generated by vectors a and b is quite different from the vector $a + b$, but at the austere level of sets, they are both $\{a, b\}$. Thus the same set concept of a subset $\{a, b\}$ (depending on whether it is viewed as $\{a, b\} = f^{-1}(r)$ or as $f \upharpoonright \{a, b\} = r\{a, b\}$) lifts to quite different vector space concepts: the subspace $[a, b]$ or the vector $a + b$. This is one of the reasons that the lifting program cannot be reduced to a simple mapping.

Moreover, the same vector space concept, viewed from different angles, may "delift" to quite different set concepts. Consider the vector space concept of a projection operator $P : V \rightarrow V$ that projects to the subspace $P(V) = W$. As a linear operator with the eigenvalues 0 and 1, a projection operator is the lift of a characteristic function $\chi_S : U \rightarrow \mathbb{R}$ as an attribute. The projection operator assigns the eigenvalues 1 and 0 to the two blocks $P(V)$ and $\ker(P)$ of its eigenspace partition, just as the attribute χ_S assigns the two values to the two blocks $\chi_S(1)$ and $\chi_S(0)$ of its set partition. But a projection operator also serves to project an arbitrary vector $v \in V$ to the part of v, namely $P(v)$, that is in the range-space W. Since the delift of vectors $v \in V$ are subsets $S \subseteq U$ (viewed as single indefinite elements), the delift of the projecting operation would be a mapping from arbitrary subsets to the part of each subset that is in the "range eigenspace" $\chi_S^{-1}(1)$. That "projection" is the idempotent mapping:

$$\chi_S^{-1}(1) \cap () : \wp(U) \rightarrow \wp(U).$$

Thus the same vector space concept of a projection operator delifts to two quite different set concepts: the set attribute $\chi_S : U \rightarrow \mathbb{R}$ and the subset operator $\chi_S^{-1}(1) \cap () : \wp(U) \rightarrow \wp(U)$.

[15] Perhaps the lifting program is akin to a type of mathematical "pornography"–it is hard to define exactly but you know it when you see it.

The subset operator treatment of a projection allows another type of "spectral decomposition" associated with an attribute $f : U \to \mathbb{R}$. The previous statement for $S \subseteq f^{-1}(r)$ that $f \restriction S = rS$ can now be written $r\left[f^{-1}(r) \cap S\right] = rS$ so that the action of f on subsets can be symbolically represented as:

$$f \restriction () = \sum_r r\left[f^{-1}(r) \cap ()\right]$$

that identifies the "eigenvectors" and "eigenvalues" in the set case and thus could be taken as the set operator analogue of $L = \sum_\lambda \lambda P_\lambda$.

6 The Delifting Program: "Quantum mechanics" on sets

6.1 Probabilities in "quantum mechanics" on sets

The lifting program establishes a relationship between concepts and operations for sets and those for vector spaces. We have so far started with set concepts, like the concept of a set partition, and then developed the corresponding concept for vector spaces (direct sum decomposition). However the relation between set and vector space concepts can also be established by going the other way, by delifting quantum mechanical concepts from vector spaces to sets. By delifting QM concepts to sets, we can develop a toy model called *"quantum mechanics"* on sets–which shows the logical structure of QM in a pedagogically simple and understandable context.

The connection between sets and the complex vector spaces of QM can be facilitated by considering an intermediate stage. A power set $\wp(U)$ can be considered as a vector space over $\mathbb{Z}_2 = \{0, 1\}$ with the *symmetric difference* of subsets, i.e., $S \Delta T = S \cup T - S \cap T$ for $S, T \subseteq U$, as the vector addition operation. Thus set concepts can be first translated into sets-as-vectors concepts for vector spaces over $\mathbb{Z}_2$ and then lifted to vector spaces over $\mathbb{C}$ (or vice-versa for delifting). One of the key pieces of machinery in QM, namely the inner product, does not exist in vector spaces over finite fields but a norm can be defined to play a similar role in the probability algorithm.

Seeing $\wp(U)$ as the vector space $\mathbb{Z}_2^{|U|}$ allows different bases in which the vectors can be expressed (as well as the basis-free notion of a vector as a ket). Consider the simple case of $U = \{a, b, c\}$ where the U-basis is $\{a\}$, $\{b\}$, and $\{c\}$. But the three subsets $\{a, b\}$, $\{b, c\}$, and $\{a, b, c\}$ also form a basis since: $\{a, b\} + \{a, b, c\} = \{c\}$; $\{b, c\} + \{c\} = \{b\}$; and $\{a, b\} + \{b\} = \{a\}$. These new basis vectors could be considered as the basis-singletons in another equicardinal universe $U' = \{a', b', c'\}$ where $a' = \{a, b\}$, $b' = \{b, c\}$, and $c' = \{a, b, c\}$. In the following table, each row is a ket of $V = \mathbb{Z}_2^3$ expressed in the U-basis and the U'-basis.

$U = \{a, b, c\}$	$U' = \{a', b', c'\}$
$\{a, b, c\}$	$\{c'\}$
$\{a, b\}$	$\{a'\}$
$\{b, c\}$	$\{b'\}$
$\{a, c\}$	$\{a', b'\}$
$\{a\}$	$\{b', c'\}$
$\{b\}$	$\{a', b', c'\}$
$\{c\}$	$\{a', c'\}$
$\emptyset$	$\emptyset$

Vector space isomorphism (i.e., preserves $+$) $\mathbb{Z}_2^3 \cong \wp(U) \cong \wp(U')$: row = ket.

In a Hilbert space, the inner product is used to define the norm $\|v\| = \sqrt{\langle v|v\rangle}$, and the probability algorithm can be formulated using this norm. In a vector space over $\mathbb{Z}_2$, the Dirac notation can still be used to define a real-valued norm even though there is no inner product. The kets $|S\rangle$ for $S \in \wp(U)$ are basis-free but the corresponding bras are basis-dependent. For $u \in U$, the *bra* $\langle\{u\}|_U : \wp(U) \to \mathbb{R}$ is defined: $\langle\{u\}|_U S\rangle = 1$ if $u \in S$ and 0 otherwise so that $\langle\{\}|_U S\rangle = \chi_S : U \to \{0,1\}$. Assuming a finite U, the bra can also be defined in a more general basis-dependent form:

$$\langle T|_U S\rangle = |T \cap S| \text{ for } T, S \subseteq U.$$

Note that for $u, u' \in U$, $\langle\{u'\}|_U \{u\}\rangle = \delta_{u'u}$ taking the distinct elements of U as being paired with the vectors in an orthonormal basis in the lift-delift relationship. In fact, this delifting of the Dirac bracket is easily motivated by considering an orthonormal basis set $\{|u\rangle\}$ in a finite dimensional Hilbert space. Given two subsets $T, S \subseteq \{|u\rangle\}$, consider the unnormalized vector $\psi_T = \sum_{|u\rangle \in T} |u\rangle$ and similarly for ψ_S. Then their inner product in the Hilbert space is $\langle\psi_T|\psi_S\rangle = |T \cap S|$, which "delifts" (running the basis principle in reverse) to $\langle T|_U S\rangle = |T \cap S|$ for subsets $T, S \subseteq U$.

Then the *U-norm* $\|S\|_U : \wp(U) \to \mathbb{R}$ is defined, as usual, as the square root of the bracket:

$$\|S\|_U = \sqrt{\langle S|_U S\rangle} = \sqrt{|S|}$$

for $S \in \wp(U)$ which is the delift of the basis-free norm $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$ (since the inner product does not depend on the basis). Note that a ket has to be expressed in the U-basis to apply the basis-dependent definition so in the above example, $\|\{a'\}\|_U = \sqrt{2}$ since $\{a'\} = \{a, b\}$ in the U-basis.

For a specific basis $\{|v_i\rangle\}$ and for any nonzero vector v in a finite dimensional vector space, $\|v\| = \sqrt{\sum_i \langle v_i|v\rangle \langle v_i|v\rangle^*}$ whose delifted version would be: $\|S\|_U = \sqrt{\sum_{u\in U} \langle\{u\}|_U S\rangle^2}$. Thus squaring both sides, we also have:

$$\sum_i \frac{\langle v_i|v\rangle\langle v_i|v\rangle^*}{\|v\|^2} = 1 \text{ and } \sum_u \frac{\langle\{u\}|_U S\rangle^2}{\|S\|_U^2} = \sum_u \frac{|\{u\}\cap S|}{|S|} = 1$$

where $\frac{\langle v_i|v\rangle\langle v_i|v\rangle^*}{\|v\|^2}$ is a 'mysterious' quantum probability while $\frac{|\{u\}\cap S|}{|S|}$ is the unmysterious probability $\Pr(\{u\}|S)$ of getting u when sampling S (equiprobable elements of U). We previously saw that a subset $S \subseteq U$ as a block in a partition could be interpreted as a single indefinite element rather than a subset of definite elements. In like manner, we can interpret a subset of outcomes (an event) in a finite probability space as a single indefinite outcome where the conditional probability $\Pr(\{u\}|S)$ is the objective probability of a "U-measurement" of S yielding the definite outcome $\{u\}$.

An observable, i.e., a Hermitian operator, on a Hilbert space determines its home basis set of orthonormal eigenvectors. In a similar manner, an attribute $f : U \to \mathbb{R}$ defined on U has the U-basis as its "home basis set." Then given a Hermitian operator $L = \sum_\lambda \lambda P_\lambda$ and a U-attribute $f : U \to \mathbb{R}$, we have:

$$\|v\| = \sqrt{\sum_\lambda \|P_\lambda(v)\|} \text{ and } \|S\|_U = \sqrt{\sum_r \|f^{-1}(r) \cap S\|_U^2}$$

where $f^{-1}(r) \cap S$ is the "projection operator" $f^{-1}(r) \cap ()$ applied to S, the delift of applying the projection operator P_λ to v.[16] This can also be written as:

[16] Since $\wp(U)$ is now interpreted as a vector space, it should be noted that the projection operator $S \cap () : \wp(U) \to \wp(U)$ is linear, i.e., $(S \cap S_1)\,\Delta(S \cap S_2) = S \cap (S_1 \Delta S_2)$. Indeed, this is the distributive law when $\wp(U)$ is interpreted as a Boolean ring.

$$\sum_\lambda \frac{\|P_\lambda(v)\|^2}{\|v\|^2} = 1 \text{ and } \sum_r \frac{\left\|f^{-1}(r)\cap S\right\|_U^2}{\|S\|_U^2} = \sum_r \frac{\left|f^{-1}(r)\cap S\right|}{|S|} = 1$$

where $\frac{\|P_\lambda(v)\|^2}{\|v\|^2}$ is the quantum probability of getting λ in an L-measurement of v while $\frac{\left|f^{-1}(r)\cap S\right|}{|S|}$ has the rather unmysterious interpretation of the probability $\Pr(r|S)$ of the random variable $f : U \to \mathbb{R}$ having the value r when sampling $S \subseteq U$. Under the set version of the objective indefiniteness interpretation, i.e., "quantum mechanics" on sets, the indefinite element S is being "measured" using the "observable" f and the probability $\Pr(r|S)$ of getting the "eigenvalue" r is $\frac{\left|f^{-1}(r)\cap S\right|}{|S|}$ with the "projected resultant state" as $f^{-1}(r) \cap S$.

