A Theory of Adjoint Functors - with some Thoughts about their Philosophical Significance

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Abstract. The question "What is category theory" is approached by focusing on universal mapping properties and adjoint functors. Category theory organizes mathematics using morphisms that transmit structure and determination. Structures of mathematical interest are usually characterized by some universal mapping property so the general thesis is that category theory is about determination through universals. In recent decades, the notion of adjoint functors has moved to center-stage as category theory's primary tool to characterize what is important and universal in mathematics. Hence our focus here is to present a theory of adjoint functors, a theory which shows that all adjunctions arise from the birepresentations of "chimeras" or "heteromorphisms" between the objects of different categories. Since representations provide universal mapping properties, this theory places adjoints within the framework of determination through universals. The conclusion considers some unreasonably effective analogies between these mathematical concepts and some central philosophical themes.

1 Introduction: What is Category Theory?

How might a question like "What is category theory?" be approached? Technically, the answer is well-known so that cannot be the point of the question.[1] The

[1]Some familiarity with basic category theory is assumed. Whenever possible, I will follow MacLane [28] on notation and terminology. Proofs will be avoided in favor of citations to the literature unless "the pudding is in the proof."

sense of the question is more about the philosophical or foundational importance of category theory (CT).

One proposed answer might be that CT provides the language in which to formulate topos theory which, in turn, provides a massive topologically-flavored generalization of the set theoretic foundations of mathematics. Although topos theory has been of great importance to release set theory's 'death grip' on foundations (and although my 1971 dissertation [10] was on generalizing ultraproducts to sheaves), I do not believe that the foundational importance of CT is exhausted by providing generalizations of set theory.

To understand category theory in its own way, we must return to the basic idea that CT sees the world of mathematics not just in terms of objects but in terms of morphisms. Morphisms express the transmission of structure or, broadly, the transmission of determination horizontally between objects. The morphism-view of the mathematical world gives quite a different perspective than set theory.

Both theories involve universals. But there are two types of universals, the non-self-participating, vertical, or "abstract" universals of set theory and the self-participating, horizontal, or "concrete" universals which are given by the universal mapping properties (UMPs) of category theory (see Ellerman [11], [13]). The abstract or non-self-participating universals of set theory collect together instances of a property but involve no machinery about the determination of the instances having the property. In contrast, a self-participating universal has the property itself and every other instance of the property has it by participating in (e.g., uniquely factoring through) the universal. The determination that the other instances have the property 'flows through' the universal to the instances.

In brief, our proposed answer to the question of "What is CT?" is that category theory is about determination represented by morphisms and the central structure is determination through the universals expressed by the UMPs.

Universals seem to always occur as part of an adjunction. Hence this research programme leads from the "What is CT?" question to a focus on adjoint functors. It is now widely recognized that adjoint functors characterize the structures that have importance and universality in mathematics. Our purpose here is to give a theory of "what adjoint functors are all about" that will sustain and deepen the thesis that category theory is about determination through universals.

Others have been lead to the focus on adjoints by different routes. Saunders MacLane and Samuel Eilenberg famously said that categories were defined in order to define functors, and functors were defined in order to define natural transformations. Their original paper [9] was named not "General Theory of Categories" but *General Theory of Natural Equivalences*. Adjoints were (surprisingly) only defined later [21] but the realization of their foundational importance has steadily increased over time [26, 24]. Now it would perhaps be not too much of an exaggeration to see categories, functors, and natural transformations as the prelude to defining adjoint functors. As Steven Awodey put it in his (forthcoming) text:

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> The notion of adjoint functor applies everything that we've learned up to now to unify and subsume all the different universal mapping properties that we have encountered, from free groups to limits to exponentials. But more importantly, it also captures an important mathematical phenomenon that is invisible without the lens of category theory. Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics. [2]

Other category theorists have given similar testimonials.

> To some, including this writer, adjunction is the most important concept in category theory. [35, p. 6]

> Nowadays, every user of category theory agrees that [adjunction] is the concept which justifies the fundamental position of the subject in mathematics. [34, p. 367]

Hence a theory of adjoint functors should help to elucidate the foundational importance of category theory.

2 Overview of the Theory of Adjoints

It might be helpful to begin with a brief outline of the argument. The basic building blocks of category theory are categories where a category consists of objects and (homo-)morphisms between the objects *within* the category, functors as homomorphisms between categories, and natural transformations as morphisms between functors. But there is another closely related type of entity that is routinely used in mathematical practice and is not 'officially' recognized in category theory, namely morphisms directly between objects in different categories such as the insertion of the generators $x \Rightarrow Fx$ from a set into the free group generated by the set. These cross-category object morphisms will be indicated by double arrows $\Rightarrow$ and will be called *chimera morphisms* (since their tail is in one category, e.g., a set, and their head is in another category, e.g., a group) or *heteromorphisms* in contrast with homomorphisms.

Since the heteromorphisms do not reside within a category, the usual categorical machinery does not define how they might compose. But that is not necessary. Chimera do not need to 'mate' with other chimera to form a 'species' or category; they only need to mate with the intra-category morphisms on each side to form other chimera. The appropriate mathematical machinery to describe that is the generalization of a group acting on a set to a generalized monoid or category acting on a set (where each element of the set has a "domain" and a "codomain" to determine when composition is defined). In this case, it is two categories acting on a set, one on the left and one on the right. Given a chimera morphism $c : x \Rightarrow a$ from an object in a category $\mathbf{X}$ to an object in a category $\mathbf{A}$ and morphisms $h : x' \to x$ in $\mathbf{X}$ and $k : a \to a'$ in $\mathbf{A}$, the composition

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$ch : x' \to x \Rightarrow a$ is another chimera $x' \Rightarrow a$ and the composition $kc : x \Rightarrow a \to a'$ is another chimera $x \Rightarrow a'$ with the usual identity, composition, and associativity properties. Such an action of two categories acting on a set on the left and on the right is exactly described by a het-bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ where $Het(x, a) = \{x \Rightarrow a\}$ and where **Set** is the category of sets and set functions. Thus the natural machinery to treat object-to-object chimera morphisms between categories are het-bifunctors $Het : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ that generalize the hom-bifunctors $Hom : \mathbf{X}^{op} \times \mathbf{X} \to \mathbf{Set}$ used to treat object-to-object morphisms within a category.

How might the categorical properties of the het-bifunctors be expressed without overtly recognizing chimera? Represent the het-bifunctors using hom-bifunctors on the left and on the right! Any bifunctor $\mathcal{D} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ is *represented on the left*[2] if for each x there is an object Fx in $\mathbf{A}$ and an isomorphism $Hom_{\mathbf{A}}(Fx, a) \cong \mathcal{D}(x, a)$ natural in a. It is a standard result that the assignment $x \mapsto Fx$ extends to a functor F and that the isomorphism is also natural in x. Similarly, $\mathcal{D}$ is *represented on the right* if for each a there is an object Ga in $\mathbf{X}$ and an isomorphism $\mathcal{D}(x, a) \cong Hom_{\mathbf{X}}(x, Ga)$ natural in x. And similarly, the assignment $a \mapsto Ga$ extends to a functor G and that the isomorphism is also natural in a.

If a het-bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ is represented on both the left and the right, then we have two functors $F : \mathbf{X} \to \mathbf{A}$ and $G : \mathbf{A} \to \mathbf{X}$ and the isomorphisms natural in x and in a:

$$Hom_{\mathbf{A}}(Fx, a) \cong Het(x, a) \cong Hom_{\mathbf{X}}(x, Ga).$$

It only remains to drop out the middle term $Het(x, a)$ to arrive at the wonderful *pas de deux* of the 'official' definition of a pair of adjoint functors—without any mention of heteromorphisms. That, in short, is our theory of "what adjoint functors are *really* about." Adjoint functors are of foundational relevance because of their ubiquity in picking out important structures in ordinary mathematics. For such concretely occurring adjoints, the heteromorphisms can be easily recovered (e.g., in all of our examples). But when an adjunction is abstractly defined—as always *sans* middle term—then where are the chimeras?

Hence to round out the theory, we give an "adjunction representation theorem" which shows how, given any adjunction $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$, heteromorphisms can be defined between (isomorphic copies of) the categories $\mathbf{X}$ and $\mathbf{A}$ so that (isomorphic copies of) the adjoints arise from the representations on the left and right of the het-bifunctor. The category $\mathbf{X}$ is embedded in the product category $\mathbf{X} \times \mathbf{A}$ by the assignment $x \mapsto (x, Fx)$ to obtain the isomorphic copy $\widehat{\mathbf{X}}$, and $\mathbf{A}$ is embedded in the product category by $a \mapsto (Ga, a)$ to yield the isomorphic copy $\widehat{\mathbf{A}}$. Then the properties of the adjunction can be nicely expressed by the commutativity within $\mathbf{X} \times \mathbf{A}$ of "adjunctive squares" of the form:

[2]This terminology "represented on the left" or "on the right" is used to agree with the terminology for left and right adjoints.

$$\begin{array}{ccccc} & (x, Fx) & \stackrel{(f,Ff)}{\longrightarrow} & (Ga, FGa) & \\ (\eta_x, 1_{Fx}) & \downarrow & & \downarrow & (1_{Ga}, \varepsilon_a) \\ & (GFx, Fx) & \stackrel{(Gg,g)}{\longrightarrow} & (Ga, a) & \end{array}$$

where the main diagonal (f, g) in a commutative adjunctive square pairs together maps that are images of one another in the adjunction isomorphism $g = f^*$ and $f = g^*$ (i.e., are adjoint transposes of one another). Since the maps on top are always in $\widehat{\mathbf{X}}$ and the maps on the bottom are in $\widehat{\mathbf{A}}$, the main diagonal pairs of maps (including the vertical maps)—which are ordinary morphisms in the product category—have all the categorical properties of chimera morphisms from objects in $\mathbf{X} \cong \widehat{\mathbf{X}}$ to objects in $\mathbf{A} \cong \widehat{\mathbf{A}}$. Hence the heteromorphisms are abstractly defined as the pairs of adjoint transposes, $Het(x, a) = \{(x, Fx) \stackrel{(f,f^*)}{\longrightarrow} (Ga, a)\}$, and the adjunction representation theorem is that (isomorphic copies of) the original adjoints F and G arise from the representations on the left and right of this het-bifunctor.

Another bifunctor $\mathcal{Z}(Fx, Ga)$ of chimeras is later defined whose elements are heteromorphisms in the opposite direction (from $\mathbf{A}$ to $\mathbf{X}$). They are defined only on the images Fx and Ga of the pair of adjoint functors. These heteromorphisms would be represented by the southwest-to-northeast anti-diagonal maps (Gg, Ff) in an adjunctive square. Moreover, this fourth bifunctor is naturally isomorphic to the other three bifunctors.

$$Hom_{\mathbf{A}}(Fx, a) \cong Het(x, a) \cong \mathcal{Z}(Fx, Ga) \cong Hom_{\mathbf{X}}(x, Ga).$$

The universals of the unit η_x and counit ε_a are associated in the isomorphism respectively with the identity maps 1_{Fx} and 1_{Ga}. The elements of the other two chimera bifunctors associated with these identities also have universality properties. The identity 1_{Fx} is associated with h_x in $Het(x, Fx)$ and h_{x2} in $\mathcal{Z}(Fx, GFx)$, and 1_{Ga} is associated with e_a in $Het(Ga, a)$ and with e_{a1} in $\mathcal{Z}(FGa, Ga)$. The two h-universals provide an over-and-back factorization of the unit: $x \stackrel{h_x}{\Longrightarrow} Fx \stackrel{h_{x2}}{\Longrightarrow} GFx = x \stackrel{\eta_x}{\longrightarrow} GFx$, and the two e-universals give an over-and-back factorization of the counit: $FGa \stackrel{e_{a1}}{\Longrightarrow} Ga \stackrel{e_a}{\Longrightarrow} a = FGa \stackrel{\varepsilon_a}{\longrightarrow} a$. These chimera universals provide another factorization of any $f : x \to Ga$ in addition to the usual one as well as another factorization of any $g : Fx \to a$. There is also another over-and-back factorization of 1_{Fx} and of 1_{Ga} in addition to the triangular identities. Moreover, there is a new type of all-chimera factorization. Given any heteromorphism $x \stackrel{c}{\Rightarrow} a$, there is a unique $\mathbf{A}$-to-$\mathbf{X}$ chimera morphism $Fx \stackrel{z(c)}{\Longrightarrow} Ga$ that factors c through the chimera version of the unit, i.e., h_x, *and* through the chimera version of the counit, i.e., e_a, in the over-back-and-over-again or zig-zag factorization:

$$x \stackrel{c}{\Rightarrow} a = x \stackrel{h_x}{\Longrightarrow} Fx \stackrel{z(c)}{\Longrightarrow} Ga \stackrel{e_a}{\Longrightarrow} a.$$

Roughly speaking, the chimeras show their hybrid vigor by more than doubling the number of factorizations and identities associated with an adjunction.

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What may be new and what isn't new in this theory of adjoints? The theory contains no strikingly new formal results; the level of the category theory involved is all quite basic. The heteromorphisms are formally treated using bifunctors of the form $Het : \mathbf{X}^{op} \times \mathbf{A} \rightarrow \mathbf{Set}$. Such bifunctors and generalizations replacing **Set** by other categories have been studied by the Australian school under the name of *profunctors* [23], by the French school under the name of *distributors* [4], and by William Lawvere under the name of *bimodules* [27].[3] However, the guiding interpretation has been interestingly different. "Roughly speaking, a distributor is to a functor what a relation is to a mapping" [5, p. 308] (and hence the name "profunctor" in the Australian school). For instance, if **Set** was replaced by **2**, then the bifunctor would just be the characteristic function of a relation from **X** to **A**. Hence in the general context of enriched category theory, a "bimodule" $Y^{op} \otimes X \stackrel{\varphi}{\longrightarrow} \mathcal{V}$ would be interpreted as a "$\mathcal{V}$-valued relation" and an element of $\varphi(y, x)$ would be interpreted as the "truth-value of the φ-relatedness of y to x" [27, p. 158 (p. 28 of reprint)]. The subsequent development of profunctors-distributors-bimodules has been along the lines suggested by that guiding interpretation.

In the approach taken here, the elements $x \Rightarrow a$ in $Het(x, a)$ are interpreted as heteromorphisms from an object x in **X** to an object a in **A** on par with the morphisms within **X** or **A**, not as an element in a 'relational' generalization of a functor from **X** to **A**. Such chimeras exist in the wild (i.e., in mathematical practice) but are not in the 'official' ontological zoo of category theory that sees object-to-object morphisms as only existing *within* a category. The principal novelty here (to my knowledge) is the use of the chimera morphism interpretation of these bifunctors to carry out a whole program of interpretation for adjunctions, i.e., a *theory* of adjoint functors. In the concrete examples, chimera morphisms have to be "found" as is done in the broad classes of examples treated here. However, in general, the adjunction representation theorem uses a very simple construction to show how 'abstract' heteromorphisms can always be found so that any adjunction arises (up to isomorphism) out of the representations on the left and right of the het-bifunctor of such heteromorphisms.

These conceptual structures suggest various applications discussed in the conclusions.[4] Following this overview, we can now turn to a more leisurely development of the concepts. Universal mapping properties and representations are two ways in which "universals" appear in category theory so we next turn to the contrasting universals of set theory and category theory.

3 Universals

The general notion of a universal for a property is ancient. In Plato's Theory of Ideas or Forms ($\epsilon\iota\delta\eta$), a property F has an entity associated with it, the *universal*

[3] Thanks to John Baez for these connections with the literature on enriched categories.
[4] See Lambek [24] for rather different philosophical speculations about adjoint functors.

u_F, which uniquely represents the property. An object x has the property F, i.e., $F(x)$, if and only if (iff) the object x *participates* in the universal u_F. Let μ (from $\mu\epsilon\theta\epsilon\xi\iota\sigma$ or methexis) represent the participation relation so "$x\,\mu\,u_F$" reads as "x participates in u_F". Given a relation μ, an entity u_F is said to be a *universal for the property F* (with respect to μ) if it satisfies the following *universality condition*: for any x, $x\,\mu\,u_F$ if and only if $F(x)$.

A universal representing a property should be in some sense unique. Hence there should be an equivalence relation ($\cong$) so that universals satisfy a *uniqueness condition*: if u_F and u'_F are universals for the same F, then $u_F \cong u'_F$. These might be taken as the bare essentials for any notion of a universal for a property.

Set theory defines "participation" as membership represented by the $\in$ taken from $\epsilon\iota\delta\eta$. The universal for a property is set of objects with that property $\{x|F(x)\}$, the "extension" of the property, so the universality condition becomes the comprehension scheme:

$$x \in \{x|F(x)\} \text{ iff } F(x).$$

The equivalence of universals for the same property takes the strong form of identity of sets if they have the same members (extensionality). The universals of set theory collect together objects or entities with the property in question to form a new entity; there is no machinery or structure for the universal to 'determine' that the instances have the property. But 'if' they have the property, then they are included in the set-universal for that property.

The Greeks not only had the notion of a universal for a property; they also developed the notion of hubris. Frege's hubris was to try to have a general theory of universals that could be either self-participating or not-self-participating. This led to the paradoxes such as Russell's paradox of the universal R for all and only the universals that are not self-participating. If R does not participate in itself, then it must participate in itself. And if R does participate in itself then it must be non-self-participating.[5]

Russell's paradox drove set theory out of Frege's Paradise. As set theory was reconstructed to escape the paradoxes, the set-universal for a property was always non-self-participating. Thus set theory became not "*the* theory of universals" but the theory of non-self-participating universals.

The reformulation of set theory cleared the ground for a separate theory of always-self-participating universals. That idea was realized in category theory [11] by the objects having universal mapping properties. The self-participating universal for a property (if it exists) is the paradigmatic or archetypical example of the property. All instances of the property are determined to have the property by a morphism "participating" in that paradigmatic instance (where the universal "participates" in itself by the identity morphism). The logic of category theory is

[5]Note that Russell's paradox was formulated using the general notion of participation, not simply for the case where participation was set membership.

the immanent logic of determination through morphisms—particularly through universal morphisms (see example below).

The notion of determination-by-morphisms plays the conceptually primitive role in category theory analogous to the primitive notion of collection in set theory [16]. The set universal for a property collects together the instances of the property but does not have the property itself. The category-universal for a property has the property itself and determines the instances of the property by morphisms.

The intuitive notion of a concrete universal[6] occurs in ordinary language (any archetypical or paradigmatic reference such as the "all-American boy" or the "perfect" example of something), in the arts and literature (the old idea that great art uses a concrete instance to universally exemplify certain human conditions, e.g., Shakespeare's *Romeo and Juliet* exemplifies romantic tragedies), and in philosophy (the pure example of F-ness with no imperfections, no junk, and no noise; only those attributes necessary for F-ness). Some properties are even defined by means of the concrete universal such as the property of being Lincolnesqe. Abraham Lincoln is the concrete universal for the property and all other persons with the property have it by virtue of resembling the concrete universal. The vague intuitive notion of a concrete or self-participating universal becomes quite precise in the universal mapping properties of CT.

Perhaps the breakthrough was MacLane's characterization of the direct product $X \times Y$ of (say) two sets X and Y by a UMP. The property in question is being "a pair of maps, one to X and one to Y, with a common domain." The self-participating universal is the universal object $X \times Y$ and the pair of projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$. Given any other pair with the property, $f : W \to X$ and $g : W \to Y$, there is a unique factor map $h : W \to X \times Y$ such that $p_X h = f$ and $p_Y h = g$. The pair (f, g) are said to *uniquely factor through* the projections (p_X, p_Y) by the factor map h. Thus a pair (f, g) has the property if and only if it participates in (uniquely factors through) the self-participating universal (p_X, p_Y). The UMP description of the product characterizes it up to isomorphism but it does not show existence.

I will give a conceptual description of its construction that will be of use later. Maps carry determination from one object to another. In this case, we are considering determinations in both X and Y by some common domain set. At the most 'atomic' level, the determiner would be a one-point set which would pick out 'determinees', a point x in X and a point y in Y. Thus the most atomic determinees are the ordered pairs (x, y) of elements from X and Y. What would

[6]I am ignoring some rather woolly notions of "concrete universal" that exist in the philosophical literature other than this notion of the concrete universal as the paradigm or exemplary instance of a property. Also since the adjectives "concrete" and "abstract" are already freighted with other meanings in category theory, I will refer to the concrete universals in that context as self-participating universals. Such a universal is "concrete" only in the sense that it is one among the instances of the property. Certainly all the examples we meet in category theory are otherwise quite abstract mathematical entities.

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be the most universal determiner of (x, y)? Universality is achieved by making the conceptual move of reconceptualizing the atoms (x, y) from being just a determinee to being its own determiner, i.e., self-determination. Thus the universal determiner of X and Y is the set of all the ordered pairs (x, y) from X and Y, denoted $X \times Y$. It has the UMP for the product and all other sets with that UMP would be isomorphic to it.

"Determination through self-determining universals" sounds off-puttingly 'philosophical' but it nevertheless provides the abstract conceptual framework for the theory of universals and adjoint functors, and thus for the view of category theory presented here. For instance, a determinative relation has both a determiner or sending end and a determinee or receiving end so there should be a pair of universals, one at each end of the relationship, and that is precisely the conceptual source of the pair of universals in an adjunction. But that is getting ahead of the story.

Consider the dual case of the coproduct or disjoint union of sets. What is the self-participating universal for two sets X and Y to determine a common codomain set by a map from X and a map from Y? At the most atomic level, what is needed to determine a single point in the codomain set? A single point x in X or a single point y in Y mapped to that single point would be sufficient to determine it. Hence those single points from X or Y are the atomic determiners or "germs" to determine a single point. What would be the most universal determinee of x or of y? Universality is achieved by making the conceptual move of reconceptualizing x and y from being determiners to being their own determinees, i.e., self-determination. Thus the universal determinee of X and Y is the set of all the x from X and the y from Y, denoted $X + Y$. Since the two maps from X and Y can be defined separately, any elements that might be common to the two sets (i.e., in the intersection of the sets) are separate determiners and thus would be separate determinees in $X + Y$. Thus it is not the union but the disjoint union of X and Y.

The universal instance of a pair of maps from X and Y to a common codomain set is the pair of injections $i_X : X \to X + Y$ and $i_Y : Y \to X + Y$ whereby each determiner x in X or y in Y determine themselves as determinees in $X + Y$. For any other pair of maps $f : X \to Z$ and $g : Y \to Z$ from X and Y to a common codomain set Z, there is a unique factor map $h : X + Y \to Z$ such that $hi_X = f$ and $hi_Y = g$. All determination of a common codomain set by X and Y flows or factors through its own self-determination $i_X : X \to X + Y$ and $i_Y : Y \to X + Y$.

The product (or coproduct) universal could be used to illustrate the point about the logic of category theory being the immanent logic of determination through morphisms and particularly through universals. Consider an inference in conventional logic: all roses are beautiful, r is a rose, and therefore r is beautiful. This could be formulated in the *language* of category theory as follows: there is an inclusion of the set of roses R in the set of beautiful things B, i.e., $R \hookrightarrow B$, r is an element of R, i.e., $1 \xrightarrow{r} R$, so by composition r is an element of B, i.e.,

$1 \xrightarrow{r} R \hookrightarrow B = 1 \xrightarrow{r} B$. This type of reformulation can be highly generalized in category theory but my point is that this is only using the powerful concepts of category theory to formulate and generalize conventional logic.

The immanent logic of category theory is different. Speaking in 'philosophical mode' for the moment, suppose there are two (self-participating or 'concrete') universals "The Rose" U_R and "The Beautiful" U_B and that the The Rose participates in The Beautiful by a morphism $U_R \to U_B$ which determines that The Rose is beautiful. A particular entity r participates in The Rose by a morphism $r \to U_R$ which determines that r is a rose. By composition, the particular rose r participates in The Beautiful by a morphism $r \to U_R \to U_B = r \to U_B$ which determines that the rose is beautiful. The logical inference is immanent in the determinative morphisms.

For a categorical version, replace the property of being beautiful by the property for which the product projections (p_X, p_Y) were the universal, i.e., the property of being "a pair of maps (f, g), one to X and one to Y, with a common domain, e.g., $f : W \to X$ and $g : W \to Y$." Suppose we have another map $h : X \to Y$. Replace the property of being a rose with the property of being "a pair of maps (f, g), one to X and one to Y, with a common domain such that $hf = g$, i.e., $W \xrightarrow{f} X \xrightarrow{h} Y = W \xrightarrow{g} Y$. The universal for that property ("The Rose") is given by a pair of projections (π_X, π_Y) with a common domain $Graph(h)$, the *graph* of h. There is a unique map $Graph(h) \xrightarrow{p} X \times Y$ so that the graph participates in the product, i.e., $Graph(h) \xrightarrow{p} X \times Y \xrightarrow{p_X} X = Graph(h) \xrightarrow{\pi_X} X$ and similarly for the other projection. Now consider a particular "rose", namely a pair of maps $r_X : W \to X$ and $r_Y : W \to Y$ that commute with h ($hr_X = r_Y$) and thus participate in the graph, i.e., there is a unique map $r : W \to Graph(h)$ such that $W \xrightarrow{r} Graph(h) \xrightarrow{\pi_X} X = W \xrightarrow{r_X} X$ and $W \xrightarrow{r} Graph(h) \xrightarrow{\pi_Y} Y = W \xrightarrow{r_Y} Y$. Then by composition, the particular pair of maps (r_X, r_Y) also participates in the product, i.e., pr is the unique map such that $p_X pr = r_X$ and $p_Y pr = r_Y$. Not only does having the property imply participation in the universal, any entity that participates in the universal is thereby forced to have the property. For instance, because (r_X, r_Y) participates in the product, it has to be a pair of maps to X and to Y with a common domain (the property represented by the product).

This example of the immanent logic of category theory, where maps play a determinative role and the determination is through universals, could be contrasted to formulating and generalizing the set treatment of the inference in the language of categories. Let P be the set of pairs of maps to X and Y with a common domain, let G be the set of pairs of such maps to X and Y that also commute with h, and let (r_X, r_Y) be a particular pair of such maps. Then $1 \xrightarrow{(r_X, r_Y)} G \hookrightarrow P = 1 \to P$. That uses the language of category theory to formulate the set treatment with the set-universals for those properties and it ignores the category-universals for those properties.

Our purpose here is not to further develop the (immanent) logic of category theory but to show how adjoint functors fit within that general framework of determination through universals.

4 Definition and Directionality of Adjoints

There are many equivalent definitions of adjoint functors (see MacLane [28]), but the most 'official' one seems to be the one using a natural isomorphism of hom-sets. Let $\mathbf{X}$ and $\mathbf{A}$ be categories and $F : \mathbf{X} \to \mathbf{A}$ and $G : \mathbf{A} \to \mathbf{X}$ functors between them. Then F and G are said to be a pair of *adjoint functors* or an *adjunction*, written $F \dashv G$, if for any x in $\mathbf{X}$ and a in $\mathbf{A}$, there is an isomorphism ϕ natural in x and in a:

$$\phi_{x,a} : Hom_{\mathbf{A}}(Fx, a) \cong Hom_{\mathbf{X}}(x, Ga).$$

With this standard way of writing the isomorphism of hom-sets, the functor F on the left is called the *left adjoint* and the functor G on the right is the *right adjoint*. Maps associated with each other by the adjunction isomorphism ("adjoint transposes" of one another) are indicated by an asterisk so if $g : Fx \to a$ then $g^* : x \to Ga$ is the associated map $\phi_{x,a}(g) = g^*$ and similarly if $f : x \to Ga$ then $\phi_{x,a}^{-1}(f) = f^* : Fx \to a$ is the associated map.

In much of the literature, adjoints are presented in a seemingly symmetrical fashion so that there appears to be no directionality of the adjoints between the categories $\mathbf{X}$ and $\mathbf{A}$. But there is a directionality and it is important in understanding adjoints. Both the maps that appear in the adjunction isomorphism, $Fx \to a$ and $x \to Ga$, go from the "x-thing" (i.e., either x or the image Fx) to the "a-thing" (either the image Ga or a itself), so we see a direction emerging from $\mathbf{X}$ to $\mathbf{A}$. That direction of an adjunction is the direction of the left adjoint (which goes from $\mathbf{X}$ to $\mathbf{A}$). Then $\mathbf{X}$ might called the *sending* category and $\mathbf{A}$ the *receiving* category.[7]

Bidirectionality of determination through adjoints occurs when a functor has both a left and right adjoint. We will later see such an example of bidirectionality of determination between sets and diagram functors where the limit and colimit functors are respectively right and left adjoints to the same constant functor.[8]

In the theory of adjoints presented here, the directionality of adjoints results from being representations of heteromorphisms which have that directionality. Such morphisms can be exhibited in concrete examples of adjoints (see the later examples). To abstractly define chimera morphisms or heteromorphisms that

[7] Sometimes adjunctions are written with this direction as in the notation $\langle F, G, \phi\rangle : \mathbf{X} \rightharpoonup \mathbf{A}$ (MacLane [28, p.78]). This also allows the composition of adjoints to be defined in a straightforward manner (MacLane [28, p.101]).

[8] Also any adjunction can be restricted to subcategories where the unit and counit are isomorphisms so that each adjoint is both a left and right adjoint of the other—and thus determination is bidirectional—on those subcategories (see [24] or [3]).

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work for all adjunctions, we turn to the presentation of adjoints using adjunctive squares.

5 Adjunctive Squares

5.1 Embedding Adjunctions in a Product Category

Our approach to a theory of adjoints uses a certain "adjunctive square" diagram that is in the product category $\mathbf{X} \times \mathbf{A}$ associated with an adjunction $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$. With each object x in the category $\mathbf{X}$, we associate the element $\widehat{x} = (x, Fx)$ in the product category $\mathbf{X} \times \mathbf{A}$ so that Ga would have associated with it $\widehat{Ga} = (Ga, FGa)$. With each morphism in $\mathbf{X}$ with the form $h : x' \to x$, we associate the morphism $\widehat{h} = (h, Fh) : \widehat{x'} = (x', Fx') \to \widehat{x} = (x, Fx)$ in the product category $\mathbf{X} \times \mathbf{A}$ (maps compose and diagrams commute component-wise). Thus the mapping of x to (x, Fx) extends to an embedding $(1_{\mathbf{X}}, F) : \mathbf{X} \to \mathbf{X} \times \mathbf{A}$ whose image $\widehat{\mathbf{X}}$ is isomorphic with $\mathbf{X}$.

With each object a in the category $\mathbf{A}$, we associate the element $\widehat{a} = (Ga, a)$ in the product category $\mathbf{X} \times \mathbf{A}$ so that Fx would have associated with it $\widehat{Fx} = (GFx, Fx)$. With each morphism in $\mathbf{A}$ with the form $k : a \to a'$,we associate the morphism $\widehat{k} = (Gk, k) : (Ga, a) \to (Ga', a')$ in the product category $\mathbf{X} \times \mathbf{A}$. The mapping of a to (Ga, a) extends to an embedding $(G, 1_{\mathbf{A}}) : \mathbf{A} \to \mathbf{X} \times \mathbf{A}$ whose image $\widehat{\mathbf{A}}$ is isomorphic to $\mathbf{A}$.