These delifts are summarized in the following table for a finite U and a finite dimensional Hilbert space V.

Set Case	Vector space case
Projection $B \cap () : \wp(U) \to \wp(U)$	Projection $P : V \to V$
$f \upharpoonright () = \sum_r r\left(f^{-1}(r) \cap ()\right)$	Herm. $L = \sum_\lambda \lambda P_\lambda$
$\Delta_{B\in\pi} B \cap () = I : \wp(U) \to \wp(U)$	$\sum_\lambda P_\lambda = I$
$\langle S\|_U T\rangle = \|S \cap T\|$ where $S, T \subseteq U$	$\langle\psi\|\varphi\rangle =$ "overlap" of ψ and φ
$\\|S\\|_U = \sqrt{\langle S\|_U S\rangle} = \sqrt{\|S\|}$ where $S \subseteq U$	$\\|\psi\\| = \sqrt{\langle\psi\|\psi\rangle}$
$\\|S\\|_U = \sqrt{\sum_{u\in U} \\|\{u\} \cap S\\|_U^2}$	$\\|\psi\\| = \sqrt{\sum_i \langle v_i\|\psi\rangle \langle v_i\|\psi\rangle^*}$
$S \neq \emptyset$, $\sum_{u\in U} \frac{\\|\{u\}\cap S\\|_U^2}{\\|S\\|_U^2} = \sum_{u\in S} \frac{1}{\|S\|} = 1$	$\|\psi\rangle \neq 0$, $\sum_i \frac{\langle v_i\|\psi\rangle\langle v_i\|\psi\rangle^*}{\\|\psi\\|^2} = 1$
$\\|S\\|_U = \sqrt{\sum_r \\|f^{-1}(r) \cap S\\|_U^2}$	$\\|\psi\\| = \sqrt{\sum_\lambda \\|P_\lambda(\psi)\\|^2}$
$S \neq \emptyset$, $\sum_r \frac{\\|f^{-1}(r)\cap S\\|_U^2}{\\|S\\|_U^2} = \sum_r \frac{\|f^{-1}(r)\cap S\|}{\|S\|} = 1$	$\|\psi\rangle \neq 0$, $\sum_\lambda \frac{\\|P_\lambda(\psi)\\|^2}{\\|\psi\\|^2} = 1$
Given S, prob. of r is $\frac{\\|f^{-1}(r)\cap S\\|_U^2}{\\|S\\|_U^2} = \frac{\|f^{-1}(r)\cap S\|}{\|S\|}$	Given ψ, prob. of λ is $\frac{\\|P_\lambda(\psi)\\|^2}{\\|\psi\\|^2}$

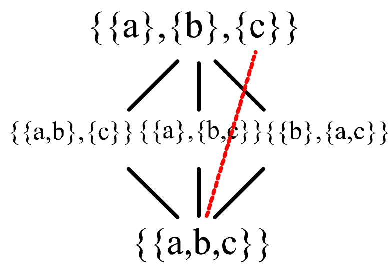
Demystifying quantum probabilities using "quantum mechanics" on sets

6.2 Measurement in "quantum mechanics" on sets

Certainly the notion of measurement is one of the most opaque notions of QM so let's consider a set version of (projective) measurement starting at some block (the "state") in a partition in a partition lattice. In the simple example illustrated below we start at the one block or "state" of the indiscrete partition or blob which is the completely indistinct element $\{a, b, c\}$. A measurement always uses some attribute that defines an inverse-image partition on $U = \{a, b, c\}$. In the case at hand, there are "essentially" four possible attributes that could be used to "measure" the indefinite element $\{a, b, c\}$ (since there are four partitions that refine the blob).

For an example of a "nondegenerate measurement," consider any attribute $f : U \to \mathbb{R}$ which has the discrete partition as its inverse image, such as the ordinal number of a letter in the alphabet: $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$. This attribute or "observable" has three "eigenvectors": $f \upharpoonright \{a\} = 1\{a\}$, $f \upharpoonright \{b\} = 2\{b\}$, and $f \upharpoonright \{c\} = 3\{c\}$ with the corresponding "eigenvalues." The "eigenspaces" in the inverse image are also $\{a\}$, $\{b\}$, and $\{c\}$, the blocks in the discrete partition of U all of which have "dimension" (i.e., cardinality) one. Starting in the "state" $S = \{a, b, c\}$, a U-measurement with this observable would yield the "eigenvalue" r with the probability of $\Pr(r|S) =$

$\frac{|f^{-1}(r)\cap S|}{|S|}=\frac{1}{3}$. A "projective measurement" makes distinctions in the measured "state" that are sufficient to induce the "quantum jump" or "projection" to the "eigenvector" associated with the observed "eigenvalue." If the observed "eigenvalue" was 3, then the "state" $\{a,b,c\}$ "projects" to $f^{-1}(3)\cap\{a,b,c\}=\{c\}\cap\{a,b,c\}=\{c\}$ as pictured below.



"Nondegenerate measurement"

It might be emphasized that this is an objective state reduction (or "collapse of the wave packet") from the single indefinite element $\{a,b,c\}$ to the single definite element $\{c\}$, not a subjective removal of ignorance as if the "state" had all along been $\{c\}$. For instance, Pascual Jordan in 1934 argued that:

> the electron is forced to a decision. We compel it to assume a definite position; previously, in general, it was neither here nor there; it had not yet made its decision for a definite position... . ... [W]e ourselves produce the results of the measurement. (quoted in [19, p. 161])

For an example of a "degenerate measurement," we choose an attribute with a non-discrete inverse-image partition such as $\{\{a\},\{b,c\}\}$, which could, for instance, just be the characteristic function $\chi_{\{b,c\}}$ with the two "eigenspaces" $\{a\}$ and $\{b,c\}$ and the two "eigenvalues" 0 and 1 respectively. Since one of the two "eigenspaces" is not a singleton of an eigen-element, the "eigenvalue" of 1 is a set version of a "degenerate eigenvalue." This attribute $\chi_{\{b,c\}}$ has four "eigenvectors": $\chi_{\{b,c\}}\restriction\{b,c\}=1\{b,c\}$, $\chi_{\{b,c\}}\restriction\{b\}=1\{b\}$, $\chi_{\{b,c\}}\restriction\{c\}=1\{c\}$, and $\chi_{\{b,c\}}\restriction\{a\}=0\{a\}$.

The "measuring apparatus" makes distinctions that further distinguishes the indefinite element $S=\{a,b,c\}$ but the measurement returns one of "eigenvalues" with certain probabilities:

$$\Pr(0|S)=\frac{|\{a\}\cap\{a,b,c\}|}{|\{a,b,c\}|}=\frac{1}{3}\text{ and }\Pr(1|S)=\frac{|\{b,c\}\cap\{a,b,c\}|}{|\{a,b,c\}|}=\frac{2}{3}.$$

Suppose it returns the "eigenvalue" 1. Then the indefinite element $\{a,b,c\}$ "jumps" to the "projection" $\chi_{\{b,c\}}^{-1}(1)\cap\{a,b,c\}=\{b,c\}$ of the "state" $\{a,b,c\}$ to that "eigenspace" [5, p. 221].

Since this is a "degenerate" result (i.e., the "eigenspace" of 1 does not have "dimension" one), another measurement is needed to make more distinctions. Measurements by attributes that give either of the other two middle partitions, $\{\{a,b\},\{c\}\}$ or $\{\{b\},\{a,c\}\}$, suffice to distinguish $\{b,c\}$ into $\{b\}$ or $\{c\}$, so either attribute together with the attribute $\chi_{\{b,c\}}$ would form a *complete set of compatible attributes* (i.e., the set version of a CSCO). The join of the two attributes' partitions gives the discrete partition. Taking the other attribute as $\chi_{\{a,b\}}$, the join of the two attributes' "eigenspace" partitions is discrete:

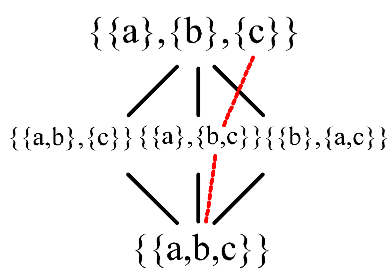
$$\{\{a\},\{b,c\}\} \vee \{\{a,b\},\{c\}\} = \{\{a\},\{b\},\{c\}\} = \mathbf{1}.$$

Hence all the singletons can be characterized by the ordered pairs of the "eigenvalues" of these two attributes: $\{a\} = |0,1\rangle$, $\{b\} = |1,1\rangle$, and $\{c\} = |1,0\rangle$ (using Dirac's kets to give the ordered pairs).

The second "projective measurement" of the indefinite "superposition" element $\{b,c\}$ using the attribute $\chi_{\{a,b\}}$ with the "eigenspace" partition $\{\{a,b\},\{c\}\}$ would induce a jump to either $\{b\}$ or $\{c\}$ with the probabilities:

$$\Pr(1|\{b,c\}) = \frac{|\{a,b\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2} \text{ and } \Pr(0|\{b,c\}) = \frac{|\{c\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2}.$$

If the measured "eigenvalue" is 0, then the "state" $\{b,c\}$ "projects" to $\chi^{-1}_{\{a,b\}}(0) \cap \{b,c\} = \{c\}$ as pictured below.



"Degenerate measurement"

The two "projective measurements" of $\{a,b,c\}$ using the complete set of compatible (both defined on U) attributes $\chi_{\{b,c\}}$ and $\chi_{\{a,b\}}$ produced the respective "eigenvalues" 1 and 0, and the resulting "eigenstate" was characterized by the "eigenket" $|1,0\rangle = \{c\}$.

In this manner, the toy model of "quantum mechanics" on sets provides a set version of "nondegenerate measurement" by an "observable," a "degenerate measurement," "projections" associated with "eigenvalues" that "project" to "eigenvectors," and characterizations of "eigenvectors" by "eigenkets" of "eigenvalues"–all of which shows the bare bones logical structure of QM measurement in the simple context of sets.

6.3 The indeterminacy principle in "quantum mechanics" on sets

Behind Heisenberg's indeterminacy principle, the basic idea (not the numerical formula) is that a vector space can have quite different bases so that a ket that is a definite state in one basis is an indefinite superposition in another basis. And that basic idea can be well illustrated at the set level by interpreting $\wp(U)$ as a vector space $\mathbb{Z}_2^n$ (where $|U| = n$) which has many bases. In our previous (simplified) treatment of attributes $f : U \to \mathbb{R}$ and $g : U' \to \mathbb{R}$ not using $\mathbb{Z}_2^n$, the attributes were compatible if $U = U'$. Now we can give a more sophisticated treatment of the set case using $\mathbb{Z}_2^n$, but with the similar result that attributes are compatible, i.e., "commute," if and only if there is a common basis set of "simultaneous eigenvectors" on which both attributes can be defined. The lifted version is the same; two observable operators are compatible if there is a basis of simultaneous eigenvectors, and that holds if and only if the operators commute–which is also equivalent to all the projection operators in the two spectral decompositions commuting.

We are given two basis sets $\{\{a\},\{b\},...\mid a,b,...\in U\}$ and $\{\{a'\},\{b'\},...\mid a',b',...\in U'\}$ for $\mathbb{Z}_2^n$ such as in the previous example where $n=3$ and the U'-basis was the three kets $\{a'\}=\{a,b\}$, $\{b'\}=\{b,c\}$, and $\{c'\}=\{a,b,c\}$. Then we have two real-valued set attributes defined on the different bases, $f:U\to\mathbb{R}$ and $g:U'\to\mathbb{R}$, and we want to investigate their compatibility.