The mapping of x to (GFx, Fx) in $\widehat{\mathbf{A}}$ extends to the functor $(GF, F) : \mathbf{X} \to \mathbf{X} \times \mathbf{A}$. Given that F and G are adjoints, the unit natural transformation $\eta : 1_{\mathbf{X}} \to GF$ of the adjunction and the identity natural transformation 1_F give a natural transformation $(\eta, 1_F) : (1_{\mathbf{X}}, F) \to (GF, F)$ with the component at x being $(\eta_x, 1_{Fx}) : (x, Fx) \to (GFx, Fx)$.

The mapping of a to (Ga, FGa) in $\widehat{\mathbf{X}}$ extends to a functor $(G, FG) : \mathbf{A} \to \mathbf{X} \times \mathbf{A}$. The counit natural transformation ε of the adjunction and the identity natural transformation 1_G give a natural transformation $(1_G, \varepsilon) : (G, FG) \to (G, 1_{\mathbf{A}})$ with the component at a being $(1_G, \varepsilon_a) : (Ga, FGa) \to (Ga, a)$.

The (F, G) twist functor, which carries (x, a) to (Ga, Fx), is an endo-functor on $\mathbf{X} \times \mathbf{A}$ which carries $\widehat{\mathbf{X}}$ to $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{A}}$ to $\widehat{\mathbf{X}}$ to reproduce the adjunction between those two subcategories.

These various parts can then be collected together in the adjunctive square diagram.

$$\begin{array}{ccccc} & (x, Fx) & \stackrel{(f,Ff)}{\to} & (Ga, FGa) & \\ (\eta_x, 1_{Fx}) & \downarrow & & \downarrow & (1_{Ga}, \varepsilon_a) \\ & (GFx, Fx) & \stackrel{(Gg,g)}{\longrightarrow} & (Ga, a) & \end{array}$$

Adjunctive Square Diagram

Any diagrams of this form for some given $f : x \to Ga$ or for some given $g : Fx \to a$ will be called *adjunctive squares*. The purpose of the adjunctive square diagram is to conveniently represent the properties of an adjunction in the format of commutative squares. The map on the top is in $\widehat{\mathbf{X}}$ (the "top category") and the map on the bottom is in $\widehat{\mathbf{A}}$ (the "bottom category") and the vertical maps as well as the main diagonal (f, g) in a commutative adjunctive square are morphisms from $\widehat{\mathbf{X}}$-objects to $\widehat{\mathbf{A}}$-objects. The (F, G) twist functor carries the main diagonal map (f, g) to the (southwest to northeast) anti-diagonal map (Gg, Ff).

Given $f : x \to Ga$, the rest of the diagram is determined by the requirement that the square commutes. Commutativity in the second component uniquely determines that $g = g1_{Fx} = \varepsilon_a Ff$ so $g = f^* = \varepsilon_a Ff$ is the map associated with f in the adjunction isomorphism. Commutativity in the first component is the universal mapping property factorization of the given $f : x \to Ga$ through the unit $x \xrightarrow{\eta_x} GFx \xrightarrow{Gf^*} Ga = x \xrightarrow{f} Ga$. Similarly, if we were given $g : Fx \to a$, then commutativity in the first component implies that $f = 1_{Ga} f = Gg\eta_x = g^*$. And commutativity in the second component is the UMP factorization of $g : Fx \to a$ through the counit $Fx \xrightarrow{Fg^*} FGa \xrightarrow{\varepsilon_a} a = Fx \xrightarrow{g} a$.

The adjunctive square diagram also brings out the directionality of the adjunction. In a commutative adjunctive square, the main diagonal map, which goes from $\widehat{x} = (x, Fx)$ in $\widehat{\mathbf{X}}$ to $\widehat{a} = (Ga, a)$ in $\widehat{\mathbf{A}}$, is (f, g) where $g = f^*$ and $f = g^*$. Each (x, Fx)-to-(Ga, a) determination crosses one of the "bridges" represented by the natural transformations $(\eta, 1_F)$ or $(1_G, \varepsilon)$. Suppose that a determination went from (x, Fx) to (Ga, a) as follows: $(x, Fx) \to (x', Fx') \to (GFx', Fx') \to (Ga, a)$.

$$\begin{array}{ccccc} (x, Fx) & \longrightarrow & (x', Fx') & & \\ \downarrow & & \downarrow & & \\ (GFx, Fx) & \longrightarrow & (GFx', Fx') & \longrightarrow & (Ga, a) \end{array}$$

By the commutativity of the square from the naturality of $(\eta, 1_F)$, the determination (x, Fx) to (Ga, a) crossing the bridge at $(x, Fx) \to (GFx, Fx)$ is the same. Hence any determination from (x, Fx) and crossing a $(\eta, 1_F)$-bridge can be taken as crossing the bridge at (x, Fx).

Suppose a determination from (x, Fx) to (Ga, a) crossed a $(1_G, \varepsilon)$-bridge: $(x, Fx) \to (Ga', FGa') \to (Ga', a') \to (Ga, a)$.

$$\begin{array}{ccccc} (x, Fx) & \longrightarrow & (Ga', FGa') & \longrightarrow & (Ga, FGa) \\ & & \downarrow & & \downarrow \\ & & (Ga', a') & \longrightarrow & (Ga, a) \end{array}$$

By the commutativity of the square from the naturality of $(1_G, \varepsilon)$, the determination crossing the bridge at $(Ga, FGa) \to (Ga, a)$ is the same. Hence any determination to (Ga, a) and crossing a $(1_G, \varepsilon)$-bridge can be taken as crossing the bridge at (Ga, a). Thus the adjunctive square represents the general case of possible (x, Fx)-to-(Ga, a) determinations using functors F and G.

5.2 Factorization Systems of Maps

Consider the generic form for an adjunctive square diagram where we are assuming that F and G are adjoint.

$$\begin{array}{ccccc} & (x, Fx) & \stackrel{(f,Ff)}{\rightarrow} & (Ga, FGa) & \\ (\eta_x, 1_{Fx}) & \downarrow & \nearrow & \downarrow & (1_{Ga}, \varepsilon_a) \\ & (GFx, Fx) & \stackrel{(Gg,g)}{\longrightarrow} & (Ga, a) & \end{array}$$

Given any $f : x \to Ga$ in the first component on top, there is a unique $f^* : Fx \to a$ such that the anti-diagonal map (Gf^*, Ff) factors (f, Ff) through the universal $(\eta_x, 1_{Fx})$, i.e., such that the upper triangle commutes. And that anti-diagonal map composes with $(1_{Ga}, \varepsilon_a)$ to determine the bottom map

$$(GFx, Fx) \stackrel{(Gf^*,Ff)}{\longrightarrow} (Ga, FGa) \stackrel{(1_{Ga},\varepsilon_a)}{\longrightarrow} (Ga, a) = (GFx, Fx) \stackrel{(Gf^*,f^*)}{\longrightarrow} (Ga, a)$$

that makes the lower triangle and thus the square commutes. Or starting with $g = f^* : Fx \to a$ in the second component on the bottom, there is a unique $g^* = f : x \to Ga$ such that the anti-diagonal map (Gf^*, Ff) factors $(Gg, g) = (Gf^*, f^*)$ through the universal $(1_{Ga}, \varepsilon_a)$, i.e., such that the lower triangle commutes. And that anti-diagonal map composes with $(\eta_x, 1_{Fx})$ to determine the upper map

$$(x, Fx) \stackrel{(\eta_x,1_{Fx})}{\longrightarrow} (GFx, Fx) \stackrel{(Gf^*,Ff)}{\longrightarrow} (Ga, FGa) = (x, Fx) \stackrel{(f,Ff)}{\longrightarrow} (Ga, FGa)$$

that makes the upper triangle and thus the square commutes.

Adjointness is closely related to the notion of a factorization system for orthogonal sets of maps such as epis and monos (see [18] or [34]). The motivating example for factorization systems of maps was the example of epimorphisms and monomorphisms. The unit and counit in any adjunction have a closely related property. For instance, suppose there are morphisms $g,\ g' : Fx \to a$ in the bottom category $\mathbf{A}$ such that $x \stackrel{\eta_x}{\longrightarrow} GFx \stackrel{Gg}{\longrightarrow} Ga = x \stackrel{\eta_x}{\longrightarrow} GFx \stackrel{Gg'}{\longrightarrow} Ga$ holds in the top category $\mathbf{X}$. By the unique factorization of any morphism $f : x \to Ga$ through the unit η_x it follows that $g = g'$. Since the uniqueness of the morphisms is in $\mathbf{A}$ while the G-image maps Gg and Gg' post-composed with η_x are in $\mathbf{X}$, this is not the same as η_x being epi. But it is a closely related property of being "epi with respect to G-images of $Fx \to a$ morphisms." Similarly, the counit ε_a is "mono with respect to F-images of $x \to Ga$ morphisms." Thus it is not surprising that the maps $(\eta_x, 1_{Fx})$ and $(1_{Ga}, \varepsilon_a)$ have an anti-diagonal factorization in the adjunctive square diagrams in a manner analogous to epis and monos in general commutative squares.

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6 Adjoints = Birepresentations of Het-bifunctors

6.1 Chimera Morphisms and Het-bifunctors

We argue that an adjunction has to do with morphisms between objects that are in general in different categories, e.g., from an object x in $\mathbf{X}$ to an object a in $\mathbf{A}$. These "heteromorphisms" (in contrast to homomorphisms) are like mongrels or chimeras that do not fit into either of the two categories. For instance, in the context of the free-group-underlying-set adjunction, we might consider any mapping $x \stackrel{c}{\Rightarrow} a$ whose tail is a set and head is a group. Some adjunctions, such as the free-underlying adjunctions, have this sort of concrete realization of the cross-category determination which is then represented by the two adjoint transposes $x \stackrel{f(c)}{\rightarrow} Ga$ and $Fx \stackrel{g(c)}{\rightarrow} a$. Since the cross-category heteromorphisms are not morphisms in either of the categories, what can we say about them?

The one thing we can reasonably say is that chimera morphisms can be precomposed or postcomposed with morphisms within the categories (i.e., intra-category morphisms) to obtain other chimera morphisms.[9] This is easily formalized. Suppose we have heteromorphisms $x \stackrel{c}{\Rightarrow} a$ as in the case of sets and groups from objects in $\mathbf{X}$ to objects in $\mathbf{A}$ (another example analyzed below would be "cones" which can be seen as chimera morphisms from sets to diagram functors). Let $Het(x, a) = \{x \stackrel{c}{\Rightarrow} a\}$ be the set of heteromorphisms from x to a. For any $\mathbf{A}$-morphism $k : a \rightarrow a'$ and any chimera morphism $x \stackrel{c}{\Rightarrow} a$, intuitively there is a composite chimera morphism $x \stackrel{c}{\Rightarrow} a \stackrel{k}{\rightarrow} a' = x \stackrel{kc}{\Rightarrow} a'$, i.e., k induces a map $Het(x, k) : Het(x, a) \rightarrow Het(x, a')$. For any $\mathbf{X}$-morphism $h : x' \rightarrow x$ and chimera morphism $x \stackrel{c}{\Rightarrow} a$, intuitively there is the composite chimera morphism $x' \stackrel{h}{\rightarrow} x \stackrel{c}{\Rightarrow} a = x' \stackrel{ch}{\Rightarrow} a$, i.e., h induces a map $Het(h, a) : Het(x, a) \rightarrow Het(x', a)$ (note the reversal of direction). Taking the sets-to-groups example to guide intuitions, the induced maps would respect identity and composite morphisms in each category. Moreover, composition is associative in the sense that $(kc)h = k(ch)$. This means that the assignments of sets of chimera morphisms $Het(x, a) = \{x \stackrel{c}{\Rightarrow} a\}$ and the induced maps between them constitute a bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \rightarrow \mathbf{Set}$ (contravariant in the first variable and covariant in the second).[10] The composition properties we would axiomatize for "chimera morphisms" or "heteromorphisms" from $\mathbf{X}$-objects to $\mathbf{A}$-objects are precisely those of the elements in the values of such a bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \rightarrow \mathbf{Set}$.

An adjunction is not simply about heteromorphisms from $\mathbf{X}$ to $\mathbf{A}$; it is

[9] The chimera genes are dominant in these mongrel matings. While mules cannot mate with mules, it is 'as if' mules could mate with either horses or donkeys to produce other mules.

[10] One conventional treatment of chimera morphisms such as cones 'misses' the bifunctor formulation by treating the cones as objects in a category (rather than as morphisms between categories). The compositions are then viewed as defining morphisms between these objects. For instance, $h : x' \rightarrow x$ would define a morphism from a cone $ch : x' \Rightarrow a$ to the cone $c : x \Rightarrow a$. The terminal object in this category would be the limit cone—and dually for cocones. These are slices of the chimera comma category defined in the next section.

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about such determinations through universals. In other words, an adjunction arises from a het-bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \rightarrow \mathbf{Set}$ that is "birepresentable" in the sense of being representable on both the left and right.

Given any bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \rightarrow \mathbf{Set}$, it is *representable on the left* if for each $\mathbf{X}$-object x, there is an $\mathbf{A}$-object Fx that represents the functor $Het(x, -)$, i.e., there is an isomorphism $\psi_{x,a} : Hom_{\mathbf{A}}(Fx, a) \cong Het(x, a)$ natural in a. For each x, let h_x be the image of the identity on Fx, i.e., $\psi_{x,Fx}(1_{Fx}) = h_x \in Het(x, Fx)$. We first show that h_x is a universal element for the functor $Het(x, -)$ and then use that to complete the construction of F as a functor. For any $c \in Het(x, a)$, let $g(c) = \psi_{x,a}^{-1}(c) : Fx \rightarrow a$. Then naturality in a means that the following diagram commutes.

$$\begin{array}{cccccc} & Hom_{\mathbf{A}}(Fx, Fx) & \cong & Het(x, Fx) & \\ {\scriptstyle Hom(Fx,g(c))} & \downarrow & & \downarrow & {\scriptstyle Het(x,g(c))} \\ & Hom_{\mathbf{A}}(Fx, a) & \cong & Het(x, a) & \end{array}$$

Chasing 1_{Fx} around the diagram yields that $c = Het(x, g(c))(h_x)$ which can be written as $c = g(c)h_x$. Since the horizontal maps are isomorphisms, $g(c)$ is the unique map $g : Fx \rightarrow a$ such that $c = gh_x$. Then (Fx, h_x) is a *universal element* (in MacLane's sense [28, p. 57]) for the functor $Het(x, -)$ or equivalently $1 \xrightarrow{h_x} Het(x, Fx)$ is a *universal arrow* [28, p. 58] from 1 (the one point set) to $Het(x, -)$. Then for any $\mathbf{X}$-morphism $j : x \rightarrow x'$, $Fj : Fx \rightarrow Fx'$ is the unique $\mathbf{A}$-morphism such that $Het(x, Fj)$ fills in the right vertical arrow in the following diagram.

$$\begin{array}{cccccc} & 1 & \xrightarrow{h_x} & Het(x, Fx) & \\ {\scriptstyle h_{x'}} & \downarrow & & \downarrow & {\scriptstyle Het(x,Fj)} \\ & Het(x', Fx') & \xrightarrow{Het(j,Fx')} & Het(x, Fx') & \end{array}$$

It is easily checked that such a definition of $Fj : Fx \rightarrow Fx'$ preserves identities and composition using the functoriality of $Het(x, -)$ so we have a functor $F : X \rightarrow A$. It is a further standard result that the isomorphism is also natural in x (e.g., [28, p. 81] or the "parameter theorem" [29, p. 525]).

Given a bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \rightarrow \mathbf{Set}$, it is *representable on the right* if for each $\mathbf{A}$-object a, there is an $\mathbf{X}$-object Ga that represents the functor $Het(-, a)$, i.e., there is an isomorphism $\varphi_{x,a} : Het(x, a) \cong Hom_X(x, Ga)$ natural in x. For each a, let e_a be the inverse image of the identity on Ga, i.e., $\varphi_{Ga,a}^{-1}(1_{Ga}) = e_a \in Het(Ga, a)$. For any $c \in Het(x, a)$, let $f(c) = \varphi_{x,a}(c) : x \rightarrow Ga$. Then naturality in x means that the following diagram commutes.

$$\begin{array}{cccccc} & Het(Ga, a) & \cong & Hom_{\mathbf{X}}(Ga, Ga) & \\ {\scriptstyle Het(f(c),a)} & \downarrow & & \downarrow & {\scriptstyle Hom(f(c),Ga)} \\ & Het(x, a) & \cong & Hom_{\mathbf{X}}(x, Ga) & \end{array}$$

Chasing 1_{Ga} around the diagram yields that $c = Het(f(c),a)(e_a) = e_a f(c)$ so (Ga, e_a) is a universal element for the functor $Het(-,a)$ and that $1 \xrightarrow{e_a} Het(Ga,a)$ is a universal arrow from 1 to $Het(-,a)$. Then for any **A**-morphism $k : a' \to a$, $Gk : Ga' \to Ga$ is the unique **X**-morphism such that $Het(Gk,a)$ fills in the right vertical arrow in the following diagram.

$$\begin{array}{ccccc} & 1 & \xrightarrow{e_a} & Het(Ga,a) & \\ e_{a'} & \downarrow & & \downarrow & Het(Gk,a) \\ & Het(Ga',a') & \xrightarrow{Het(Ga',k)} & Het(Ga',a) & \end{array}$$

In a similar manner, it is easily checked that the functoriality of G follows from the functoriality of $Het(-,a)$. Thus we have a functor $G : \mathbf{A} \to \mathbf{X}$ such that Ga represents the functor $Het(-,a)$, i.e., there is a natural isomorphism $\varphi_{x,a} : Het(x,a) \cong Hom_{\mathbf{X}}(x, Ga)$ natural in x. And in a similar manner, it can be shown that the isomorphism is natural in both variables. Thus given a bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ representable on both sides, we have an adjunction isomorphism:

$$Hom_{\mathbf{A}}(Fx,a) \cong Het(x,a) \cong Hom_{\mathbf{X}}(x,Ga).$$

The morphisms mapped to one another by these isomorphisms are *adjoint correlates* of one another. When the two representations are thus combined, the universal element $h_x \in Het(x, Fx)$ induced by 1_{Fx} is the *chimera unit* and is the adjoint correlate of the ordinary unit $\eta_x : x \to GFx$. The universal element $e_a \in Het(Ga,a)$ induced by 1_{Ga} is the *chimera counit* and is the adjoint correlate of the ordinary counit $\varepsilon_a : FGa \to a$. The two factorizations $g(c)h_x = c = e_a f(c)$ combine to give what we will later call the "chimera adjunctive square" with c as the main diagonal.

6.2 Comma Categories and Bifunctors

The above treatment uses the 'official' hom-set definition of an adjunction. It might be useful to briefly restate the ideas using William Lawvere's comma category definition of an adjunction in his famous 1963 thesis [25]. Given three categories **A**, **B**, and **C** and two functors

$$A : \mathbf{A} \to \mathbf{C} \leftarrow \mathbf{B} : B$$

the *comma category* (A,B) has as objects the morphism in **C** from a value $A(a)$ of the functor $A : \mathbf{A} \to \mathbf{C}$ to a value $B(b)$ of the functor $B : \mathbf{B} \to \mathbf{C}$. A morphism from an object $A(a) \to B(b)$ to an object $A(a') \to B(b')$ is an **A**-morphism $k : a \to a'$ and a **B**-morphism $h : b \to b'$ such that the following diagram commutes:

$$\begin{array}{ccc} A(a) & \xrightarrow{A(k)} & A(a') \\ \downarrow & & \downarrow \\ B(b) & \xrightarrow{B(h)} & B(b'). \end{array}$$

There are two projection functors $\pi_0 : (A, B) \to \mathbf{A}$ which takes the object $A(a) \to B(b)$ to the object a in $\mathbf{A}$ and a morphism (k, h) to k, and $\pi_1 : (A, B) \to \mathbf{B}$ with the analogous definition.

Then with the adjunctive setup $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$, there are two associated comma categories $(F, 1_{\mathbf{A}})$ defined by $F : \mathbf{X} \to \mathbf{A} \leftarrow \mathbf{A} : 1_{\mathbf{A}}$, and $(1_{\mathbf{X}}, G)$ defined by $1_{\mathbf{X}} : \mathbf{X} \to \mathbf{X} \leftarrow \mathbf{A} : G$. Then Lawvere's definition of an adjunction (e.g., [28, p. 84], [31, p. 92], or [34, p. 389]) is that F is the left adjoint to G if there is an isomorphism $(F, 1_{\mathbf{A}}) \cong (1_{\mathbf{X}}, G)$ over the projections to the product category $\mathbf{X} \times \mathbf{A}$, i.e., such that the following diagram commutes.

$$\begin{array}{ccccc} & (F, 1_{\mathbf{A}}) & \cong & (1_{\mathbf{X}}, G) & \\ (\pi_0, \pi_1) & \downarrow & & \downarrow & (\pi_0, \pi_1) \\ & \mathbf{X} \times \mathbf{A} & = & \mathbf{X} \times \mathbf{A} & \end{array}$$

To connect the comma category approach to the bifunctor treatment, we might note that the data $A : \mathbf{A} \to \mathbf{C} \leftarrow \mathbf{B} : B$ used to define the comma category also defines a bifunctor $Hom_{\mathbf{C}}(A(-), B(-)) : \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$. But we might start with any bifunctor $\mathcal{D} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ and then mimic the definition of a comma category to arrive at a category that we will denote $\mathcal{D}(1_{\mathbf{X}}, 1_{\mathbf{A}})$. The objects of $\mathcal{D}(1_{\mathbf{X}}, 1_{\mathbf{A}})$ are elements of any value $\mathcal{D}(x, a)$ of the bifunctor which we will denote $x \overset{c}{\Rightarrow} a$. A morphism from $x \overset{c}{\Rightarrow} a$ to $x' \overset{c'}{\Rightarrow} a'$ is given by an $\mathbf{X}$-morphism $j : x \to x'$ and an $\mathbf{A}$-morphism $k : a \to a'$ such that the following diagram commutes:

$$\begin{array}{cccc} & x & \xrightarrow{j} & x' & \\ c & \Downarrow & & \Downarrow & c' \\ & a & \xrightarrow{k} & a' & \end{array}$$

which means in terms of the bifunctor that the following diagram commutes:

$$\begin{array}{ccccc} & 1 & \xrightarrow{c'} & \mathcal{D}(x', a') & \\ c & \downarrow & & \downarrow & \mathcal{D}(j, a') \\ & \mathcal{D}(x, a) & \xrightarrow{\mathcal{D}(x,k)} & \mathcal{D}(x, a'). & \end{array}$$

The projection functors to $\mathbf{X}$ and to $\mathbf{A}$ are defined in the obvious manner.

There is a somewhat pedantic question of whether or not $\mathcal{D}(1_{\mathbf{X}}, 1_{\mathbf{A}})$ should be called a "comma category." Technically it is not since its objects are elements of an arbitrary bifunctor $\mathcal{D} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$, not morphisms within any category $\mathbf{C}$. This point is not entirely pedantic for our purposes since when the elements of $\mathcal{D}(x, a)$ are taken to be concrete chimera morphisms, then they indeed are not morphisms within a category but are (object-to-object) heteromorphisms between categories. For instance, there is no comma category $(1_{\mathbf{X}}, 1_{\mathbf{A}})$ with identity functors on both sides unless $\mathbf{X}$ and $\mathbf{A}$ are the same category. However, $\mathcal{D}(1_{\mathbf{X}}, 1_{\mathbf{A}})$ will often be isomorphic to a comma category (e.g., when $\mathcal{D}$ is representable on one side or the other) and Lawvere only considered that comma categories were defined "up to isomorphism." Moreover, I think that ordinary usage by category theorists would call $\mathcal{D}(1_{\mathbf{X}}, 1_{\mathbf{A}})$ a "comma category" even though its objects are not necessarily morphisms within any category $\mathbf{C}$, and I will follow that general usage. Thus $\mathcal{D}(1_{\mathbf{X}}, 1_{\mathbf{A}})$ is the *comma category of the bifunctor* $\mathcal{D}$. This broader notion of "comma category" is simply another way of describing bifunctors $\mathcal{D} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$.[11] Starting with the special case of $\mathcal{D} = Hom_{\mathbf{C}}(A(-), B(-))$, we get back the comma category (A, B) as $Hom_{\mathbf{C}}(A, B)$ (in the notation that makes the bifunctor explicit). Starting with the hom-bifunctors $Hom_{\mathbf{A}}(F(-), 1_{\mathbf{A}}(-))$ and $Hom_{\mathbf{X}}(1_{\mathbf{X}}(-), G(-))$, this construction would reproduce Lawvere's comma categories $Hom_{\mathbf{A}}(F, 1_{\mathbf{A}}) = (F, 1_{\mathbf{A}})$ and $Hom_{\mathbf{X}}(1_{\mathbf{X}}, G) = (1_{\mathbf{X}}, G)$.

Suppose that $Het(x, a)$ is represented on the left so that we have an isomorphism $Hom_{\mathbf{A}}(Fx, a) \cong Het(x, a)$ natural in x and in a. Then the claim is that there is an isomorphism of comma categories $Hom_{\mathbf{A}}(F, 1_{\mathbf{A}}) \cong Het(1_{\mathbf{X}}, 1_{\mathbf{A}})$ (using the notation with the bifunctor explicit) over the projections into the product category. The representation associates a morphism $g(c) : Fx \to a$ with each element $x \overset{c}{\Rightarrow} a$ in $Het(x, a)$ so that it provides the correspondence between the objects of the two comma categories. Consider the two diagrams associated with morphisms in the two comma categories.

$$\begin{array}{ccccc} & Fx & \xrightarrow{Fj} & Fx' & \\ g(c) & \downarrow & & \downarrow & g(c') \\ & a & \xrightarrow{k} & a' & \end{array} \qquad\qquad \begin{array}{ccccc} & x & \xrightarrow{j} & x' & \\ c & \Downarrow & & \Downarrow & c' \\ & a & \xrightarrow{k} & a' & \end{array}$$

To see that the representation implies that one square will commute if and only if the other does, consider the following commutative diagram given by the representation.

$$\begin{array}{ccccc} g(c) \in & Hom_{\mathbf{A}}(Fx, a) & \cong & Het(x, a) & \ni c \\ Hom(Fx, k) & \downarrow & & \downarrow & Het(x, k) \\ & Hom_{\mathbf{A}}(Fx, a') & \cong & Het(x, a') & \\ Hom(Fj, a') & \uparrow & & \uparrow & Het(j, a') \\ g(c') \in & Hom_{\mathbf{A}}(Fx', a') & \cong & Het(x', a') & \ni c' \end{array}$$

Then $g(c)$ is carried by $Hom(Fx, k)$ to the same element in $Hom_{\mathbf{A}}(Fx, a')$ as $g(c')$ is carried by $Hom(Fj, a')$ if and only if the square on the left above in the comma category $Hom_{\mathbf{A}}(F, 1_{\mathbf{A}}) = (F, 1_{\mathbf{A}})$ commutes. Similarly, c is carried by $Het(x, k)$ to the same element in $Het(x, a')$ as c' is carried by $Het(j, a')$ if and only if the square on the right above in the comma category $Het(1_{\mathbf{X}}, 1_{\mathbf{A}})$ commutes. Thus given the representation isomorphisms, the square in the one comma category commutes if and only if the other one does. With some more

[11] Thus there are at least three ways to think of the elements of $\mathcal{D}(x, a)$: as objects in the comma category $\mathcal{D}(1_{\mathbf{X}}, 1_{\mathbf{A}})$, as heteromorphisms $x \Rightarrow a$, or as elements in a set-valued "categorical relation" from $\mathbf{X}$ to $\mathbf{A}$ [5, p. 308] as in the profunctors-distributors-bimodules.

checking, it can be verified that the two comma categories are then isomorphic over the projections.

$$\begin{array}{ccccc} & Hom_{\mathbf{A}}(F,1_{\mathbf{A}}) & \cong & Het(1_{\mathbf{X}},1_{\mathbf{A}}) & \\ (\pi_0,\pi_1) & \downarrow & & \downarrow & (\pi_0,\pi_1) \\ & \mathbf{X}\times\mathbf{A} & = & \mathbf{X}\times\mathbf{A} & \end{array}$$

Thus the 'half-adjunction' in the language of representations is equivalent to the 'half-adjunction' in the language of comma categories. With the other representation of $Het(x,a)$ on the right, we would then have the comma category version of the three naturally isomorphic bifunctors.[12]

$$\begin{array}{ccccc} Hom_{\mathbf{A}}(F,1_{\mathbf{A}}) & \cong & Het(1_{\mathbf{X}},1_{\mathbf{A}}) & \cong & Hom_{\mathbf{X}}(1_{\mathbf{X}},G) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{X}\times\mathbf{A} & = & \mathbf{X}\times\mathbf{A} & = & \mathbf{X}\times\mathbf{A} \end{array}$$

6.3 Adjunction Representation Theorem

Adjunctions (particularly ones that are only abstractly specified) may not have *concretely* defined het-bifunctors that yield the adjunction via the two representations. However, given any adjunction, there is always an "abstract" associated het-bifunctor given by the main diagonal maps in the commutative adjunctive squares:

$$Het(\widehat{x},\widehat{a}) = \{\widehat{x} = (x,Fx) \stackrel{(f,f^*)}{\longrightarrow} (Ga,a) = \widehat{a}\}.$$

The diagonal maps are closed under precomposition with maps from $\widehat{\mathbf{X}}$ and postcomposition with maps from $\widehat{\mathbf{A}}$. Associativity follows from the associativity in the ambient category $\mathbf{X}\times\mathbf{A}$.