The set attributes define set partitions $\left\{f^{-1}(r)\right\}$ and $\left\{g^{-1}(s)\right\}$ respectively on U and U'. These set partitions on the basis sets define, as usual, vector space partitions $\left\{\wp\left(f^{-1}(r)\right)\right\}$ and $\left\{\wp\left(g^{-1}(s)\right)\right\}$ on $\mathbb{Z}_2^n$. But those vector space partitions cannot in general be obtained as the eigenspace partitions of Hermitian operators on $\mathbb{Z}_2^n$ since the only available eigenvalues are 0 and 1. But any set attribute that is the characteristic function $\chi_S:U\to\{0,1\}\subseteq\mathbb{R}$ of a subset $S\subseteq U$ can represented by an operator, indeed a projection operator, whose action on $\wp(U)\cong\mathbb{Z}_2^n$ is given by the "projection operator" $S\cap():\wp(U)\to\wp(U)$, and similarly for U'. The properties of the real-valued attributes f and g can then stated in terms of these projection operators for subsets $S=f^{-1}(r)\subseteq U$ and $S'=g^{-1}(s)\subseteq U'$.

Consider first the above example and the simple case where the attributes are just characteristic functions $f=\chi_{\{b,c\}}:U\to\{0,1\}\subseteq\mathbb{R}$ so $f^{-1}(1)=\{b,c\}$ and $g=\chi_{\{a',b'\}}:U'\to\{0,1\}\subseteq\mathbb{R}$ so $g^{-1}(1)=\{a',b'\}$. The two projection operators are $\{b,c\}\cap():\wp(U)\to\wp(U)$ and $\{a',b'\}\cap():\wp(U')\to\wp(U')$. Note that this representation of the projection operators is basis-dependent. For instance, $\{a',b'\}=\{a,c\}$ but the operator $\{a,c\}\cap()$ operating on $\wp(U)$ is a very different operator than $\{a',b'\}\cap()$ operating on $\wp(U')$. The following ket table computes the two projection operators and checks if they commute.

U	U'	$f\restriction=\{b,c\}\cap()$	$g\restriction=\{a',b'\}\cap()$	$g\restriction f\restriction$	$f\restriction g\restriction$
$\{a,b,c\}$	$\{c'\}$	$\{b,c\}$	0	$\{b,c\}$	0
$\{a,b\}$	$\{a'\}$	$\{b\}$	$\{a'\}=\{a,b\}$	$\{a,c\}$	$\{b\}$
$\{b,c\}$	$\{b'\}$	$\{b,c\}$	$\{b'\}=\{b,c\}$	$\{b,c\}$	$\{b,c\}$
$\{a,c\}$	$\{a',b'\}$	$\{c\}$	$\{a',b'\}=\{a,c\}$	$\{a,b\}$	$\{c\}$
$\{a\}$	$\{b',c'\}$	0	$\{b'\}=\{b,c\}$	0	$\{b,c\}$
$\{b\}$	$\{a',b',c'\}$	$\{b\}$	$\{a',b'\}=\{a,c\}$	$\{a,c\}$	$\{a,c\}$
$\{c\}$	$\{a',c'\}$	$\{c\}$	$\{a'\}=\{a,b\}$	$\{a,b\}$	$\{b\}$
$\emptyset$	$\emptyset$	0	0	0	0

Non-commutativity of the projections $\{b,c\}\cap()$ and $\{a',b'\}\cap()$.

We can move even closer to QM mathematics by using matrices in $\mathbb{Z}_2^n$ to represent the operators. The U-basis vectors $\{a\}$, $\{b\}$, and $\{c\}$ are represented by the respective column vectors:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}_U,\ \begin{bmatrix}0\\1\\0\end{bmatrix}_U,\text{ and }\begin{bmatrix}0\\0\\1\end{bmatrix}_U$$

where the subscripts indicate the basis. The projection operator $\{b,c\}\cap()$ would be represented by the matrix whose columns give the result of applying the operator to the basis vectors:

$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}_U$$

$\{b,c\}\cap()$ projection matrix in U-basis.

In the U'-basis (with the corresponding basis vectors using the U' subscript), the $\{a',b'\} \cap ()$ projection operator is represented by the projection matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{U'}$$

$\{a',b'\} \cap ()$ projection matrix in U'-basis.

These matrices cannot be meaningfully multiplied since they are in different bases but we can convert them into the same basis to see if they commute. Since $\{a'\} = \{a,b\}$, $\{b'\} = \{b,c\}$, and $\{c'\} = \{a,b,c\}$, the conversion matrix $\mathcal{C}_{U \leftarrow U'}$ to convert U'-basis vectors to U-basis vectors is given by the entries such as $\langle \{a\} |_U \{a'\} \rangle = 1$:

$$\mathcal{C}_{U \leftarrow U'} = \begin{bmatrix} \langle \{a\} |_U \{a'\} \rangle & \langle \{a\} |_U \{b'\} \rangle & \langle \{a\} |_U \{c'\} \rangle \\ \langle \{b\} |_U \{a'\} \rangle & \langle \{b\} |_U \{b'\} \rangle & \langle \{b\} |_U \{c'\} \rangle \\ \langle \{c\} |_U \{a'\} \rangle & \langle \{c\} |_U \{b'\} \rangle & \langle \{c\} |_U \{c'\} \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{U \leftarrow U'}.$$

The conversion the other way is given by the inverse matrix (remember $\bmod (2)$ arithmetic):

$$\mathcal{C}_{U' \leftarrow U} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}_{U' \leftarrow U} = \mathcal{C}_{U \leftarrow U'}^{-1}$$

which could also be directly seen from the ket table since $\{a\} = \{b',c'\}$, $\{b\} = \{a,b,c\}$, and $\{c\} = \{a',c'\}$.

The projection matrix for $\{a',b'\} \cap ()$ in the U'-basis can be converted to the U-basis by computing the matrix that starting with any U-basis vector will convert it to the U'-basis, then apply the projection matrix in that U'-basis and then convert the result back to the U-basis:

$$\begin{aligned} & \mathcal{C}_{U \leftarrow U'} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{U'} \mathcal{C}_{U' \leftarrow U} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{U \leftarrow U'} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{U'} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}_{U' \leftarrow U} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{U} \end{aligned}$$

$\{a',b'\} \cap ()$ projection operator in the U-basis.

Now the two projection operators are represented as projection matrices in the same U-basis so they can be multiplied to see if they commute:

$$g \upharpoonright f \upharpoonright () = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{U} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{U} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{U}$$

$$f \upharpoonright g \upharpoonright () = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{U} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{U} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{U}$$

so the two projection matrices do not commute, as we previously saw in the table computation.

There is a standard theorem of linear algebra:

Proposition 2 *For two diagonalizable (i.e., eigenvectors span the space) linear operators on a finite dimensional space: the operators commute if and only if there is a basis of simultaneous eigenvectors [17, p. 177].*

In the above example of non-commuting projection operators, there is no basis of simultaneous eigenvectors (in fact $\{b,c\} = \{b'\}$ is the only common eigenvector).

In the following example of a third U''-basis where $U'' = \{a'', b'', c''\}$ with the set attributes $f = \chi_{\{b,c\}} : U \to \{0,1\}$ and $g = \chi_{\{a'',b''\}} : U'' \to \{0,1\}$, the projections $\{b,c\} \cap ()$ and $\{a'',b''\} \cap ()$ commute as we see from the last two columns.

U	U''	$f \upharpoonright = \{b,c\} \cap ()$	$g \upharpoonright = \{a'',b''\} \cap ()$	$g \upharpoonright f \upharpoonright$	$f \upharpoonright g \upharpoonright$
$\{a,b,c\}$	$\{a'',b'',c''\}$	$\{b,c\}$	$\{a'',b''\} = \{a,c\}$	$\{c\}$	$\{c\}$
$\{a,b\}$	$\{b'',c''\}$	$\{b\}$	$\{b''\} = \{a\}$	$\emptyset$	$\emptyset$
$\{b,c\}$	$\{a'',c''\}$	$\{b,c\}$	$\{a''\} = \{c\}$	$\{c\}$	$\{c\}$
$\{a,c\}$	$\{a'',b''\}$	$\{c\}$	$\{a'',b''\} = \{a,c\}$	$\{c\}$	$\{c\}$
$\{a\}$	$\{b''\}$	0	$\{b''\} = \{a\}$	$\emptyset$	$\emptyset$
$\{b\}$	$\{c''\}$	$\{b\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{c\}$	$\{a''\}$	$\{c\}$	$\{a''\} = \{c\}$	$\{c\}$	$\{c\}$
$\emptyset$	$\emptyset$	0	0	0	0

Commuting projection operators $\{b,c\} \cap ()$ and $\{a'',b''\} \cap ()$.

Hence in this case, there is a basis of simultaneous eigenvectors $\{a\} = \{b''\}$, $\{b\} = \{c''\}$, and $\{c\} = \{a''\}$, so that f and g are defined on the same set (which we could take to be either U or U'').

Returning to the two basis sets $\{\{a\}, \{b\}, ... \mid a, b, ... \in U\}$ and $\{\{a'\}, \{b'\}, ... \mid a', b', ... \in U'\}$ for $\mathbb{Z}_2^n$ with two real-valued set attributes $f : U \to \mathbb{R}$ and $g : U' \to \mathbb{R}$, the attributes cannot be represented as operators on $\mathbb{Z}_2^n$ but each block $f^{-1}(r)$ and $g^{-1}(s)$ can be analyzed using the projection operators $f^{-1}(r) \cap ()$ and $g^{-1}(s) \cap ()$ for those subsets. Thus instead of the criterion of operators commuting, we define that attributes f and g *"commute"* if all their projection operators $f^{-1}(r) \cap ()$ and $g^{-1}(s) \cap ()$ commute. Then the above proposition about commuting operators can be applied to the commuting operators to yield the result: f and g "commute" if and only if they are *compatible* in the sense that there is a basis set $\{\{a''\}, \{b''\}, ...\}$ for $\mathbb{Z}_2^n$ whose subsets (vectors) are "simultaneous eigenvectors" for all the projection operators–so that f and g can be taken as being defined on the same basis set of n vectors. This result also justifies our earlier simplification that f and g were defined as compatible if they were defined on the same set $U = U'$.

If the two set attributes f and g could be defined on the same set, then they could have definite values at the same time, and that holds if and only if the attributes "commute." But in the non-commutative case, f and g cannot always have definite values in any state. A definite value for one means an indefinite value for the other. In the first example, we have $f = \chi_{\{b,c\}}$ and $g = \chi_{\{a',b'\}}$ so, for example, in the state $\{c\} = \{a', c'\}$, f has the definite value $f(c) = 1$ while g is indefinite between the values of $g(a') = 1$ and $g(c') = 0$. In this manner, we see how the essential points (but not the numerical formulas) of Heisenberg's indeterminacy principle, i.e., when two observables can or cannot have simultaneous definite values, are evidenced in the model of "quantum mechanics" on sets.

6.4 Entanglement in "quantum mechanics" on sets

Another QM concept that also generates much mystery is entanglement. Hence it might be useful to consider entanglement in "quantum mechanics" on sets.

First we need to lift the set notion of the direct (or Cartesian) product $X \times Y$ of two sets X and Y. Using the basis principle, we apply the set concept to the two basis sets $\{v_1, ..., v_m\}$ and $\{w_1, ..., w_n\}$ of two vector spaces V and W (over the same base field) and then we see what it generates. The set direct product of the two basis sets is the set of all ordered pairs (v_i, w_j), which we will write as $v_i \otimes w_j$, and then we generate the vector space, denoted $V \otimes W$, over the same base field from those basis elements $v_i \otimes w_j$. That vector space is the *tensor product*, and it not the direct product $V \times W$ of the vector spaces. The cardinality of $X \times Y$ is the product of the cardinalities of the two sets, and the dimension of the tensor product $V \otimes W$ is the product of the dimensions of the two spaces (while the dimension of the direct product $V \times W$ is the sum of the two dimensions).

A vector $z \in V \otimes W$ is said to be *separated* if there are vectors $v \in V$ and $w \in W$ such that $z = v \otimes w$; otherwise, z is said to be *entangled*. Since vectors delift to subsets, a subset $S \subseteq X \times Y$ is said to be *"separated"* or a *product* if there exists subsets $S_X \subseteq X$ and $S_Y \subseteq Y$ such that $S = S_X \times S_Y$; otherwise $S \subseteq X \times Y$ is said to be *"entangled."* In general, let S_X be the support or projection of S on X, i.e., $S_X = \{x : \exists y \in Y, (x, y) \in S\}$ and similarly for S_Y. Then S is "separated" iff $S = S_X \times S_Y$.

For any subset $S \subseteq X \times Y$, where X and Y are finite sets, a natural measure of its "entanglement" can be constructed by first viewing S as the support of the equiprobable or Laplacian joint probability distribution on S. If $|S| = N$, then define $\Pr(x, y) = \frac{1}{N}$ if $(x, y) \in S$ and $\Pr(x, y) = 0$ otherwise.

The marginal distributions[17] are defined in the usual way:

$$\Pr(x) = \sum_y \Pr(x, y)$$
$$\Pr(y) = \sum_x \Pr(x, y).$$

A joint probability distribution $\Pr(x, y)$ on $X \times Y$ is *independent* if for all $(x, y) \in X \times Y$,

$$\Pr(x, y) = \Pr(x) \Pr(y).$$

Independent distribution

Otherwise $\Pr(x, y)$ is said to be *correlated*.

Proposition 3 *A subset $S \subseteq X \times Y$ is "entangled" iff the equiprobable distribution on S is correlated.*

Proof: If S is "separated", i.e., $S = S_X \times S_Y$, then $\Pr(x) = |S_Y|/N$ for $x \in S_X$ and $\Pr(y) = |S_X|/N$ for $y \in S_Y$ where $|S_X||S_Y| = N$. Then for $(x, y) \in S$,

$$\Pr(x, y) = \frac{1}{N} = \frac{N}{N^2} = \frac{|S_X||S_Y|}{N^2} = \Pr(x) \Pr(y)$$

[17] The marginal distributions are the set versions of the reduced density matrices of QM.

and $\Pr(x,y) = 0 = \Pr(x)\Pr(y)$ for $(x,y) \notin S$ so the equiprobable distribution is independent. If S is "entangled," i.e., $S \neq S_X \times S_Y$, then $S \subsetneqq S_X \times S_Y$ so let $(x,y) \in S_X \times S_Y - S$. Then $\Pr(x), \Pr(y) > 0$ but $\Pr(x,y) = 0$ so it is not independent, i.e., is correlated. $\square$

Consider the set version of one qubit space where $U = \{a,b\}$. The product set $U \times U$ has 15 nonempty subsets. Each factor U and U has 3 nonempty subsets so $3 \times 3 = 9$ of the 15 subsets are "separated" subsets leaving 6 "entangled" subsets.

$S \subseteq U \times U$
$\{(a,a),(b,b)\}$
$\{(a,b),(b,a)\}$
$\{(a,a),(a,b),(b,a)\}$
$\{(a,a),(a,b),(b,b)\}$
$\{(a,b),(b,a),(b,b)\}$
$\{(a,a),(b,a),(b,b)\}$

The six entangled subsets

The first two are the "Bell states" which are the two graphs of bijections $U \longleftrightarrow U$ and have the maximum entanglement if entanglement is measured by the logical divergence $d(\Pr(x,y)||\Pr(x)\Pr(y))$[8]. All the 9 "separated" states have zero "entanglement" by the same measure.

For an "entangled" subset S, a sampling x of left-hand system will change the probability distribution for a sampling of the right-hand system y, $\Pr(y|x) \neq \Pr(y)$. In the case of maximal "entanglement" (e.g., the "Bell states"), when S is the graph of a bijection between U and U, the value of y is determined by the value of x (and vice-versa).

In this manner, we see that many of the basic ideas and relationships of quantum mechanical entanglement (e.g., "entangled states," "reduced density matrices," maximally "entangled states," and "Bell states"), can be reproduced in "quantum mechanics" on sets.

The two-slit experiment and the Bell inequality for "quantum mechanics" on sets are developed in Appendices 2 and 3.

7 Waving good-by to waves

7.1 Wave-particle duality = indistinct-distinct particle duality

States that are indistinct for an observable are represented as weighted vector sums or superpositions of the eigenstates that might be actualized by further distinctions. This indistinctness-represented-as-superpositions is usually interpreted as "wave-like aspects" of the particles in the indefinite state. Hence the distinction-making measurements take away the indistinctness–which is usually interpreted as taking away the "wave-like aspects," i.e., "collapse of the wave packet." But there are no actual physical waves in quantum mechanics (e.g., the "wave amplitudes" are complex numbers); only particles with indistinct attributes for certain observables. Thus the "collapse of the wave packet" is better described as the "collapse of indefiniteness" to achieve definiteness. And the "wave-particle duality" is actually the *indistinct-distinct particle duality* or *complementarity*.

We have provided the back-story to objective indefiniteness by building the notion of distinctions from the ground up starting with partition logic and logical information theory. But the importance of distinctions and indistinguishability has been there all along in quantum mechanics.

Consider the standard double-slit experiment. When there is no distinction between the two slits, then the position attribute of the traversing particle is indefinite, neither top slit nor bottom slit (not "going through both slits"), which is usually interpreted as the "wave-like aspects" that show interference. But when a distinction is made between the slits, e.g., inserting a detector in one slit or closing one slit, then the distinction reduces the indefiniteness to definiteness so the indefiniteness disappears, i.e., the "wave-like aspects" disappear. For instance, Feynman makes this point about distinctions in terms of *distinguishing* the alternatives or "final states" (such as "traversing top slit" or "traversing bottom slit").

> If you could, *in principle*, distinguish the alternative *final* states (even though you do not bother to do so), the total, final probability is obtained by calculating the *probability* for each state (not the amplitude) and then adding them together. If you *cannot* distinguish the final states *even in principle*, then the probability amplitudes must be summed before taking the absolute square to find the actual probability.[13, p. 3-9]

Moreover, when the properties of entities are carved out by distinctions (starting at the blob), then it is perfectly possible to have two entities that result from the same distinctions but with no other distinctions so they are *in principle* indistinguishable (unlike two twins who are "hard to tell apart"). In QM, this has enormous consequences as in the distinction between bosons and fermions, the Pauli exclusion principle, and the chemical properties of the elements. This sort of in-principle indistinguishability is a feature of the micro-reality envisaged by partition logic, but is not possible under the "properties all the way down" vision of subset logic.

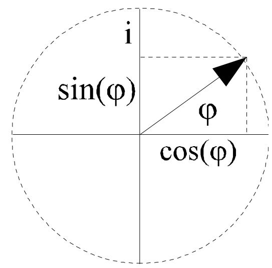
7.2 Wave math without waves = indistinctness-preserving mathematics

What about the Schrödinger *wave* equation? Since measurements, or, more generally, interactions between a quantum system and the environment, may make distinctions (measurement and decoherence), we might ask the following question. What is the evolution of a quantum system that is isolated so that not only are no distinctions made, but even the degree of indistinctness between state vectors is not changed? Two states ψ and φ in a Hilbert space are *fully distinct* if they are orthogonal, i.e., $\langle\psi|\varphi\rangle = 0$. Two states are *fully indistinct* if $\langle\psi|\varphi\rangle = 1$. In between, the *degree of indistinctness* can be measured by $\langle\psi|\varphi\rangle$, the inner product of the state vectors. Hence the evolution of as isolated quantum system where the degree of indistinctness does not change is described by a linear transformation that preserves inner products, i.e., a unitary transformation.[18]

The connection between unitary transformations and the solutions to the Schrödinger "wave" equation is given by Stone's Theorem [28]: there is a one-to-one correspondence between strongly continuous 1-parameter unitary groups $\{U_t\}_{t\in\mathbb{R}}$ and Hermitian operators A on the Hilbert space so that $U_t = e^{itA}$.

In simplest terms, a unitary transformation describes a rotation such as the rotation of the unit vector in the complex plane.

[18] A unitary transformation is an isomorphism of inner product vector spaces. In the model of "quantum mechanics" on sets where there is no inner product to be preserved, the delift would just be an isomorphism of vector spaces over $\mathbb{Z}_2$.



Rotating vector

The rotating unit vector traces out the cosine and sine functions on the two axes, and the position of the arrow can be compactly described as a function of φ using Euler's formula:

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi).$$

Such complex exponentials and their superpositions are the "wave functions" of QM. The "wave functions" describe the evolution of particles in indefinite states in isolated systems where there are no interactions to change the degree of indistinctness between states, i.e., the context where Schrödinger's equation holds. Previously it has been assumed that the mathematics of waves must describe physical waves of some sort, and thus the puzzlement about the "waves" of QM having complex amplitudes and no corresponding physical waves. But we have supplied *another* interpretation; wave mathematics is the mathematics of indefiniteness, e.g., superposition represents indefiniteness and unitary evolution represents the indistinctness-preserving evolution of an isolated system. Thus the objective indefiniteness approach to interpreting QM provides an explanation for the appearance of the wave mathematics (which implies interference as well as the quantized solutions to the "wave" equation that gave QM its name) when, in fact, there are no actual physical waves involved.

8 Logical entropy measures measurement

8.1 Logical entropy as the total distinction probability

The notion of logical entropy of a probability distribution $p = (p_1, ..., p_n)$, $h(p) = 1 - \sum_i p_i^2$, generalizes to the *quantum logical entropy* of a density matrix ρ [10],

$$h(\rho) = 1 - \mathrm{tr}\left[\rho^2\right].$$

Given a state vector $|\psi\rangle = \sum_i \alpha_i |i\rangle$ expressed in the orthonormal basis $\{|i\rangle\}_{i=1,...,n}$, the density matrix

$$\rho = |\psi\rangle\langle\psi| = [\rho_{ij}] = \left[\alpha_i \alpha_j^*\right]$$

(where α_j^* is the complex conjugate of α_j) is a *pure state* density matrix. For a pure state density matrix:

$$h(\rho) = 1 - \mathrm{tr}\left[\rho^2\right] = 1 - \sum_i \sum_j \alpha_i \alpha_j^* \alpha_j \alpha_i^* = 1 - \sum_i \alpha_i \alpha_i^* \sum_j \alpha_j \alpha_j^* = 1 - 1 = 0.$$

Otherwise, a density matrix ρ is said to represent a *mixed state*, and its logical entropy is positive.