The representation is accomplished essentially by putting a $\widehat{hat}$ on objects and morphisms embedded in $\mathbf{X}\times\mathbf{A}$. The categories $\mathbf{X}$ and $\mathbf{A}$ are represented respectively by the subcategory $\widehat{\mathbf{X}}$ with objects $\widehat{x} = (x,Fx)$ and morphisms $\widehat{f} = (f,Ff)$ and by the subcategory $\widehat{\mathbf{A}}$ with objects $\widehat{a} = (Ga,a)$ and morphisms $\widehat{g} = (Gg,g)$. The (F,G) twist functor restricted to $\widehat{\mathbf{X}} \cong \mathbf{X}$ is $\widehat{F}$ which has the action of F, i.e., $\widehat{F}\widehat{x} = (F,G)(x,Fx) = (GFx,Fx) = \widehat{Fx}$ and similarly for morphisms. The twist functor restricted to $\widehat{\mathbf{A}} \cong \mathbf{A}$ yields $\widehat{G}$ which has the action of G, i.e., $\widehat{G}\widehat{a} = (F,G)(Ga,a) = (Ga,FGa) = \widehat{Ga}$ and similarly for morphisms. These functors provide representations on the left and right of the abstract het-bifunctor $Het(\widehat{x},\widehat{a}) = \{\widehat{x} \stackrel{(f,f^*)}{\longrightarrow} \widehat{a}\}$, i.e., the natural isomorphism

$$Hom_{\widehat{\mathbf{A}}}(\widehat{F}\widehat{x},\widehat{a}) \cong Het(\widehat{x},\widehat{a}) \cong Hom_{\widehat{\mathbf{X}}}(\widehat{x},\widehat{G}\widehat{a}).$$

[12] I owe to Colin McLarty the observation that the objects of the comma categories $(F,1_{\mathbf{A}})$ and $(1_{\mathbf{X}},G)$ are "essentially the same" as the chimera morphisms—as is shown by these isomorphisms.

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This birepresentation of the abstract het-bifunctor gives an isomorphic copy of the original adjunction between the isomorphic copies $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{A}}$ of the original categories. This theory of adjoints may be summarized in the following:

Adjunction Representation Theorem: Every adjunction $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$ can be represented (up to isomorphism) as arising from the left and right representing universals of a het-bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ giving the chimera morphisms from the objects in a category $\mathbf{X} \cong \widehat{\mathbf{X}}$ to the objects in a category $\mathbf{A} \cong \widehat{\mathbf{A}}$.[13]

6.4 Chimera Natural Transformations

The concrete heteromorphisms, say, from sets to groups are just as 'real' in any relevant mathematical sense as the set-to-set functions or group-to-group homomorphisms. At first, it might seem that the chimera morphisms do not fit into a category theoretic framework since they do not compose with one another like morphisms within a category. For instance, a set-to-functor cone cannot be composed with another set-to-functor cone. But the proper sort of composition was defined by having the morphisms of a category act on chimera morphisms to yield other chimera morphisms. For instance, a set-to-set function acts on a set-to-functor cone (as always, when composition is defined) to yield another set-to-functor cone, and similarly for functor-to-functor natural transformations composed on the other side. That action on each side of a set of category-bridging heteromorphisms is described by a het-bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$. Thus the transition from the ordinary composition of morphisms within a category to the sort of composition appropriate to morphisms between categories is no more of a conceptual leap than going from one bifunctor $Hom : \mathbf{X}^{op} \times \mathbf{X} \to \mathbf{Set}$ to another $Het : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$. All adjunctions arise as the birepresentations of such het-bifunctors.

Ordinary hom-bifunctors are (by definition) represented on the left and on the right by the identity functor so the self-adjoint identity functor on any category could be thought of as the "ur-adjunction" that expresses intra-category determination in an adjunctive framework. Replace the birepresentable hom-bifunctor $Hom : \mathbf{X}^{op} \times \mathbf{X} \to \mathbf{Set}$ with a birepresentable het-bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ and we have the smooth transition from intra-category determination to cross-category determination via adjunctions.

Given the ubiquity of such adjoints in mathematics, there seems to be a good case to stop treating chimera morphisms as some sort of 'dark matter' invisible to category theory; chimeras should be admitted into the 'zoo' of category

[13]In a historical note [28, p. 103], MacLane noted that Bourbaki "missed" the notion of an adjunction because Bourbaki focused on the left representations of bifunctors $W : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Sets}$. MacLane remarks that given $G : \mathbf{A} \to \mathbf{X}$, they should have taken $W(x,a) = Hom_{\mathbf{X}}(x, Ga)$ and then focused on "the symmetry of the adjunction problem" to find Fx so that $Hom_{\mathbf{A}}(Fx, a) \cong Hom_{\mathbf{X}}(x, Ga)$. Thus MacLane missed the completely symmetrical adjunction problem which is to find Ga and Fx such that $Hom_{\mathbf{A}}(Fx, a) \cong W(x,a) \cong Hom_{\mathbf{X}}(x, Ga)$.

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theoretic creatures. Heteromorphisms (defined by the properties necessary to be the elements of a het-bifunctor) between the objects of different categories should be taken as entities in the ontology of category theory at the same level as the morphisms between the objects within a category (morphisms defined by the properties necessary to be the elements of a hom-bifunctor).

New possibilities arise. For instance, the notion of a natural transformation immediately generalizes to functors with different codomains by taking the components to be heteromorphisms. Given functors $F : \mathbf{X} \to \mathbf{A}$ and $H : \mathbf{X} \to \mathbf{B}$ and a het-bifunctor $Het : \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$, a *chimera natural transformation* (*relative to* Het), $\varphi : F \Rightarrow H$, is given by set of morphisms $\{\varphi_x \in Het(Fx, Hx)\}$ indexed by the objects of $\mathbf{X}$ such that for any $j : x \to x'$ the following diagram commutes

$$\begin{array}{ccccc} & Fx & \stackrel{\varphi_x}{\Longrightarrow} & Hx & \\ Fj & \downarrow & & \downarrow & Hj \\ & Fx' & \stackrel{\varphi_{x'}}{\Longrightarrow} & Hx' & \end{array}$$

[the composition $Fx \stackrel{\varphi_x}{\Longrightarrow} Hx \stackrel{Hj}{\longrightarrow} Hx'$ is $Het(Fx, Hj)(\varphi_x) \in Het(Fx, Hx')$, the composition $Fx \stackrel{Fj}{\longrightarrow} Fx' \stackrel{\varphi_{x'}}{\Longrightarrow} Hx'$ is $Het(Fj, Hx')(\varphi_{x'}) \in Het(Fx, Hx')$, and commutativity means they are the same element of $Het(Fx, Hx')$]. These chimera natural transformations do not compose like the morphisms in a functor category but they are acted upon by the natural transformations in the functor categories on each side to yield another bifunctor $Het^X(F, H) = \{F \Rightarrow H\}$.

There are chimera natural transformations each way between any functor and the identity on its domain if the functor itself is used to define the appropriate bifunctor Het. Given any functor $F : \mathbf{X} \to \mathbf{A}$, there is a chimera natural transformation $1_{\mathbf{X}} \Rightarrow F$ relative to the bifunctor defined as $Het(x, a) = Hom_{\mathbf{A}}(Fx, a)$ as well as a chimera natural transformation $F \Rightarrow 1_{\mathbf{X}}$ relative to $Het(a, x) = Hom_{\mathbf{A}}(a, Fx)$.

Chimera natural transformations 'in effect' already occur with reflective (or coreflective) subcategories. A subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is a *reflective subcategory* if the inclusion functor $K : \mathbf{A} \to \mathbf{B}$ has a left adjoint. For any such reflective adjunctions, the heteromorphisms $Het(b, a)$ are the $\mathbf{B}$-morphisms with their heads in the subcategory $\mathbf{A}$ so the representation on the right $Het(b, a) \cong Hom_{\mathbf{B}}(b, Ka)$ is trivial. The left adjoint $F : \mathbf{B} \to \mathbf{A}$ gives the representation on the left: $Hom_{\mathbf{A}}(Fb, a) \cong Het(b, a) \cong Hom_{\mathbf{B}}(b, Ka)$. Then it is perfectly 'natural' to see the unit of the adjunction as defining a natural transformation $\eta : 1_{\mathbf{B}} \Rightarrow F$ but that is actually a chimera natural transformation (since the codomain of F is $\mathbf{A}$). Hence the conventional treatment (e.g., [28, p. 89]) is to define another functor R with the same domain and values on objects and morphisms as F except that its codomain is taken to be $\mathbf{B}$ so that we can then 'legally' have a natural transformation $\eta : 1_{\mathbf{X}} \to R$ between two functors with the same codomain. Similar remarks hold for the dual coreflective case where the inclusion functor has a right adjoint and where the heteromorphisms are turned around, i.e., are $\mathbf{B}$-morphisms with their tail in the subcategory $\mathbf{A}$.

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The insertion of the generators maps $\{x \Rightarrow Fx\}$ define another chimera natural transformation $h : 1_{\mathbf{X}} \Rightarrow F$ from the identity functor on the category of sets to the free-group functor (the chimera version of the unit $\eta : 1_{\mathbf{X}} \to GF$). A category theory without chimeras can explain the naturality of maps such as $x \to GFx$ but not the naturality of maps $x \Rightarrow Fx$!

For functors that are part of adjunctions, we can consider the embedding of the adjunction representation theorem in the product category $\mathbf{X} \times \mathbf{A}$. Even if the adjunction is only abstractly given (so that we have no concrete chimera maps), the representation theorem shows the maps $(\eta_x, 1_{Fx}) : \widehat{x} = (x, Fx) \to (GFx, Fx) = \widehat{Fx}$ (left vertical arrows in adjunctive squares which are later shown to be "sending universals") have the role of the chimera units $h_x : x \Rightarrow Fx$. Moreover, they define a chimera natural transformation $1_{\widehat{\mathbf{X}}} \Rightarrow \widehat{F}$ from the identity functor on $\widehat{\mathbf{X}}$ to the functor $\widehat{F} : \widehat{\mathbf{X}} \to \widehat{\mathbf{A}}$ (which is the (F, G) twist functor restricted to $\widehat{\mathbf{X}}$). Dually even if chimera counits $e_a : Ga \Rightarrow a$ are not available, we still have the chimera natural transformation $\widehat{G} \Rightarrow 1_{\widehat{\mathbf{A}}}$ given by the maps $(1_{Ga}, \varepsilon_a) : \widehat{G}\widehat{a} = \widehat{Ga} = (Ga, FGa) \to (Ga, a) = \widehat{a}$ (right vertical arrows in adjunctive squares which are later shown to be "receiving universals") where $\widehat{G} : \widehat{\mathbf{A}} \to \widehat{\mathbf{X}}$ is the (F, G) twist functor restricted to $\widehat{\mathbf{A}}$. Thus every adjunction yields two chimera natural transformations, the chimera versions of the unit and counit (we will see two more later).

For the adjunctions of MacLane's 'working mathematician,' there 'should' be concrete chimera morphisms that can be used to define these two chimera natural transformations without resort to the representation theorem. In the case of the limit adjunction, the chimera morphisms are "cones" from a set w to a functor $D : \mathbf{D} \to \mathbf{Set}$ (where $\mathbf{D}$ is a diagram category). As with every adjunction, there are two chimera natural transformations associated with the unit and counit which in this case are $h : 1_{\mathbf{Set}} \Rightarrow \Delta$ and $e : Lim \Rightarrow 1_{\mathbf{Set}^{\mathbf{D}}}$ with the chimera components given respectively by the two "universal cones" $w \Rightarrow \Delta w$ and $LimD \Rightarrow D$ (see the section on limits in sets for more explanation).

While not 'officially' acknowledged, the chimera morphisms or heteromorphisms between objects of different categories are quite visible under ordinary circumstances, once one acquires an eye to 'see' them. This non-acknowledgement is facilitated by the common practice of passing effortlessly between the chimera morphism and its representation on one side or the other. For instance in the last example, the chimera cone $w \Rightarrow D$ from a set to a functor is often treated interchangeably with a natural transformation $\Delta w \to D$ (its representation on the left) and both are called "cones." We have argued that chimera morphisms occur under quite ordinary circumstances, they fit easily into a category-theoretic framework, and they spawn some new creatures themselves—such as the chimera natural transformations.[14]

[14]This might also have implications for n-category theory. The basic motivating example from 1-category theory sees object-to-object morphisms only within a category so that connections between categories are only by functors. And natural transformations are seen only as going

7 Chimera Factorizations

7.1 Abstract Adjunctive Squares

These relationships can be conveniently and suggestively restated in the adjunctive squares framework. Given $f : x \to Ga$ or $g : Fx \to a$, the rest of the adjunctive square is determined so that it commutes and $f = g^*$ and $g = f^*$. The two associated determinations are the two components of the main diagonal (f, g) so it can be thought of as one abstract heteromorphism from $\widehat{x} = (x, Fx)$ to $\widehat{a} = (Ga, a)$.

$$\begin{array}{ccccc} & (x, Fx) & \stackrel{(f,Ff)}{\rightarrow} & (Ga, FGa) & \\ (\eta_x, 1_{Fx}) & \downarrow & \nearrow & \downarrow & (1_{Ga}, \varepsilon_a) \\ & (GFx, Fx) & \stackrel{(Gg,g)}{\longrightarrow} & (Ga, a) & \end{array}$$

Of all the determinations $(x, Fx) \to (Ga, a)$ *to* (Ga, a), the one that represents 'self-determination' is the receiving universal $(1_{Ga}, \varepsilon_a) : (Ga, FGa) \to (Ga, a)$ where $x = Ga$. All other instances of a determination to (Ga, a), e.g., $(GFx, Fx) \stackrel{(Gg,g)}{\longrightarrow} (Ga, a)$, factor uniquely through the receiving universal by the anti-diagonal map (Gg, Fg^*), i.e.,

$$(GFx, Fx) \stackrel{(Gg,g)}{\longrightarrow} (Ga, a) = (GFx, Fx) \stackrel{(Gg,Fg^*)}{\longrightarrow} (Ga, FGa) \stackrel{(1_{Ga},\varepsilon_a)}{\longrightarrow} (Ga, a).$$

Of all the determinations $(x, Fx) \to (Ga, a)$ *from* (x, Fx), the one that represents 'self-determination' is the sending universal $(\eta_x, 1_{Fx}) : (x, Fx) \to (GFx, Fx)$ where $a = Fx$. All other instances of a determination from (x, Fx), e.g., $(x, Fx) \stackrel{(f,Ff)}{\longrightarrow} (Ga, FGa)$, factor uniquely through the sending universal by the anti-diagonal map (Gf^*, Ff), i.e.,

$$(x, Fx) \stackrel{(f,Ff)}{\rightarrow} (Ga, FGa) = (x, Fx) \stackrel{(\eta_x,1_{Fx})}{\longrightarrow} (GFx, Fx) \stackrel{(Gf^*,Ff)}{\longrightarrow} (Ga, FGa).$$

Since each of these factorizations of a top or bottom arrow goes over to the other subcategory (e.g., $\widehat{\mathbf{X}}$ or $\widehat{\mathbf{A}}$) and then back, they might be called the *over-and-back factorizations.*

Hence any (x, Fx)-to-(Ga, a) heteromorphism given by the main diagonal (f, g) in a commutative adjunctive square (where $g = f^*$ and $f = g^*$) factors through both the self-determination universals at the sending and receiving ends by the anti-diagonal map (Gg, Ff) obtained by applying the (F, G) twist functor to (f, g). This factorization through the two universals will be called the *zig-zag factorization.* The "zig" ($\downarrow\nearrow$) followed by the "zag" ($\nearrow\downarrow$) give the zig-zag ($\downarrow\nearrow\downarrow$) factorization.

between two functors pinched together at both ends to form a lens-shaped area. With chimera natural transformations, the two functors only need to be pinched together on the domain side.

Zig-Zag Factorization: Any (x, Fx)-to-(Ga, a) heteromorphism in an adjunction factors uniquely through the sending and receiving universals by the anti-diagonal map.