In the set case, the logical entropy $h(\pi)$ of a partition π was interpreted as the probability that two independent draws from U (equiprobable elements) would give a distinction of π. For a probability distribution $p = (p_1, ..., p_n)$, the logical entropy $h(p) = 1 - \sum_i p_i^2$ is the probability that two independent samples from the distribution will give distinct outcomes $i \neq j$. The probability of the distinct outcomes (i, j) for $i \neq j$ is $p_i p_j$. Since $1 = (p_1 + ... + p_n)(p_1 + ... + p_n) = \sum_{i,j} p_i p_j$, we have:

$$h(p) = 1 - \sum_i p_i^2 = \sum_{i,j} p_i p_j - \sum_i p_i^2 = \sum_{i \neq j} p_i p_j$$

which is the sum of all the distinction (i.e., distinct indices) probabilities.

This interpretation generalizes to the quantum logical entropy $h(\rho)$. The diagonal terms $\{p_i\}$ in a density matrix:

$$\rho = \begin{bmatrix} p_1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & p_2 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & p_n \end{bmatrix}$$

are the probabilities of getting the i^{th} eigenvector $|i\rangle$ in a projective measurement of a system in the state ρ (using $\{|i\rangle\}$ as the measurement basis). The off-diagonal terms ρ_{ij} give the amplitude that the eigenstates $|i\rangle$ and $|j\rangle$ cohere, i.e., are indistinct, in the state ρ so the absolute square $|\rho_{ij}|^2$ is the *indistinction probability*. Since $p_i p_j$ is the probability of getting $|i\rangle$ and $|j\rangle$ in two independent measurements, the difference $p_i p_j - |\rho_{ij}|^2$, is the *distinction probability*. But $1 = \sum_{i,j} p_i p_j$ so we see that the interpretation of the logical entropy as the total distinction probability carries over to the quantum case:

$$h(\rho) = 1 - \mathrm{tr}\left[\rho^2\right] = 1 - \sum_{ij} |\rho_{ij}|^2 = \sum_{ij} \left[p_i p_j - |\rho_{ij}|^2\right] = \sum_{i \neq j} \left[p_i p_j - |\rho_{ij}|^2\right]$$
$$\text{Quantum logical entropy} = \text{sum of distinction probabilities}$$

where the last step follows since $p_i p_i - |\rho_{ii}|^2 = 0$.

8.2 Measuring measurement

Since $h(\rho) = 0$ for a pure state ρ, that means that all the eigenstates $|i\rangle$ and $|j\rangle$ cohere together, i.e., are indistinct, in a pure state, like the indiscrete partition or blob in the set case. For set partitions, the transition, $\mathbf{0} \to \mathbf{1}$, from the blob to the discrete partition turns all the indistinctions (u, u') (where $u \neq u'$) into distinctions, and the logical entropy increases from 0 to $1 - \sum_i p_i^2 = 1 - \frac{1}{n}$ where $p_i = \frac{1}{n}$ for $|U| = n$.

Similarly in quantum mechanics, a nondegenerate measurement turns a pure state density matrix ρ into the mixed state diagonal matrix $\hat{\rho}$ with the same diagonal entries $p_1, ..., p_n$:

$$\rho = \begin{bmatrix} p_1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & p_2 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & p_n \end{bmatrix} \overset{measurement}{\Rightarrow} \hat{\rho} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix}.$$

Hence the quantum logical entropy similarly goes from $h(\rho) = 0$ to $h(\hat{\rho}) = 1 - \sum_i p_i^2$. This is usually described by saying that all the off-diagonal coherence terms are decohered in a nondegenerate measurement–which means that all the indistinctions $(|i\rangle, |j\rangle)$ where $|i\rangle \neq |j\rangle$ of the pure state (like a mini-blob) are distinguished by the measurement. And the sum of all those new distinction probabilities for the decohered off-diagonal terms is precisely the quantum logical entropy since $h(\hat{\rho}) = \sum_{i \neq j} \left[p_i p_j - |\hat{\rho}_{ij}|^2\right] = \sum_{i \neq j} p_i p_j$. For any measurement (degenerate or not), the increase in logical entropy

$$h(\hat{\rho}) - h(\rho) = \sum_{new} |\rho_{ij}|^2 = \text{sum of } \textit{new} \text{ distinction probabilities}$$

where the sum is over the zeroed or decohered coherence terms $|\rho_{ij}|^2$ that gave indistinction probabilities in the pure state ρ. Thus we see how quantum logical entropy interprets the off-diagonal entries in the pure state density matrices and how the change in the quantum logical entropy measures precisely the decoherence, i.e., the distinctions, made by a measurement.[19]

9 Lifting to the axioms of quantum mechanics

We have now reached the point where the program of lifting partition logic and logical information theory to the quantum concepts of Hilbert spaces essentially yields the axioms of quantum mechanics.

Using axioms based on [21], the first axiom gives the vector space endpoint of the lifting program.

Axiom 1: *An isolated system is represented by a complex inner product vector space (i.e., a Hilbert space) where the complete description of a state of the system is given by a state vector, a unit vector in the system's space.*

Two fully distinct states would be orthogonal (thinking of them as eigenstates of an observable), and a state indefinite between them would be represented as a weighted vector sum or superposition of the two states.

We previously saw that the evolution of a closed system that preserves the degree of indistinction between states would be a unitary transformation.

Axiom 2: *The evolution of a closed quantum system is described by a unitary transformation.*

In the last section, we saw how a projective measurement would zero some or all of the off-diagonal coherence terms in a pure state ρ to give a mixed state $\hat{\rho}$ (and how the sum of the absolute squares of the zeroed coherence terms gave the change in quantum logical entropy).

Axiom 3: *A projective measurement for an observable (Hermitian operator) $M = \sum_m m P_m$ (spectral decomposition using projection operators P_m) on a pure state ρ has an outcome m with probability $p_m = \rho_{mm}$ giving the mixed state $\hat{\rho} = \sum_m P_m \rho P_m$.*

And finally we saw how the lifting program lifted the notion of combining sets with the direct product of sets $X \times Y$ to the notion of representing combined quantum systems with the vector space generated by taking the direct product of two bases of the state spaces.

Axiom 4: *The state space of a composite system is the tensor product of the state spaces of the component systems.*

[19] In contrast, the standard notion of entropy currently used in quantum information theory, the von Neumann entropy, is only qualitatively related to measurement, i.e., projective measurement increases von Neumann entropy [21, p. 515].

10 Conclusion

The objective indefiniteness interpretation of quantum mechanics is based on using partition logic, logical information theory, and the lifting program to fill out the back story to the old notion of "objective indefiniteness" ([25], [26]). In Appendix 1, the lifting program is further applied to lift set representations of groups to vector space representations, and thus to explain the fundamental importance of group representation theory in quantum mechanics (not to mention particle physics). But enough of the research program has already been presented to point to some conclusions.

At the level of sets, if we start with a universe set U as representing our common-sense macroscopic world, then there are *only two* logics, the logics of subsets and quotient sets (i.e., partitions), to envisage the "creation story" for U. Increase the size of subsets or increase the refinement of quotient sets until reaching the universe U. That is, starting with the empty subset of U, take larger and larger subsets of well-defined fully definite elements until finally reaching all the fully definite elements of U. Or starting with the indiscrete partition on U, take more and more refined partitions, each block interpreted as an indefinite element, until finally reaching all the fully definite elements of U. Those are the two dual options.

Classical mechanics was fully compatible with the subset creation story, where elements were always fully propertied ("properties all the way down"). But almost from the beginning, quantum mechanics was seen not to be compatible with that world view of always fully definite entities; QM seems to envisage entities at the micro-level that are objectively indefinite. Within the framework of the two logics given by subset-partition duality, the "obvious" thing to do is to elaborate on the dual creation story to try to build *the* other interpretation of QM.

With the development of the logic of partitions (dual to the logic of subsets) and logical information theory built on top of it, the foundation was in place to lift those set concepts to the richer mathematical environment of vector spaces (Hilbert spaces in particular). In that manner, *the* other interpretation of QM was constructed. Unlike the interpretation based on entities with fully definite properties expressed by Boolean subset logic, *the* dual interpretation works. That is, the result reproduces the basic ideas and mathematical machinery of quantum mechanics, e.g., as expressed in four axioms given above. That completes an outline of the vision of micro-reality that provides the objective indefiniteness interpretation of quantum mechanics.

11 Appendix 1: Lifting in group representation theory

11.1 Group representations define partitions

Given a *set* G of mappings $R = \{R_g : U \to U\}_{g \in G}$ on a set U, what are the conditions on the set of mappings so that it is a set representation of a group? Define the binary relation on $U \times U$:

$$u \sim u' \text{ if } \exists g \in G \text{ such that } R_g(u) = u'.$$

Then the conditions that make R into a group representation are the conditions that imply $u \sim u'$ is an equivalence relation:

1. existence of the identity $1_U \in U$ implies reflexivity of $\sim$;

2. existence of inverses implies symmetry of $\sim$; and

3. closure under products, i.e., for $g, g' \in G$, $\exists g'' \in G$ such that $R_{g''} = R_{g'} R_g$, implies transitivity of $\sim$.

Hence a set representation of a group might be seen as a "dynamic" way to define an equivalence relation and thus a partition on the set. Given this intimate connection between groups and partitions, it is then no surprise that group representation theory has a basic role to play in quantum mechanics and in the partition-based objective indefiniteness interpretation of QM.

11.2 Where do the fully distinct eigen-alternatives come from?

In the vector space case, we may be *given* the observable with its distinct eigenstates so the indefinite states are linear combinations of those eigenstates.

In the set case, we are *given* the universe U of distinct eigen-alternatives $u \in U$, and then the indistinct entities are the subsets such as the blocks $B \in \pi$ in a partition of U. A "measurement" is some distinction-making operation that reduces an indistinct state B down to a more distinct state $B' \subseteq B$ or, in the nondegenerate case, to a fully distinct singleton $\{u\}$ for some $u \in B$.

But where do the fully distinct elements $\{u\}$ come from? Answer:

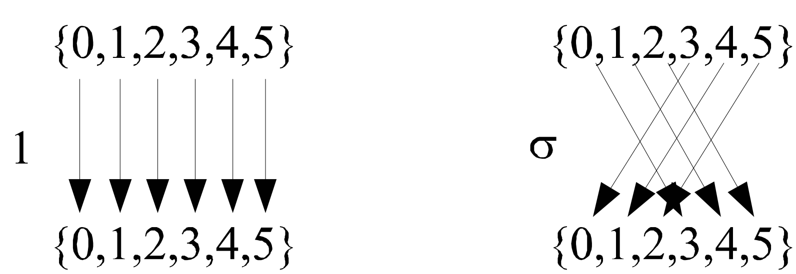
fully distinct elements $\approx$ orbits of a symmetry group representation.

Let U be a set and $S(U)$ the group of all permutations of U. Then a *set representation* of a group G is an assignment $R : G \rightarrow S(U)$ where for $g \in G$, $g \longmapsto R_g \in S(U)$ such that R_1 is the identity on U and for any $g, g' \in G$, $R_{g'} R_g = R_{g'g}$. Equivalently, a *group action* is a binary operation $G \times U \rightarrow U$ such that $1u = u$ and $g'(gu) = (g'g)u$ for all $u \in U$.

Defining $u \sim u'$ if $\exists g \in G$ such that $R_g(u) = u'$ [or $gu = u'$ using the group action notation], we have an equivalence relation on U where the blocks are called the *orbits*.

How are the ultimate distinct eigen-alternatives, the distinct "eigen-forms" of "substance," defined in the set case? Instead of just assuming U as the set of eigen-alternatives, we start with U as the carrier for a set representation of the group G as a group of symmetries. What are the smallest subsets (forming the blocks B in a set partition) that still have the symmetries, i.e., that are *invariant* in the sense that $R_g(B) \subseteq B$ for all $g \in G$? Those minimal invariant subsets are the orbits, and all invariant subsets are unions of orbits. Thus the orbits, thought of as points in the quotient set U/G (set of orbits), are the eigen-alternatives, the "eigen-forms" of "substance," defined by the symmetry group G in the set case.

Example 1: Let $U = \{0, 1, 2, 3, 4, 5\}$ and let $G = S_2 = \{1, \sigma\}$ (symmetric group on two elements) where $R_1 = 1_U$ and $R_\sigma(u) = u + 3 \bmod 6$.



Action of S_2 on $U = \{0, 1, 2, 3, 4, 5\}$

There are 3 orbits: $\{0,3\}, \{1,4\}$, and $\{2,5\}$. Those three orbits are the "points" in the quotient set U/G, i.e., they are the distinct eigen-alternatives defined by the symmetry group's S_2 action on U.

A *vector space representation* of a group G on a vector space V is a mapping $g \longmapsto R_g : V \to V$ from G to invertible linear transformations on V such that $R_{g'}R_g = R_{g'g}$.

The lifts to the vector space representations of groups are;

- minimal invariant subset = orbits $\overset{Lifts}{\longrightarrow}$ minimal invariant subspaces = *irreducible subspaces*,
- representation restricted to orbits $\overset{Lifts}{\longrightarrow}$ representation restricted to irreducible subspaces which gives the *irreducible representations* (the eigen-forms of substance in the vector space case[20]), and
- set partition of orbits $\overset{Lifts}{\longrightarrow}$ vector space partition of irreducible subspaces.

The "irreducible representations" in the set case are just the restrictions of the representation to the orbits, e.g., $R \upharpoonright \{0,3\} : S_2 \to S(\{0,3\})$, as their carriers. A set representation is said to be *transitive*, if for any $u, u' \in U$, $\exists g \in G$ such that $R_g(u) = u'$. A transitive set representation has only one orbit, all of U. Any set "irreducible representation" is transitive.

We are accustomed to thinking of some distinction-making operation as reducing a whole partition to a more refined partition, and thus breaking up a block B into distinguishable non-overlapping subsets $B', B'', ... \subseteq B$. Now we are working at the more basic level of determining the distinct eigen-alternatives, i.e., the orbits of a set representation of a symmetry group. Here we might also consider how distinctions are made to move to a more refined partition of orbits. Since the group operations identify elements, $u \sim u'$ if $\exists g \in G$ such that $R_g(u) = u'$, we would further *distinguish* elements by moving to a subgroup. The symmetry operations in the larger group are "broken," so the remaining group of symmetries is a subgroup.

Example 1 revisited: the group S_2 has only one subgroup, the trivial subgroup of the identity operation, and its orbits are clearly the singletons $\{u\}$ for $u \in U$. That is the simplest example of *symmetry-breaking* that gives a more distinct set of eigen-alternatives.

In any set representation, the maximum distinctions are made by the smallest symmetry subgroup which is always the identity subgroup, so that is always the waste case that takes us back to the singleton orbits in U, i.e., the distinct elements of U.

Thus we see that symmetry-breaking is analogous to measurement but at this more fundamental level where the distinct eigen-forms are determined in the first place by symmetry considerations.

[20] For a certain group of particle physics, "an elementary particle 'is' an irreducible unitary representation of the group."[27, p. 149] In Heisenberg's philosophical terms, the irreducible representations of certain symmetry groups of particle physics determine the fundamental eigen-forms that the substance (energy) can take.

> The elementary particles are therefore the fundamental forms that the substance energy must take in order to become matter, and these basic forms must in some way be determined by a fundamental law expressible in mathematical terms. ... The real conceptual core of the fundamental law must, however, be formed by the mathematical properties of the symmetry it represents.[16, pp. 16-17]

11.3 Attributes and observables

An (real-valued) *attribute* on a set U is a function $f : U \rightarrow \mathbb{R}$. An attribute induces a set partition $\left\{f^{-1}(r)\right\}$ on U. An attribute $f : U \rightarrow \mathbb{R}$ *commutes* with a set representation $R : G \rightarrow S(U)$ if for any R_g, the following diagram commutes in the sense that $fR_g = f$:

$$\begin{array}{ccc} U & \xrightarrow{R_g} & U \\ & \searrow^{f} & \downarrow^{f} \\ & & \mathbb{R} \end{array}$$

Commuting attribute.

The lifts to vector space representations are immediate:

- a real-valued attribute on a set $\xrightarrow{Lifts}$ an observable represented by a Hermitian operator on a complex vector space; and
- the commutativity condition on a set-attribute $\xrightarrow{Lifts}$ an observable operator H (like the Hamiltonian) commuting with a symmetry group in the sense that $HR_g = R_gH$ for all $g \in G$.

The commutativity-condition in the set case means that whenever $R_g(u) = u'$ then $f(u) = f(u')$, i.e., that f is an *invariant* of the group. Recall that each orbit of a set representation is transitive so for any u, u' in the same orbit, $\exists R_g$ such that $R_g(u) = u'$ so $f(u) = f(u')$ for any two u, u' in the same orbit. In other words:

"Schur's Lemma" (set version): a commuting attribute restricted to an orbit is constant.

The lift to vector space representations is one version of the usual

Schur's Lemma (vector space version): An operator H commuting with G restricted to irreducible subspace is a constant operator.

This also means that the inverse-image partition $\left\{f^{-1}(r)\right\}$ of a commuting attribute is refined by the orbit partition. If an orbit $B \subseteq f^{-1}(r)$, then the "eigenvalue" r of the attribute f is associated with that orbit. Every commuting attribute $f : U \rightarrow \mathbb{R}$ can be uniquely expressed as a decomposition:

$$f = \sum_{o \in Orbits} r_o \chi_o,$$

where r_o is the constant value on the orbit $o \subseteq U$ and $\chi_o : U \rightarrow \mathbb{R}$ is the characteristic function of the orbit o.

There may be other orbits with the same "eigenvalue." Then we would need another commuting attribute $g : U \rightarrow \mathbb{R}$ so that for each orbit B, there is an "eigenvalue" s of the attribute g such that $B \subseteq g^{-1}(s)$. Then the eigen-alternative B may be characterized by the ordered pair $|r, s\rangle$ if $B = f^{-1}(r) \cap g^{-1}(s)$. If not, we continue until we have a Complete Set of Commuting Attributes (CSCA) whose ordered n-tuples of "eigenvalues" would characterize the eigen-alternatives, the orbits of the set representation $R : G \rightarrow S(U)$.

Obviously, we are just spelling out the set version whose lift is the use of a Complete Set of Commuting Operators (CSCO) to characterize the eigenstates by kets of ordered n-tuples $|\lambda, \mu, ...\rangle$

of eigenvalues of the commuting operators.[21] But these "eigenstates" are not the singletons $\{u\}$ but are the minimal invariant subsets or orbits of the set representation of the symmetry group G.

Example 1 again: Consider the attribute $f : U = \{0,1,2,3,4,5\} \to \mathbb{R}$ where $f(n) = n \bmod 3$. This attribute commutes with the previous set representation of S_2, namely $R_1 = 1_U$ and $R_\sigma(u) = u + 3 \bmod 6$, and accordingly by "Schur's Lemma" (set version), the attribute is constant on each orbit $\{0,3\}, \{1,4\}$, and $\{2,5\}$. In this case, the blocks of the inverse-image partition $\{f^{-1}(0), f^{-1}(1), f^{-1}(2)\}$ equal the blocks of the orbit partition, so this attribute is the set version of a "nondegenerate measurement" in that its "eigenvalues" suffice to characterize the eigen-alternatives, i.e., the orbits. By itself, it forms a complete set of attributes.

Example 2: Let $U = \{0,1,...,11\}$ where $S_2 = \{1,\sigma\}$ is represented by the operations $R_1 = 1_U$ and $R_\sigma(n) = n+6 \bmod (12)$. Then the orbits are $\{0,6\}, \{1,7\}, \{2,8\}, \{3,9\}, \{4,10\}$, and $\{5,11\}$. Consider the attribute $f : U \to \mathbb{R}$ where $f(n) = n \bmod (2)$. This attribute commutes with the symmetry group and is thus constant on the orbits. But the blocks in the inverse-image partition are now larger than the orbits, i.e., $f^{-1}(0) = \{0,2,4,6,8,10\}$ and $f^{-1}(1) = \{1,3,5,7,9,11\}$ so the orbit partition strictly refines $\{f^{-1}(r)\}$. Thus this attribute corresponds to a degenerate measurement in that the two "eigenvalues" do not suffice to characterize the orbits.

Consider the attribute $g : U \to \mathbb{R}$ where $g(n) = n \bmod (3)$. This attribute commutes with the symmetry group and is thus constant on the orbits. The blocks in the inverse-image partition are: $g^{-1}(0) = \{0,3,6,9\}$, $g^{-1}(1) = \{1,4,7,10\}$, and $g^{-1}(2) = \{2,5,8,11\}$. The blocks in the join of the two partitions $\{f^{-1}(r)\}$ and $\{g^{-1}(s)\}$ are the non-empty intersections of the blocks:

$f^{-1}(r)$	$g^{-1}(s)$	$f^{-1}(r) \cap g^{-1}(s)$	$\|r,s\rangle$
$\{0,2,4,6,8,10\}$	$\{0,3,6,9\}$	$\{0,6\}$	$\|0,0\rangle$
$\{0,2,4,6,8,10\}$	$\{1,4,7,10\}$	$\{4,10\}$	$\|0,1\rangle$
$\{0,2,4,6,8,10\}$	$\{2,5,8,11\}$	$\{2,8\}$	$\|0,2\rangle$
$\{1,3,5,7,9,11\}$	$\{0,3,6,9\}$	$\{3,9\}$	$\|1,0\rangle$
$\{1,3,5,7,9,11\}$	$\{1,4,7,10\}$	$\{1,7\}$	$\|1,1\rangle$
$\{1,3,5,7,9,11\}$	$\{2,5,8,11\}$	$\{5,11\}$	$\|1,2\rangle$

f and g as a complete set of commuting attributes

Thus f and g form a Complete Set of Commuting Attributes to characterize the eigen-alternatives, the orbits, by the "kets" of ordered pairs of their "eigenvalues."