Recall that G has to be one-to-one on morphisms of the form $g : Fx \to a$ (uniqueness in the UMP for the counit) and F has to be one-to-one on morphisms of the form $f : x \to Ga$ (uniqueness in the UMP for the unit). Thus the anti-diagonal maps of the form $(Gf^*, Ff) : \widehat{Fx} \to \widehat{Ga}$ are uniquely correlated with f. Hence we can define another bifunctor $Z(\widehat{F(-)}, \widehat{G(-)}) : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ of zig-zag factorization maps:

$$\mathcal{Z}(\widehat{Fx}, \widehat{Ga}) = \{(Gf^*, Ff) : \widehat{Fx} \to \widehat{Ga}\}.$$

Since the anti-diagonal zig-zag factor map is uniquely determined by the main diagonal map in a commutative adjunctive square, the (F, G) twist functor that takes (f, f^*) to (Gf^*, Ff) is an isomorphism from Het to $\mathcal{Z}$ which is easily checked to be natural in x and in a. Bearing in mind that these chimera bifunctors play the role of hom-sets for chimera, this can be seen as another adjunction-like isomorphism between the chimera morphisms in one direction and the chimera morphisms in the other direction:

$$Het(\widehat{x}, \widehat{a}) \cong \mathcal{Z}(\widehat{Fx}, \widehat{Ga}).$$

In spite of the notation $\mathcal{Z}(\widehat{Fx}, \widehat{Ga})$, there is no implication that we have a bifunctor $\mathcal{Z}(\widehat{a}, \widehat{x})$ defined on arbitrary $\widehat{a}$ and $\widehat{x}$, and thus no implication that we are dealing with arbitrary chimera morphisms $\widehat{Fx} \Rightarrow \widehat{Ga}$. The bifunctor $\mathcal{Z}(\widehat{Fx}, \widehat{Ga})$ is here *defined* as the image of $Het(\widehat{x}, \widehat{a})$ under the (F, G) twist functor.

In the representation theorem, we saw that every adjunction has a het-bifunctor $Het(\widehat{x}, \widehat{a})$ of chimera morphisms from $\widehat{\mathbf{X}}$ to $\widehat{\mathbf{A}}$ so that the adjunction arises (up to isomorphism) as the birepresentation of that het-bifunctor $Hom_{\widehat{\mathbf{A}}}(\widehat{F}\widehat{x}, \widehat{a}) \cong Het(\widehat{x}, \widehat{a}) \cong Hom_{\widehat{\mathbf{X}}}(\widehat{x}, \widehat{G}\widehat{a})$. Now we see that every adjunction also gives rise to another bifunctor $\mathcal{Z}(\widehat{F}\widehat{x}, \widehat{G}\widehat{a})$ of heteromorphisms going in the other direction from $\widehat{\mathbf{A}}$ to $\widehat{\mathbf{X}}$ and that the two bifunctors are naturally isomorphic. Thus we now have four different bifunctors that are naturally isomorphic in x and in a.

$$Hom_{\widehat{\mathbf{A}}}(\widehat{F}\widehat{x}, \widehat{a}) \cong Het(\widehat{x}, \widehat{a}) \cong \mathcal{Z}(\widehat{F}\widehat{x}, \widehat{G}\widehat{a}) \cong Hom_{\widehat{\mathbf{X}}}(\widehat{x}, \widehat{G}\widehat{a})$$

Given $f : a \to Ga$ and thus $f^* : Fx \to a$, these isomorphisms correlate together four morphism which in the adjunctive square are the bottom map, the main diagonal, the anti-diagonal, and the top map.

In an adjunction, the operation of taking the adjoint transpose amounts to applying the appropriate functor and pre- or post-composing with the appropriate universal. For $f : x \to Ga$, $f^* = \varepsilon_a Ff$ and for $g : Fx \to a$, $g^* = Gg\eta_x$. For the chimera isomorphism, the transpose of $\widehat{x} = (x, Fx) \stackrel{(f,f^*)}{\longrightarrow} (Ga, a) = \widehat{a}$ is obtained simply by applying the (F, G) twist functor to get $\widehat{Fx} = (GFx, Fx) \stackrel{(Gf^*,Ff)}{\longrightarrow} (Ga, FGa) = \widehat{Ga}$. Taking the transpose in the other direction is done by pre- and post-composing both the universals, i.e., the zig-zag factorization. Both isomorphisms are illustrated in a commutative adjunctive square. The adjoint transposes are the top and bottom arrows and the chimera adjoint correlates are the main diagonal and the anti-diagonal.

7.2 Abstract Adjunctive-Image Squares

It was previously noted that the uniqueness requirement in the UMPs of an adjunction imply that the functor F has to be one-one on morphisms of the form $f : x \to Ga$ while G has to be one-one on morphisms $g : Fx \to a$. Thus the (F, G) twist functor has to be one-one on all the morphisms in an adjunctive square. Applying the twist functor yields the image of the (commutative) adjunctive square which we will call the *adjunctive-image square*.

$$\begin{array}{ccccc}
\widehat{Fx} = & (GFx, Fx) & \stackrel{(GFf,Ff)}{\longrightarrow} & (GFGa, FGa) & = \widehat{FGa} \\
(1_{GFx}, F\eta_x) & \downarrow & & \downarrow & (G\varepsilon_a, 1_{FGa}) \\
\widehat{GFx} = & (GFx, FGFx) & \stackrel{(Gf^*,FGf^*)}{\longrightarrow} & (Ga, FGa) & = \widehat{Ga}
\end{array}$$

Abstract Adjunctive-Image Square

The top map is in $\widehat{\mathbf{A}}$, the bottom map is in $\widehat{\mathbf{X}}$, and the main diagonal is the anti-diagonal of the adjunctive square. Again there is a unique anti-diagonal map (obtained again as the image under the (F, g) twist functor) in the adjunctive-image square, i.e., $(GFf, FGf^*) : (GFx, FGFx) \to (GFGa, FGa)$, which makes the upper and lower triangles commute. Hence the original anti-diagonal map (Gf^*, Ff) has a unique zig-zag factorization through this new anti-diagonal map using the two vertical universal maps.

The image of the hom-sets such as $Hom_{\widehat{\mathbf{X}}}(\widehat{x}, \widehat{G}\widehat{a})$ under $\widehat{F}$ will yield an isomorphic set of morphisms like the top maps in the adjunctive-image square but it is not necessarily a hom-set. This isomorphic image is a bifunctor which we could denote $H(\widehat{F}\widehat{x}, \widehat{FG}\widehat{a})$ where there is a natural isomorphism $Hom_{\widehat{\mathbf{X}}}(\widehat{x}, \widehat{G}\widehat{a}) \cong H(\widehat{F}\widehat{x}, \widehat{FG}\widehat{a})$. Similarly, we have $Hom_{\widehat{\mathbf{A}}}(\widehat{F}\widehat{x}, \widehat{a}) \cong H(\widehat{GF}\widehat{x}, \widehat{G}\widehat{a})$ concerning the bottom maps in the adjunctive-image square. This use of the (F, G) twist functor to obtain an isomorphic bifunctor from a given one was already used to derive the bifunctor of anti-diagonal maps from the bifunctor of diagonal maps: $Het(\widehat{x}, \widehat{a}) \cong \mathcal{Z}(\widehat{F}\widehat{x}, \widehat{G}\widehat{a})$.[15] In all three cases, the process can be continued to derive an infinite sequence of bifunctors all isomorphic to one another. Only the first stage in the sequence for the chimera maps $Het(\widehat{x}, \widehat{a}) \cong \mathcal{Z}(\widehat{F}\widehat{x}, \widehat{G}\widehat{a})$ will be investigated here.

[15] The anti-diagonal in the adjunctive square becomes the main diagonal in the adjunctive-image square. That process can be repeated so that "it's chimeras all the way down."

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7.3 Chimera Adjunctive Squares

In any concretely specified adjunction (i.e., not just the abstract definition used in the representation theorem) that occurs in mathematics, we would *expect* to be able to 'takes the hats off' and find concrete chimera bifunctors $Het(x,a)$ and $\mathcal{Z}(Fx,Ga)$ to give us the same natural isomorphisms:

$$Hom_{\mathbf{A}}(Fx,a) \cong Het(x,a) \cong \mathcal{Z}(Fx,Ga) \cong Hom_{\mathbf{X}}(x,Ga).$$

In this section, we give the relationships that the abstract adjunctive square tells us to expect to find for the heteromorphisms in the two chimera bifunctors. In each example, we need to find these bifunctors and demonstrate these relationships.

Suppose we have the het-bifunctor, the 'zig-zag' bifunctor and the birepresentations to give the above isomorphisms. We previously used the representations of $Het(x,a)$ to pick out universal elements, the chimera unit $h_x \in Het(x,Fx)$ and the chimera counit $e_a \in Het(Ga,a)$, as the respective correlates of 1_{Fx} and 1_{Ga} under the isomorphisms. In the isomorphisms of the four bifunctors, let $g(c) : Fx \to a$, $c : x \Rightarrow a$, $z(c) : Fx \Rightarrow Ga$, and $f(c) : x \to Ga$ be the four adjoint correlates. We showed that from the birepresentation of $Het(x,a)$, any chimera morphism $x \overset{c}{\Rightarrow} a$ in $Het(x,a)$ would have two factorizations: $g(c)h_x = c = e_a f(c)$. This two factorizations are spliced together along the main diagonal $c : x \Rightarrow a$ to form the chimera (commutative) adjunctive square.

$$\begin{array}{cccc} & x & \overset{f(c)}{\longrightarrow} & Ga & \\ h_x & \Downarrow & & \Downarrow & e_a \\ & Fx & \overset{g(c)}{\longrightarrow} & a & \end{array}$$

Chimera Adjunctive Square

In the examples, we need to find the chimera morphisms $Fx \overset{z(c)}{\Longrightarrow} Ga$ that give us $\mathcal{Z}(Fx,Ga)$ and that fit as the anti-diagonal morphisms in these chimera adjunctive squares to give commutative upper and lower triangles. The two commutative triangles formed by the anti-diagonal might be called the:

$$x \overset{h_x}{\Longrightarrow} Fx \overset{z(c)}{\Rightarrow} Ga = x \overset{f(c)}{\longrightarrow} Ga$$
$$Fx \overset{z(c)}{\Rightarrow} Ga \overset{e_a}{\Longrightarrow} a = Fx \overset{g(c)}{\longrightarrow} a$$

Over-and-back factorizations of f and g.

One of the themes of this theory of adjoints is that some of the rigmarole of the conventional treatment of adjoints (*sans* chimeras) is only necessary because of the restriction to morphisms within one category or the other. The chimera unit $x \overset{h_x}{\Longrightarrow} Fx$ only involves one of the functors and it appears already with the half-adjunction of a representation on the left, and similarly for the chimera

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counit $Ga \overset{e_a}{\Longrightarrow} a$. The importance of the unit and counit lies in their universal mapping properties for morphisms of the form $x \xrightarrow{f} Ga$ or $Fx \xrightarrow{g} a$. With the over-and-back factorizations, we again see the power of adjoints expressed in simpler terms using the underlying heteromorphisms. The same holds for the triangular identities as we will see in the next section.

7.4 Chimera Adjunctive-Image Squares

When we have the four isomorphic bifunctors, then 1_{Fx} would associate with a chimera denoted $h_{x2} = z(h_x) : Fx \Rightarrow GFx$ in $\mathcal{Z}(Fx, GFx)$ and 1_{Ga} would associate with $e_{a1} = z(e_a) : FGa \Rightarrow Ga$ in $\mathcal{Z}(FGa, Ga)$. The notation is chosen since taking $f(c) = \eta_x$ (namely $c = h_x$), we have the over-and-back factorization of the unit: $x \overset{h_x}{\Longrightarrow} Fx \overset{h_{x2}}{\Longrightarrow} GFx = x \xrightarrow{\eta_x} GFx$, so h_{x2} is the 'second part' post-composed to h_x to yield η_x. And taking $g(c) = \varepsilon_a$ (namely $c = e_a$), we have the over-and-back factorization of the counit: $FGa \overset{e_{a1}}{\Longrightarrow} Ga \overset{e_a}{\Longrightarrow} a = FGa \xrightarrow{\varepsilon_a} a$, so e_{a1} is the 'first part' pre-composed to e_a to yield ε_a.

These special chimera morphisms have universality properties since they are associated with identity maps 1_{Fx} and 1_{Ga} in the above natural isomorphism of four bifunctors (mimic the proofs of the universality properties of η_x, ε_a, h_x, and e_a). The 'second half of the unit' $h_{x2} = z(h_x)$ has the following universality property: for any anti-diagonal map from Fx, $z(c) : Fx \Rightarrow Ga$, there is unique map $g = g(c) : Fx \to a$ such that

$$Fx \overset{z(c)}{\Longrightarrow} Ga = Fx \overset{h_{x2}}{\Longrightarrow} GFx \xrightarrow{Gg} Ga.$$

Precompose $h_x : x \Rightarrow Fx$ on both sides and we have the usual UMP for the unit, i.e.,

$$x \xrightarrow{f(c)} Ga = x \overset{h_x}{\Rightarrow} Fx \overset{z(c)}{\Rightarrow} Ga = x \overset{h_x}{\Rightarrow} Fx \overset{h_{x2}}{\Rightarrow} GFx \xrightarrow{Gg(c)} Ga = x \xrightarrow{\eta_x} GFx \xrightarrow{Gf^*} Ga.$$

The same chimera morphism $z(c) : Fx \Rightarrow Ga$ also has a factorization through the other anti-diagonal universal, the first half of the counit, $e_{a1} = z(e_a)$. Given any anti-diagonal map to Ga, $z(c) : Fx \Rightarrow Ga$, there is a unique map $f = f(c) : x \to Ga$ such that

$$Fx \overset{z(c)}{\Longrightarrow} Ga = Fx \overset{Ff}{\Longrightarrow} FGa \overset{e_{a1}}{\Longrightarrow} Ga.$$

Post-composing with $e_a : Ga \Rightarrow a$ on both sides yields the usual UMP for the counit, i.e.,

$$Fx \xrightarrow{g(c)} a = Fx \overset{z(c)}{\Longrightarrow} Ga \overset{e_a}{\Rightarrow} a = Fx \overset{Ff}{\Longrightarrow} FGa \overset{e_{a1}}{\Longrightarrow} Ga \overset{e_a}{\Rightarrow} a = Fx \overset{Fg^*}{\Longrightarrow} FGa \xrightarrow{\varepsilon_a} a.$$

Splicing together the two factorizations of $z(c) : Fx \Rightarrow Ga$ as the main diagonal, we have the chimera version of the adjunctive-image square.

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$$\begin{array}{cccccc} & Fx & \xrightarrow{Ff} & FGa & \\ h_{x2} & \Downarrow & & \Downarrow & e_{a1} \\ & GFx & \xrightarrow{Gf^*} & Ga & \end{array}$$

Chimera Adjunctive-Image Square

Thus the two universal properties give the two ways the main diagonal $Fx \stackrel{z(c)}{\Longrightarrow} Ga$ factors through the two vertical universal arrows $Fx \stackrel{h_{x2}}{\Longrightarrow} GFx$ and $FGa \stackrel{e_{a1}}{\Longrightarrow} Ga$. As before with $1_{\mathbf{X}} \stackrel{h}{\Longrightarrow} F$ and $G \stackrel{e}{\Longrightarrow} 1_{\mathbf{A}}$, the chimera universals are also the components of two chimera natural transformations: $F \stackrel{h_2}{\Rightarrow} GF$ and $FG \stackrel{e_1}{\Rightarrow} G$. Thus every adjunction has associated with it four chimera natural transformations, and the two conventional natural transformations associated with an adjunction are obtained as composites of the four chimera natural transformations.

$$1_{\mathbf{X}} \stackrel{h}{\Longrightarrow} F \stackrel{h_2}{\Longrightarrow} GF = 1_{\mathbf{X}} \stackrel{\eta}{\longrightarrow} GF \text{ and } FG \stackrel{e_1}{\Longrightarrow} G \stackrel{e}{\Longrightarrow} 1_{\mathbf{A}} = FG \stackrel{\varepsilon}{\longrightarrow} 1_{\mathbf{A}}.$$

Since the anti-diagonal $z(c)$ can be factored, the over-and-back factorizations for $f : x \to Ga$ and for $g : Fx \to a$ can be factored again as can be pictured using the adjunctive-image square. Add $x \stackrel{h_x}{\Longrightarrow} Fx$ on as a pendant to the above chimera adjunctive-image square (where $z(c)$ is the main diagonal) to obtain the following diagram.

$$\begin{array}{cccccc} x & \stackrel{h_x}{\Longrightarrow} & Fx & \xrightarrow{Ff} & FGa & \\ & h_{x2} & \Downarrow & & \Downarrow & e_{a1} \\ & & GFx & \xrightarrow{Gf^*} & Ga & \end{array}$$

The pendant followed by the counter-clockwise maps gives the usual factorization $x \stackrel{\eta_x}{\longrightarrow} GFx \stackrel{Gf^*}{\longrightarrow} Ga = x \stackrel{f}{\longrightarrow} Ga$. Hence following the pendant by the clockwise maps gives a different *over-across-and-back factorization of* f. Dually, we could postcompose the pendant $Ga \stackrel{e_a}{\Longrightarrow} a$ on the bottom and obtain the over-across-and-back factorization of $g = f^* : Fx \to a$ in addition to the usual one. These four factorizations may be summarized as follows:

$$x \stackrel{h_x}{\Longrightarrow} Fx \stackrel{h_{x2}}{\Longrightarrow} GFx \stackrel{Gf^*}{\longrightarrow} Ga = x \stackrel{f}{\longrightarrow} Ga$$
$$x \stackrel{h_x}{\Longrightarrow} Fx \stackrel{Ff}{\longrightarrow} FGa \stackrel{e_{a1}}{\Rightarrow} Ga = x \stackrel{f}{\longrightarrow} Ga$$
$$Fx \stackrel{Fg^*}{\longrightarrow} FGa \stackrel{e_{a1}}{\Rightarrow} Ga \stackrel{e_a}{\Longrightarrow} a = Fx \stackrel{g}{\longrightarrow} a$$
$$Fx \stackrel{h_{x2}}{\Longrightarrow} GFx \stackrel{Gg}{\longrightarrow} Ga \stackrel{e_a}{\Longrightarrow} a = Fx \stackrel{g}{\longrightarrow} a.$$

Specializing $f = 1_{Ga}$ gives one triangular identity in the first equation. But in the second equation, it gives the first *over-and-back identity* (a 'short form' of the triangular identity):

$$Ga \stackrel{h_{Ga}}{\Longrightarrow} FGa \stackrel{e_{a1}}{\Longrightarrow} Ga = Ga \stackrel{1_{Ga}}{\longrightarrow} Ga.$$

Specializing $g = 1_{Fx}$ gives the other triangular identity in the third equation. But in the fourth equation, it gives the other over-and-back identity:

$$Fx \stackrel{h_{x2}}{\Longrightarrow} GFx \stackrel{e_{Fx}}{\Longrightarrow} Fx = Fx \stackrel{1_{Fx}}{\longrightarrow} Fx.$$

Here again we see chimera natural transformations composing to yield conventional natural transformations:

$$G \stackrel{h_G}{\Longrightarrow} FG \stackrel{e_1}{\Longrightarrow} G = G \stackrel{1_G}{\longrightarrow} G$$
$$F \stackrel{h_2}{\Longrightarrow} GF \stackrel{e_F}{\Longrightarrow} F = F \stackrel{1_F}{\longrightarrow} F.$$

Over-and-back identities

Thus on the functorial images Ga and Fx, there is a canonical heteromorphism to the other category and a canonical heteromorphism coming back so that the composition is the identity on the images.

8 Limits in Sets

Category theory is about the horizontal transmission of structure or, more generally, determination between objects. The important examples in mathematics have the structure of determination through (self-participating) universals. Such a universal has the property in question and then every other instance of the property is determined to have it by "participating in" (e.g., uniquely factoring through) the self-participating universal. Adjunction extends this theme so that the determination has symmetrically both a sending universal and a receiving universal and where all determinations factor through both universals by the anti-diagonal morphism in an adjunctive square. The 'self-determination' involved in the universals was previously illustrated using the example of the product and coproduct construction in sets. We now generalize the illustration by considering the adjunction for limits in the category of sets (and colimits in the next section).

The construction of limits in sets generalizes the example of the product. Let $\mathbf{D}$ be a small (diagram) category and $D : \mathbf{D} \rightarrow \mathbf{Set}$ a functor considered as a diagram in the category of **Set**. The diagram D is in the functor category $\mathbf{Set}^{\mathbf{D}}$ where the morphisms are natural transformation between the functors. Let $\Delta : \mathbf{Set} \rightarrow \mathbf{Set}^{\mathbf{D}}$, the diagonal functor, assign to each set w the constant functor Δw on $\mathbf{D}$ whose value for each i in $\mathbf{D}$ is w, and for each morphism $\alpha : i \rightarrow j$ in $\mathbf{D}$, the value of Δw_α is 1_w. The functor in the other direction is the limit functor Lim which assigns a set $LimD$ to each diagram D and a set function $Lim\theta : LimD \rightarrow LimD'$ to every natural transformation $\theta : D \rightarrow D'$.

The adjunction for the diagonal and limit functors is:

$$Hom(\Delta w, D) \cong Hom(w, LimD)$$

where is Δ is the left adjoint and Lim is the right adjoint. A adjunctive square for this adjunction would have the generic form:

$$\begin{array}{ccc} (w, \Delta w) & \overset{(f,\Delta f)}{\longrightarrow} & (LimD, \Delta LimD) \\ & {\scriptstyle (Limg,\Delta f)} & \\ \downarrow & \nearrow & \downarrow \\ (Lim\Delta w, \Delta w) & \overset{(Limg,g)}{\longrightarrow} & (LimD, D) \end{array}$$

Abstract Adjunctive Square for Limits Adjunction

which commutes when $g = f^*$or $f = g^*$.

For this adjunction, a chimera morphism or heteromorphism from a set w to a diagram functor D is concretely given by a *cone* $w \Rightarrow D$ which is defined as a set of maps $\{w \overset{f_i}{\longrightarrow} D_i\}$ indexed by the objects i in the diagram category $\mathbf{D}$ such that for any morphism $\alpha : i \to j$ in $\mathbf{D}$, $w \overset{f_i}{\longrightarrow} D_i \overset{D_\alpha}{\longrightarrow} D_j = w \overset{f_j}{\longrightarrow} D_j$. The adjunction is then given by the birepresentation of the het-bifunctor where $Het(w, D) = \{w \Rightarrow D\}$ is the set of cones from the set w to the diagram functor D. Instead of proceeding formally from the het-bifunctor, we will conceptually analyze the construction of the functors $G = Lim$ and $F = \Delta$.

Conceptually, start with the idea of a set function as a way for elements in the domain to determine certain 'elements' in the codomain, the root concept abstracted to form the concept of a morphism in category theory. In the case of a diagram functor D as the target for the determination, the conceptual atom or *element of* D is the maximal set of points that could be determined in the codomains D_i by functions from a one point set and that are compatible with the morphisms between the D_is. Thus an "element" of D is classically a "1-cone" which consists of a member x_i of the set D_i for each object i in the diagram category $\mathbf{D}$ such that for every morphism $\alpha : i \to j$ in $\mathbf{D}$, $D_\alpha(x_i) = x_j$. Ordinarily, a element of D would be thought of as an element in the product of the D_is that is compatible with the morphisms, and $LimD$ would be the subset of compatible elements of the direct product of all the D_is.

An element of D is the abstract 'determinee' of a point *before* any point is assigned to be the 'determiner'. Thus the notion of a 1-cone isn't quite right conceptually in that it pictures the element as having been already determined by a one point set. The notion of a global section [30, $p.47$] is a better description that is free of this connotation. The right adjoint represents these atomic determinees as an object $LimD$ in the top category, in this case **Set**. Then a morphism in that category $f : w \to LimD$ will map one-point determiners to the atomic determinees in $LimD$ and will thus give a determination from the set w to the diagram functor D (i.e., a cone).

It is a conceptual move to take a 'determinee' as the abstract 'determiner' of itself, and that 'self-determination' by the map $1_{LimD} : LimD \to LimD$ takes the set of determinees of D reconceptualized as 'determiner' elements of a set. But to be a determiner by a morphism in the bottom category, one needs a morphism in that functor category, namely a natural transformation. A natural transformation $\theta : D' \to D$ carries the determinees or elements of a functor D' to the elements of the functor D. The determinees in the set $LimD$ are repackaged in the diagram $\Delta LimD$ (see below) as determiners and the natural transformation by which they canonically determine the determinees of the diagram D (from which they came) is the counit $\varepsilon_D : \Delta LimD \to D$.

The conventional treatment of an adjunction is complicated by the need to deal only with morphisms within either of the categories. That need also accounts for the intertwining of the two functors in the unit and counit universals. The properties of the chimera versions of the unit and counit follow from each of the half-adjunctions (the representations on the left or right of the het-bifunctor) without any intertwining of the two functors. For instance, there is no composition of functors in the chimera adjunctive square (see below). The additional relationships obtained from both representations and the intertwining of the functors are illustrated in the adjunctive-image square. For instance, the chimera version of the counit ε_D is $e_D : LimD \Rightarrow D$ is simply the cone of projection maps and it does not involve the diagonal functor Δ. Moreover, the role of self-determination is much clearer. Thinking of the elements of D as the determinees, they become the determiners as the elements of $LimD$ and that self-determination is given by the projection maps $e_D : LimD \Rightarrow D$.

We need to do a similar conceptual analysis of the other functor Δ. How can the elements of a set w be determiners of some diagram functor on the diagram category $\mathbf{D}$? The constant functor Δw repackages the set w as the constant value of a functor on the diagram category. Since the maps between the values of Δw are all identity maps 1_w, the elements on each connected component of the diagram category are just the elements of w. If the diagram category is multiply connected, then the elements of any diagram D on that category will be the product of the elements of the functor restricted to each connected component. If the diagram category $\mathbf{D}$ had, say, two components $\mathbf{D}\restriction\mathbf{1}$ and $\mathbf{D}\restriction\mathbf{2}$, then the elements of the functor D would be ordered pairs of the elements of $D\restriction 1$ and $D\restriction 2$, the functor D restricted to each component of the diagram category. Thus the set $LimD$ would have the structure of a product of sets of elements of the functors $D\restriction 1$ and $D\restriction 2$, i.e., $LimD = LimD\restriction 1 \times LimD\restriction 2$.

A determination by a natural transformation $g : \Delta w \to D$ could be parsed as two natural transformations $g_k : \Delta w\restriction k \to D\restriction k$ for $k = 1, 2$, each one carrying an element of w to a determinee or element of $D\restriction k$. Thus a determination or cone $w \Rightarrow D$ represented in the form $g : \Delta w \to D$ would induce a determination $g^* : w \to LimD = LimD\restriction 1 \times LimD\restriction 2$ in the other form. The constant functor Δ repackages a set w as a diagram functor to be a determiner of diagram functors. The set w can also be self-determining by taking $g = 1_{\Delta w} : \Delta w \to \Delta w$. But for that determination to be given by a morphism in the top category, the determinees of Δw are represented in the top category by applying the limit functor.

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There is then a canonical (diagonal) map wherein w determines the determinees $Lim\Delta w$ of its own representation as a determiner Δw, namely $\eta_w : w \to Lim\Delta w$ (where with two connected components $Lim\Delta w \cong w \times w$). As will be seen, the chimera version is $h_w : w \Rightarrow \Delta w$ is simply the cone of identity maps 1_w.

In the adjunctions encountered by the working mathematician, there should be concrete chimeras or heteromorphisms. This is true for all the examples considered here. As noted above, the heteromorphisms from a set to a diagram functor are the cones $w \Rightarrow D$ (*Nota bene*, not the natural transformations $\Delta w \to D$ which represent-on-the-left the chimera cones $w \Rightarrow D$). When the concrete chimeras are available, then the adjunctive square can be developed using them (instead of only the abstract version embedded in the product category). The chimera version $h_w : w \Rightarrow \Delta w$ of the unit is the cone where each function $w \to (\Delta w)_i = w$ is 1_w. The chimera version $e_D : LimD \Rightarrow D$ of the counit is the cone of projection maps. Given any cone $c : w \Rightarrow D$, there is a unique set map $f(c) : w \to LimD$ such that $w \xrightarrow{f(c)} LimD \overset{e_D}{\Rightarrow} D = w \overset{c}{\Rightarrow} D$.[16] And there is a unique natural transformation $g(c) : \Delta w \to D$ such that $w \overset{h_w}{\Rightarrow} \Delta w \xrightarrow{g(c)} D = w \overset{c}{\Rightarrow} D$. These mappings provide the two representations:

$$Hom(\Delta w, D) \cong Het(w, D) \cong Hom(w, LimD)$$

These maps also give the chimera version of the adjunctive square where $c : w \Rightarrow D$ is the main diagonal.

$$\begin{array}{ccccc} & w & \overset{f(c)}{\longrightarrow} & LimD & \\ h_w & \Downarrow & & \Downarrow & e_D \\ & \Delta w & \overset{g(c)}{\longrightarrow} & D & \end{array}$$

Chimera Adjunctive Square for Limits Adjunction

The anti-diagonal map for the zig-zag factorization will be a chimera morphism going from the diagram Δw to the set $LimD$ (see cocones in the next section). For each i in the diagram category, $(\Delta w)_i = w$ so each component of the "cocone" $z(c) : \Delta w \Rightarrow LimD$ is $f(c) : w \to LimD$. Often the anti-diagonal chimera can be defined using a heteromorphic inverse to one of the vertical morphisms. In this case, $h_w : w \Rightarrow \Delta w$ has the inverse cocone $\Delta w \Rightarrow w$ which has the identity map 1_w as each component. Joining the cone and cocone at the open end just yields the identity: $w \overset{h_w}{\Rightarrow} \Delta w \Rightarrow w = 1_w$ and joining them at the vertexes yields the other identity $\Delta w \Rightarrow w \overset{h_w}{\Rightarrow} \Delta w = 1_{\Delta w}$.[17] Then the chimera anti-diagonal morphism could also be defined as: $z(c) = \Delta w \Rightarrow w \xrightarrow{f(c)} LimD$. There are always the

[16] We leave all the routine checking of the chimera relationships to the reader.

[17] Composing a cone and cocone (same diagram category) on the open end to yield a set map works intuitively if the diagram category is connected or if all the maps in the cone are the same and similarly in the cocone (the latter being the case here). Note that we have here the heteromorphic version of an 'isomorphism' between a set w and a functor Δw!