Example 3: Let $U = \mathbb{R}^2$ as a set and let G be the special orthogonal matrix group $SO(2,\mathbb{R})$ of matrices of the form;

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \text{ for } 0 \le \varphi < 2\pi.$$

This group is trivially represented by the rotations in $U = \mathbb{R}^2$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

[21] For a presentation of group representation theory that uses a CSCO approach to characterizing the irreducible representations, see [3].

The orbits are the circular orbits around the origin. The attribute "radius" $f:\mathbb{R}^2\to\mathbb{R}$ where $f(x,y)=\sqrt{x^2+y^2}$ commutes with the representation since:

$$\begin{aligned}
f(x',y') &= \sqrt{(x')^2+(y')^2} \\
&= \sqrt{(x\cos\varphi - y\sin\varphi)^2 + (x\sin\varphi + y\cos\varphi)^2} \\
&= \sqrt{x^2\left(\cos^2\varphi+\sin^2\varphi\right) + y^2\left(\cos^2\varphi+\sin^2\varphi\right)} \\
&= f(x,y).
\end{aligned}$$

That means that "radius" is an invariant of the rotation symmetry group. The blocks in the set partition $\left\{f^{-1}(r) : 0\le r\right\}$ of $\mathbb{R}^2$ coincide with the orbits so the "eigenvalues" of the radius attribute suffice to characterize the orbits.

Example 4: The Cayley set representation of any group G is given by permutations on $U=G$ itself defined by $R_g(g')=gg'$, which is also called the *left regular representation*. Given any $g,g'\in G$, $R_{g'g^{-1}}(g)=g'$ so the Cayley representation is always transitive, i.e., has only one orbit consisting of all of $U=G$. Since any commuting attribute $f:U=G\to\mathbb{R}$ is constant on each orbit, it can only be a constant function such as χ_G.

Thus the Cayley set representation is rather simple, but we could break some symmetry by considering a proper subgroup $H\subseteq G$. Then using only the R_h for $h\in H$, we have a representation $H\to S(G)$. The orbit-defining equivalence relation is $g\sim g'$ if $\exists h\in H$ such that $hg=g'$, i.e., the orbits are the *right cosets* of the form Hg.

Lift Program	Sets	Vector spaces
Representation	Group G represented by permutations $R_g:U\to U$	Group G represented by invertible linear ops. $R_g:V\to V$
Min. invariants	Orbits	Irreducible subspaces
Partition	U partitioned by orbits	V partitioned by irred. spaces
Irred. Reps.	Rep. R restricted to orbits	Rep. R restricted to irred. spaces
Commuting with G	Attribute $f:U\to\mathbb{R}$ commuting with R_g, i.e., $fR_g = f$, for all $g\in G$.	Operator $H:V\to V$ commuting with R_g, i.e., $HR_g = R_gH$, for all $g\in G$.
Invariant subsets (spaces)	Inverse images $f^{-1}(r)$ for commuting f are invariant.	Eigenspaces of commuting op. H are invariant.
Schur's Lemma	Attribute f commuting with G restricted to an orbit is constant function.	Operator H commuting with G restricted to irreducible subspace is a constant operator.

Summary: lifting group representations

12 Appendix 2: "Unitary evolution" and the two-slit experiment in "quantum mechanics" on sets

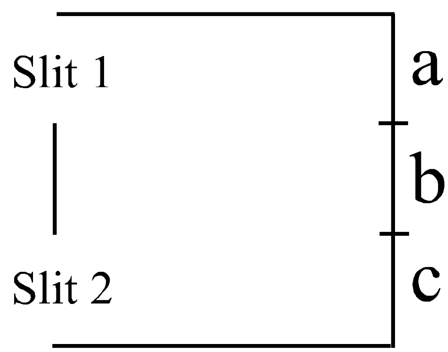
To illustrate a two-slit experiment in "quantum mechanics" on sets, we need to introduce some "dynamics." In quantum mechanics, the requirement was that the linear transformation had to preserve the degree of indistinctness $\langle\psi|\varphi\rangle$, i.e., that it preserved the inner product. Where two states are fully distinct if $\langle\psi|\varphi\rangle=0$ and fully indistinct if $\langle\psi|\varphi\rangle=1$, it is also sufficient to just require that full distinctness and indistinctness be preserved since that would imply orthonormal bases are preserved and that is equivalent to being unitary. In "quantum mechanics" on sets, we have no inner product but the idea of a linear transformation $\mathbb{Z}_2^n \to \mathbb{Z}_2^n$ preserving distinctness would simply mean being non-singular.

Hence our only requirement on the "dynamics" is that the change-of-state matrix is non-singular (so states are not merged). Consider the dynamics given in terms of the U-basis where: $\{a\} \to \{a,b\}$; $\{b\} \to \{a,b,c\}$; and $\{c\} \to \{b,c\}$ in one time period. This is represented by the non-singular one-period change of state matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The seven nonzero vectors in the vector space are divided by this "dynamics" into a 4 -orbit: $\{a\} \to \{a,b\} \to \{c\} \to \{b,c\} \to \{a\}$, a 2-orbit: $\{b\} \to \{a,b,c\} \to \{b\}$, and a 1-orbit: $\{a,c\} \to \{a,c\}$.

If we take the U-basis vectors as "vertical position" eigenstates, we can device a "quantum mechanics" version of the "two-slit experiment" which models "all of the mystery of quantum mechanics" [12, p. 130]. Taking a, b, and c as three vertical positions, we have a vertical diaphragm with slits at a and c. Then there is a screen or wall to the right of the slits so that a "particle" will travel from the diaphragm to the screen in one time period according to the A dynamics.



We start with or "prepare" the state of a particle being at the slits in the indefinite position state $\{a,c\}$. Then there are two cases.

First case of distinctions at slits: The first case is where we measure the U-state at the slits and then let the resultant position eigenstate evolve by the A-dynamics to hit the wall at the right where the position is measured again. The probability that the particle is at slit 1 or at slit 2 is:

$$\Pr(\{a\} \mid \{a,c\}) = \frac{\langle \{a\}|_U \{a,c\}\rangle^2}{\|\{a,c\}\|_U^2} = \frac{|\{a\} \cap \{a,c\}|}{|\{a,c\}|} = \frac{1}{2};$$
$$\Pr(\{c\} \mid \{a,c\}) = \frac{\langle \{c\}|_U \{a,c\}\rangle^2}{\|\{a,c\}\|_U^2} = \frac{|\{c\} \cap \{a,c\}|}{|\{a,c\}|} = \frac{1}{2}.$$

If the particle was at slit 1, i.e., was in eigenstate $\{a\}$, then it evolves in one time period by the A-dynamics to $\{a,b\}$ where the position measurements yield the probabilities of being at a or at b as:

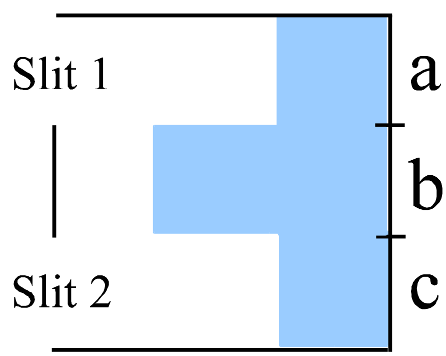
$$\Pr(\{a\} \mid \{a,b\}) = \frac{\langle\{a\}\,|_U\,\{a,b\}\rangle^2}{\|\{a,b\}\|_U^2} = \frac{|\{a\} \cap \{a,b\}|}{|\{a,b\}|} = \frac{1}{2}$$
$$\Pr(\{b\} \mid \{a,b\}) = \frac{\langle\{b\}\,|_U\,\{a,b\}\rangle^2}{\|\{a,b\}\|_U^2} = \frac{|\{b\} \cap \{a,b\}|}{|\{a,b\}|} = \frac{1}{2}.$$

If on the other hand the particle was found in the first measurement to be at slit 2, i.e., was in eigenstate $\{c\}$, then it evolved in one time period by the A-dynamics to $\{b,c\}$ where the position measurements yield the probabilities of being at b or at c as:

$$\Pr(\{b\} \mid \{b,c\}) = \tfrac{|\{b\}\cap\{b,c\}|}{|\{b,c\}|} = \tfrac{1}{2} \text{ and } \Pr(\{c\} \mid \{b,c\}) = \tfrac{|\{c\}\cap\{b,c\}|}{|\{b,c\}|} = \tfrac{1}{2}.$$

Hence we can use the laws of probability theory to compute the probabilities of the particle being measured at the three positions on the wall at the right if it starts at the slits in the superposition state $\{a,c\}$ *and* the measurements were made at the slits:

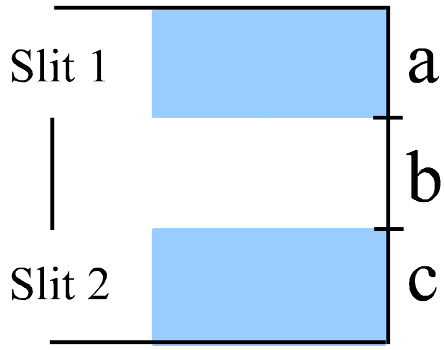
$$\begin{aligned}
&\Pr(\{a\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{4};\\
&\Pr(\{b\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \tfrac{1}{2}\tfrac{1}{2} + \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{2};\\
&\Pr(\{c\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{4}.
\end{aligned}$$



Final probability distribution with measurements at slits

Second case of no distinctions at slits: The second case is when no measurements are made at the slits and then the superposition state $\{a,c\}$ evolves by the A-dynamics to $\{a,b\} + \langle b,c\rangle = \{a,c\}$ where the superposition at $\{b\}$ cancels out. Then the final probabilities will just be probabilities of finding $\{a\}$, $\{b\}$, or $\{c\}$ when the measurement is made only at the wall on the right is:

$$\begin{aligned}
&\Pr(\{a\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \Pr(\{a\} \mid \{a,c\}) = \tfrac{|\{a\}\cap\{a,c\}|}{|\{a,c\}|} = \tfrac{1}{2};\\
&\Pr(\{b\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \Pr(\{b\} \mid \{a,c\}) = \tfrac{|\{b\}\cap\{a,c\}|}{|\{a,c\}|} = 0;\\
&\Pr(\{c\} \text{ at wall} \mid \{a,c\} \text{ at slits}) = \Pr(\{c\} \mid \{a,c\}) = \tfrac{|\{c\}\cap\{a,c\}|}{|\{a,c\}|} = \tfrac{1}{2}.
\end{aligned}$$



Final probability distribution with no measurement at slits

Since no "collapse" took place at the slits due to no distinctions being made there, the indistinct element $\{a,c\}$ evolved (rather than one or the other of the distinct elements $\{a\}$ or $\{c\}$). The action of A is the same on $\{a\}$ and $\{c\}$ as when they evolve separately since A is a linear operator but the two results are now added together *as part of the evolution*. This allows the "interference" of the two results and thus the cancellation of the $\{b\}$ term in $\{a,b\}+\langle b,c\rangle = \{a,c\}$. The addition is, of course, mod 2 (where $-1=+1$) so, in "wave language," the two "wave crests" that add at the location $\{b\}$ cancel out. When this indistinct element $\{a,c\}$ "hits the wall" on the right, there is an equal probability of that distinction yielding either of those eigenstates.

13 Appendix 3: Bell inequality in "quantum mechanics" on sets

A simple version of a Bell inequality can be derived in the case of $\mathbb{Z}_2^2$ with three bases $U=\{a,b\}$, $U'=\{a',b'\}$, and $U''=\{a'',b''\}$, and where the kets are:

kets	U-basis	U'-basis	U''-basis
$\|1\rangle$	$\{a,b\}$	$\{a'\}$	$\{a''\}$
$\|2\rangle$	$\{b\}$	$\{b'\}$	$\{a'',b''\}$
$\|3\rangle$	$\{a\}$	$\{a',b'\}$	$\{b''\}$
$\|4\rangle$	$\emptyset$	$\emptyset$	$\emptyset$

Ket table for $\wp(U)\cong\wp(U')\cong\wp(U'')\cong\mathbb{Z}_2^2$.

Attributes defined on the three universe sets U, U', and U'', such as say $\chi_{\{a\}}$, $\chi_{\{b'\}}$, and $\chi_{\{a''\}}$, are incompatible as can be seen in several ways. For instance the set partitions defined on U and U', namely $\{\{a\},\{b\}\}$ and $\{\{a'\},\{b'\}\}$, cannot be obtained as two different ways to partition the same set since $\{a\}=\{a',b'\}$ and $\{a'\}=\{a,b\}$, i.e., an "eigenstate" in one basis is a superposition in the other. The same holds in the other pairwise comparison of U and U'' and of U' and U''.

A more technical way to show incompatibility is to exploit the vector space structure of $\mathbb{Z}_2^2$ and to see if the projection matrices for $\{a\}\cap()$ and $\{b'\}\cap()$ commute. The basis conversion matrices between the U-basis and U'-basis are:

$$\mathcal{C}_{U\leftarrow U'}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \text{ and } \mathcal{C}_{U'\leftarrow U}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}.$$

The projection matrix for $\{a\} \cap ()$ in the U-basis is, of course, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and the projection matrix for $\{b'\} \cap ()$ in the U'-basis is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Converting the latter to the U-basis to check commutativity gives:

$$\begin{aligned}
[\{b'\} \cap ()]_U &= \mathcal{C}_{U \leftarrow U'} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{C}_{U' \leftarrow U} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.
\end{aligned}$$

Hence the commutativity check is:

$$\begin{aligned}
[\{a\} \cap ()]_U \, [\{b'\} \cap ()]_U &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \\
[\{b'\} \cap ()]_U \, [\{a\} \cap ()]_U &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

so the two operators for the "observables" $\chi_{\{a\}}$ and $\chi_{\{b'\}}$ do not commute. In a similar manner, it is seen that the three "observables" are mutually incompatible.

Given a ket in $\mathbb{Z}_2^2 \cong \wp(U) \cong \wp(U') \cong \wp(U'')$, and using the usual equiprobability assumption on sets, the probabilities of getting the different outcomes for the various "observables" in the different given states are given in the following table.

Given state \ Outcome of test	a	b	a'	b'	a''	b''
$\{a,b\} = \{a'\} = \{a''\}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	0
$\{b\} = \{b'\} = \{a'',b''\}$	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$
$\{a\} = \{a',b'\} = \{b''\}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1

State-outcome table.

The delift of the tensor product of vector spaces is the Cartesian or direct product of sets, and the delift of the vectors in the tensor product are the subsets of direct product of sets (as seen in the above treatment of entanglement in "quantum mechanics" on sets). Thus in the U-basis, the basis elements are the elements of $U \times U$ and the "vectors" are all the subsets in $\wp(U \times U)$. But we could obtain the same "space" as $\wp(U' \times U')$ and $\wp(U'' \times U'')$, and we can construct a ket table where each row is a ket expressed in the different bases. And these calculations in terms of sets could also be carried out in terms of vector spaces over $\mathbb{Z}_2$ where the rows of the ket table are the kets in the tensor product:

$$\mathbb{Z}_2^2 \otimes \mathbb{Z}_2^2 \cong \wp(U \times U) \cong \wp(U' \times U') \cong \wp(U'' \times U'').$$

Since $\{a\} = \{a', b'\} = \{b''\}$ and $\{b\} = \{b'\} = \{a'', b''\}$, the subset $\{a\} \times \{b\} = \{(a,b)\} \subseteq U \times U$ is expressed in the $U' \times U'$-basis as $\{a', b'\} \times \{b'\} = \{(a', b'), (b', b')\}$, and in the $U'' \times U''$-basis it is $\{b''\} \times \{a'', b''\} = \{(b'', a''), (b'', b'')\}$. Hence one row in the ket table has:

$$\{(a,b)\} = \{(a', b'), (b', b')\} = \{(b'', a''), (b'', b'')\}.$$

Since the full ket table has 16 rows, we will just give a partial table that suffices for our calculations.

$U \times U$	$U' \times U'$	$U'' \times U''$
$\{(a,a)\}$	$\{(a',a'),(a',b'),(b',a'),(b',b')\}$	$\{(b'',b'')\}$
$\{(a,b)\}$	$\{(a',b'),(b',b')\}$	$\{(b'',a''),(b'',b'')\}$
$\{(b,a)\}$	$\{(b',a'),(b',b')\}$	$\{(a'',b''),(b'',b'')\}$
$\{b,b\}$	$\{(b',b')\}$	$\{(a'',a''),(a'',b''),(b'',a''),(b'',b'')\}$
$\{(a,a),(a,b)\}$	$\{(a',a'),(b',a')\}$	$\{(b'',a'')\}$
$\{(a,a),(b,a)\}$	$\{(a',a'),(a',b')\}$	$\{(a'',b'')\}$
$\{(a,a),(b,b)\}$	$\{(a',a'),(a',b'),(b',a')\}$	$\{(a'',a''),(a'',b''),(b'',a'')\}$
$\{(a,b),(b,a)\}$	$\{(a',b'),(b',a')\}$	$\{(a'',b''),(b'',a'')\}$

Partial ket table for $\wp(U \times U) \cong \wp(U' \times U') \cong \wp(U'' \times U'')$

As before, we can classify each "vector" or subset as "separated" or "entangled" and we can furthermore see how that is independent of the basis. For instance $\{(a,a),(a,b)\}$ is "separated" since:

$$\{(a,a),(a,b)\} = \{a\} \times \{a,b\} = \{(a',a'),(b',a')\} = \{a',b'\} \times \{a'\} = \{(b'',a'')\} = \{b''\} \times \{a''\}.$$

An example of an "entangled state" is:

$$\{(a,a),(b,b)\} = \{(a',a'),(a',b'),(b',a')\} = \{(a'',a''),(a'',b''),(b'',a'')\}.$$

Taking this "entangled state" as the initial "state," there is a probability distribution on $U \times U' \times U''$ where $\Pr(a,a',a'')$ (for instance) is defined as the probability of getting the result $\{a\}$ if a U-measurement is performed on the left-hand system, and if instead a U'-measurement is performed on the left-hand system then $\{a'\}$ is obtained, and if instead a U''-measurement is performed on the left-hand system then $\{a''\}$ is obtained. Thus we would have $\Pr(a,a',a'') = \frac{1}{2}\frac{2}{3}\frac{2}{3} = \frac{2}{9}$. In this way the probability distribution $\Pr(x,y,z)$ is defined on $U \times U' \times U''$.

A Bell inequality can be obtained this joint probability distribution over the outcomes $U \times U' \times U''$ of measuring these three incompatible attributes [6]. Consider the following marginals:

$$\begin{aligned}
\Pr\left(a,a'\right) &= \Pr\left(a,a',a''\right) + \Pr\left(a,a',b''\right) \checkmark \\
\Pr\left(b',b''\right) &= \Pr\left(a,b',b''\right) \checkmark + \Pr\left(b,b',b''\right) \\
\Pr\left(a,b''\right) &= \Pr\left(a,a',b''\right) \checkmark + \Pr\left(a,b',b''\right) \checkmark .
\end{aligned}$$

The two terms in the last marginal are each contained in one of the two previous marginals (as indicated by the check marks) and all the probabilities are non-negative, so we have the following inequality:

$$\Pr(a,a') + \Pr(b',b'') \geq \Pr(a,b'')$$

Bell inequality.

All this has to do with measurements on the left-hand system. But there is an alternative interpretation to the probabilities $\Pr(x,y)$, $\Pr(y,z)$, and $\Pr(x,z)$ *if* we assume that the outcome of a measurement on the right-hand system is *independent* of the outcome of the same measurement on the left-hand system. Then $\Pr(a,a')$ is the probability of a U-measurement on the left-hand system giving $\{a\}$ and then a U'-measurement on the right-hand system giving $\{a'\}$, and so forth. Under that *independence assumption* and for this initially prepared "Bell state,"

$$\{(a,a),(b,b)\} = \{(a',a'),(a',b'),(b',a')\} = \{(a'',a''),(a'',b''),(b'',a'')\},$$

the probabilities would be the same.[22] That is, the probabilities, $\Pr(a) = \frac{1}{2} = \Pr(b)$, $\Pr(a') = \frac{2}{3} = \Pr(a'')$, and $\Pr(b') = \frac{1}{3} = \Pr(b'')$ are the same regardless of whether we are measuring the left-hand or right-hand system of that composite state. Hence the above Bell inequality would still hold. But we can use "quantum mechanics" on sets to compute the probabilities for those different measurements on the two systems to see if the independence assumption is compatible with "QM" on sets.

To compute $\Pr(a,a')$, we first measure the left-hand component in the U-basis. Since $\{(a,a),(b,b)\}$ is the given state, and (a,a) and (b,b) are equiprobable, the probability of getting $\{a\}$ (i.e., the "eigenvalue" 1 for the "observable $\chi_{\{a\}}$) is $\frac{1}{2}$. But the right-hand system is then in the state $\{a\}$ and the probability of getting $\{a'\}$ (i.e., "eigenvalue" 0 for the "observable" $\chi_{\{b'\}}$) is $\frac{1}{2}$ (as seen in the state-outcome table). Thus the probability is $\Pr(a,a') = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$.

To compute $\Pr(b',b'')$, we first perform a U'-basis "measurement" on the left-hand component of the given state $\{(a,a),(b,b)\} = \{(a',a'),(a',b'),(b',a')\}$, and we see that the probability of getting $\{b'\}$ is $\frac{1}{3}$. Then the right-hand system is in the state $\{a'\}$ and the probability of getting $\{b''\}$ in a U''-basis "measurement" of the right-hand system in the state $\{a'\}$ is 0 (as seen from the state-outcome table). Hence the probability is $\Pr(b',b'') = 0$.

Finally we compute $\Pr(a,b'')$ by first making a U-measurement on the left-hand component of the given state $\{(a,a),(b,b)\}$ and get the result $\{a\}$ with probability $\frac{1}{2}$. Then the state of the second system is $\{a\}$ so a U''-measurement will give the $\{b''\}$ result with probability 1 so the probability is $\Pr(a,b'') = \frac{1}{2}$.

Then we plug the probabilities into the Bell inequality:

$$\begin{gathered}\Pr(a,a') + \Pr(b',b'') \geq \Pr(a,b'') \\ \tfrac{1}{4} + 0 \ngeq \tfrac{1}{2} \\ \text{Violation of Bell inequality.}\end{gathered}$$

The violation of the Bell inequality shows that the independence assumption about the measurement outcomes on the left-hand and right-hand systems is incompatible with "QM" on sets so the effects of the "QM" on sets measurements are "nonlocal."

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