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over-and-back factorizations of the ordinary intra-category maps at the top and bottom of the adjunctive square (i.e., the upper and lower commutative triangles formed by the anti-diagonal). The over-and-back factorization of the top map $f(c) = z(c)h_w$ follows from the definition of the $z(c)$ just given:

$$w \overset{h_w}{\Longrightarrow} \Delta w \overset{z(c)}{\Longrightarrow} LimD = w \overset{h_w}{\Longrightarrow} \Delta w \Rightarrow w \overset{f(c)}{\longrightarrow} LimD = w \overset{f(c)}{\longrightarrow} LimD.$$

It was previously noted that some category theorists refer to the natural transformation $\Delta w \overset{g(c)}{\longrightarrow} D$ as a "cone" while others refer to the set-to-functor bundle of compatible maps—which is our chimera cone $w \overset{c}{\Rightarrow} D$—as a cone. Often writers will effortlessly switch back and forth between the two notions of a cone. The two inverse chimeras $w \overset{h_w}{\Longrightarrow} \Delta w$ and $\Delta w \Rightarrow w$ are used implicitly to switch back and forth, i.e., $\Delta w \Rightarrow w \overset{c}{\Rightarrow} D = \Delta w \overset{g(c)}{\longrightarrow} D$ and $w \overset{h_w}{\Longrightarrow} \Delta w \overset{g(c)}{\longrightarrow} D = w \overset{c}{\Rightarrow} D$, which is the back and forth action of the isomorphism for the representation on the left: $Hom(\Delta w, D) \cong Het(w, D)$. That is used in the other over-and-back factorization $g(c) = e_D z(c)$:

$$\begin{aligned}\Delta w \overset{z(c)}{\Longrightarrow} LimD \overset{e_D}{\Longrightarrow} D = \Delta w \Rightarrow w \overset{f(c)}{\longrightarrow} LimD \overset{e_D}{\Rightarrow} D \\ = \Delta w \Rightarrow w \overset{c}{\Rightarrow} D = \Delta w \overset{g(c)}{\longrightarrow} D.\end{aligned}$$

The remaining isomorphism $Het(w, D) \cong \mathcal{Z}(\Delta w, LimD)$ is the isomorphism between cones $c : w \Rightarrow D$ and such cocones $z(c) : \Delta w \Rightarrow LimD$ (as defined above). The chimera zig-zag factorization is:

$$w \overset{c}{\Rightarrow} D = w \overset{h_w}{\Rightarrow} \Delta w \overset{z(c)}{\Rightarrow} LimD \overset{e_D}{\Rightarrow} D.$$

There are also the two universal anti-diagonal chimera morphisms. The cocone $z(h_w) = h_{w2} : \Delta w \Rightarrow Lim\Delta w$ has the diagonal map $w \to Lim\Delta w$ as each component, and the cocone $z(e_D) = e_{D1} : \Delta LimD \Rightarrow LimD$ has the identity 1_{LimD} as each component. By their universality properties, given any cocone of the form $z(c) : \Delta w \Rightarrow LimD$, there are unique morphisms $g = g(c) : \Delta w \to D$ and $f = f(c) : w \to LimD$ (where $g = f^*$) that give two factorizations of the anti-diagonal morphism:

$$\Delta w \overset{h_{w2}}{\Longrightarrow} Lim\Delta w \overset{Limg}{\longrightarrow} LimD = \Delta w \overset{z(c)}{\Longrightarrow} LimD = \Delta w \overset{\Delta f}{\longrightarrow} \Delta LimD \overset{e_{D1}}{\Longrightarrow} LimD.$$

These two factorizations fit together to form the adjunctive-image square with $z(c)$ as its main diagonal. Postcomposing the first equation with e_D yields the over-across-and-back factorization of $g(c) : \Delta w \to D$:

$$\Delta w \overset{h_{w2}}{\Longrightarrow} Lim\Delta w \overset{Limg}{\longrightarrow} LimD \overset{e_D}{\Longrightarrow} D = \Delta w \overset{z(c)}{\Longrightarrow} LimD \overset{e_D}{\Longrightarrow} D = \Delta w \overset{g(c)}{\longrightarrow} D.$$

Precomposing the second equation with h_w yields the over-across-and-back factorization of $f(c) : w \to LimD$:

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$$w \overset{h_w}{\Rightarrow} \Delta w \xrightarrow{\Delta f} \Delta LimD \overset{e_{D1}}{\Longrightarrow} LimD = w \overset{h_w}{\Rightarrow} \Delta w \overset{z(c)}{\Longrightarrow} LimD = w \xrightarrow{f(c)} LimD.$$

The two anti-diagonal universals also give the two over-and-back identities on the functorial images (the 'short forms' of the triangular identities):

$$LimD \overset{h_{LimD}}{\Longrightarrow} \Delta LimD \overset{e_{D1}}{\Longrightarrow} LimD = LimD \overset{1_{LimD}}{\longrightarrow} LimD$$
$$\Delta w \overset{h_{w2}}{\Longrightarrow} Lim\Delta w \overset{e_{\Delta w}}{\Longrightarrow} \Delta w = \Delta w \overset{1_{\Delta w}}{\longrightarrow} \Delta w.$$

9 Colimits in Sets

Colimits in sets generalize the previous example of the coproduct construction. We will consider colimits in **Set** but the argument here (as with limits) would work for any other cocomplete (or complete) category of algebras replacing the category of sets. The diagonal functor $\Delta : \mathbf{Set} \to \mathbf{Set}^{\mathbf{D}}$ also has a *left* adjoint $Colim : \mathbf{Set}^{\mathbf{D}} \to \mathbf{Set}$.

For any diagram functor D and set z, the adjunction for the colimit and diagonal functors is:

$$Hom(ColimD, z) \cong Hom(D, \Delta z).$$

A adjunctive square for this adjunction has the form:

$$\begin{array}{ccc} (D, ColimD) & \overset{(f,Colimf)}{\longrightarrow} & (\Delta z, Colim\Delta z) \\ & {\scriptstyle (Colimf,\Delta g)} & \\ \downarrow & \nearrow & \downarrow \\ (\Delta ColimD, ColimD) & \overset{(\Delta g,g)}{\longrightarrow} & (\Delta z, z) \end{array}$$

which commutes when $g = f^*$ or $f = g^*$.

For this adjunction, a chimera morphism or heteromorphism from a diagram functor D to a set z is concretely given by a *cocone* $D \Rightarrow z$ which is defined as a set of maps $\{D_i \xrightarrow{g_i} z\}$ indexed by the objects i in the diagram category $\mathbf{D}$ such that for any morphism $\alpha : i \to j$ in $\mathbf{D}$, $D_i \xrightarrow{D_\alpha} D_j \xrightarrow{g_j} z = D_i \xrightarrow{g_i} z$. The adjunction is then given by the birepresentations of the het-bifunctor where $Het(D, z) = \{D \Rightarrow z\}$ is the set of cocones from the diagram functor D to the set z.

Since the role of the diagram and set are reversed from the case of limits, a new notion of "coelements" as "determiners" from D is necessary. Conceptually, if we go back to the idea of a function as a way for (co)elements in the domain to determine certain elements in the codomain, then a *coelement of* D is the minimal set in the domain D_is that are necessary for functions on those domains to compatibly determine a one point set as the codomain. This could be thought of as a minimal partially-defined 1-cocone, or better, the germ of a cocone to

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determine a one point set before such a set is selected. If the diagram category was discrete so there were no maps between the D_is, then a coelement would simply be a member x_i of one of the D_is since a function defined on that point to the one point set is sufficient to "determine" that point in the codomain. The set of coelements of D would then be the coproduct (disjoint union) of the D_is. But if there were maps between the D_is such as $D_\alpha : D_i \to D_j$ then $x_j = D_\alpha(x_i)$ would also need to be mapped to the same one point. Hence we define a compatibility equivalence relation on the disjoint union of the D_is where $x_i \sim x_j$ if $D_\alpha(x_i) = x_j$ for any morphism D_α between the D_is. Thus a *coelement* or atomic determiner ("germ of a cocone") from D would consist of an equivalence class or block in the partition of the disjoint union determined by the compatibility equivalence relation.

Each coelement of D represents a determiner (without a specified determinee point), a germ of functions on the D_is as domains to compatibly determine a single point in the codomain. As always, the left adjoint repackages an object D in the top category as the object of determiners $ColimD$ in the bottom category so that a determination would be represented by a morphism in the bottom category. Thus a chimera $D \Rightarrow z$ (i.e., a cocone) would be represented by a set morphism $g : ColimD \to z$.

As before (but dually), the basic conceptual move is to take such a conceptual atom, a determiner coelements of D, to be its own determinee, and that self-determination is represented by the identity map $1_{ColimD} : ColimD \to ColimD$, the set of determiners reconceptualized as the elements of the set of determinees. But for that self-determination from D to be represented by a morphism in the top category with domain D, the right adjoint must, as always, repackage the determinees of an object in the bottom category as an object in the top category. The right adjoint Δ repackages the determinees or coelements of the set $ColimD$ as the determinees of the functor $\Delta ColimD$ and the determination from D is expressed by the canonical morphism $\eta_D : D \to \Delta ColimD$.

This self-determination is much clearer in the one-functor treatment of the chimera version of the unit. The chimera unit $h_D : D \Rightarrow ColimD$ is simply the cocone of injection maps. The coelements of D are the determiners so if they are collected together as the determinees in a set $ColimD$, then the self-determination would be expressed by the heteromorphic unit $h_D : D \Rightarrow ColimD$, namely the injection maps which map each coelement of D to itself as an element of $ColimD$.

If the diagram category $\mathbf{D}$ is multiply connected, then the coelements (determiners or germs) of any diagram D would correspond to the coproduct of the coelements of D restricted to each component. In the two component case, the coelements of D would correspond to the members of the coproduct or disjoint sum $ColimD\restriction 1 + ColimD\restriction 2$.

Starting with a set z, its determinees are simply its elements as a set. The right adjoint Δz represents those elements as the determinees of a diagram functor so that a chimera morphism $D \Rightarrow z$ would be expressed by a morphism in the top category (natural transformation) $f : D \to \Delta z$. But as a diagram functor, Δz can also be a determiner in a determination to z by the identity morphism $1_{\Delta z} : \Delta z \to \Delta z$. For that self-determination to be expressed by a morphism in the bottom category (i.e., a set map), the determiners of Δz must be repackaged by the left adjoint as determiners $Colim\Delta z$ in the bottom category and that self-determination is realized by the canonical map $\varepsilon_z : Colim\Delta z \to z$. The colimit $Colim\Delta z$ is the coproduct of a copy of z, one for each connected component in the diagram category. Thus for the case of two components, the counit ε_z is the codiagonal or folding map $Colim\Delta z \cong z + z \to z$. The chimera version $e_z : \Delta z \Rightarrow z$ is simply the cocone of identity maps 1_z. The determinees of z are repackaged as the determiner coelement of Δz and then the heteromorphic counit $e_z : \Delta z \Rightarrow z$ gives the self-determination wherein the coelements of Δz determine themselves as elements of z.

Since the limit and colimit are respectively the right and left adjoints to the same constant functor, the two adjunctions give bidirectional determination from sets w to diagram functors D (e.g., cones) and from diagram functors D to sets z (e.g., cocones).

Here again, the concrete heteromorphisms can be used to give a chimera version of the adjunctive squares diagram. Given a cocone $c : D \Rightarrow z$, there is a unique natural transformation $f(c) : D \to \Delta z$ such that $D \overset{c}{\Rightarrow} z = D \overset{f(c)}{\to} \Delta z \overset{e_z}{\Rightarrow} z$ which gives the representation on the right: $Het(D, z) \cong Hom(D, \Delta z)$. And there is a unique set map $g(c) : ColimD \to z$ such that $D \overset{c}{\Rightarrow} z = D \overset{h_D}{\Rightarrow} ColimD \overset{g(c)}{\to} z$ which gives the representation on the left: $Hom(ColimD, z) \cong Het(D, z)$. Splicing the commutative triangles together gives the adjunctive square with $D \overset{c}{\Rightarrow} z$ as the main diagonal.

$$\begin{array}{cccc} & D & \overset{f(c)}{\longrightarrow} & \Delta z & \\ h_D & \Downarrow & & \Downarrow & e_z \\ & ColimD & \overset{g(c)}{\longrightarrow} & z & \end{array}$$

Chimera Adjunctive Square for Colimit Adjunction

The anti-diagonal map $z(c) : ColimD \Rightarrow \Delta z$ is the set-to-functor cone each of whose components is $g(c) : ColimD \to z$. It could also be constructed using the chimera cone $z \Rightarrow \Delta z$ (each component is 1_z so it is the chimera universal denoted $h_w : w \Rightarrow \Delta w$ in the adjunction for limits) that is inverse to the cocone $e_z : \Delta z \Rightarrow z$. Then $z(c) = ColimD \overset{g(c)}{\longrightarrow} z \Rightarrow \Delta z$ which also establishes the remaining isomorphism: $Hom(ColimD, z) \cong \mathcal{Z}(ColimD, \Delta z)$. The over-and-back factorization of $g(c)$ is immediate:

$$ColimD \overset{z(c)}{\Longrightarrow} \Delta z \overset{e_z}{\Longrightarrow} z = ColimD \overset{g(c)}{\longrightarrow} z \Rightarrow \Delta z \overset{e_z}{\Rightarrow} z = ColimD \overset{g(c)}{\longrightarrow} z.$$

The other over-and-back factorization is:

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$$D \overset{h_D}{\Rightarrow} ColimD \overset{z(c)}{\Rightarrow} \Delta z = D \overset{h_D}{\Rightarrow} ColimD \overset{g(c)}{\to} z \Rightarrow \Delta z$$
$$= D \overset{c}{\Rightarrow} z \Rightarrow \Delta z = D \overset{f(c)}{\to} \Delta z$$

where the last equality is the way of going from cocones as chimera to "cocones" as natural transformations.

The isomorphisms of the adjunction have been established:

$$Hom(ColimD, z) \cong Het(D, z) \cong \mathcal{Z}(ColimD, \Delta z) \cong Hom(D, \Delta z).$$

The zig-zag factorization is:

$$D \overset{c}{\Rightarrow} z = D \overset{h_D}{\Rightarrow} ColimD \overset{z(c)}{\Rightarrow} \Delta z \overset{e_z}{\Rightarrow} z.$$

As in the case of the limits, there are the two anti-diagonal chimera universals with the analogous properties.

10 Adjoints to Forgetful Functors

Perhaps the most accessible adjunctions are the free-forgetful adjunctions between $\mathbf{X} = \mathbf{Set}$ and a category of algebras such as the category of groups $\mathbf{A} = \mathbf{Grps}$. The right adjoint $G : \mathbf{A} \to \mathbf{X}$ forgets the group structure to give the underlying set Ga of a group a. The left adjoint $F : \mathbf{X} \to \mathbf{A}$ gives the free group Fx generated by a set x. The hom-set isomorphism and the adjunctive squares have the usual forms.

For this adjunction, the heteromorphisms are the set-to-group functions $x \overset{c}{\Rightarrow} a$ and the het-bifunctor is given by such functions: $Het(x, a) = \{x \Rightarrow a\}$. A chimera $c : x \Rightarrow a$ determines a set map $f = f(c) : x \to Ga$ and a group homomorphism $g(c) = f^* : Fx \to a$ so that $\widehat{x} = (x, Fx) \overset{(f,f^*)}{\longrightarrow} (Ga, a) = \widehat{a}$ is the abstract version of the concrete $x \overset{c}{\Rightarrow} a$. These associations also give us the two representations:

$$Hom(Fx, a) \cong Het(x, a) \cong Hom(x, Ga).$$

The universal element for the functor $Het(x, -)$ is the chimera $h_x : x \Rightarrow Fx$ (insertion of the generators into the free group) and the universal element for the functor $Het(-, a)$ is the chimera $e_a : Ga \Rightarrow a$ (the retracting of the elements of the underlying set back to the group).

The right adjoint always gives a representation of all the possible determinees of the target a as an $\mathbf{X}$-object Ga with maximal structure so that a determination $x \Rightarrow a$ would be represented by an $\mathbf{X}$-morphism $x \to Ga$. The underlying set functor does exactly that and a set map $x \to Ga$ gives such a determination. The left adjoint always gives a representation of the determiners of a source x as an $\mathbf{A}$-object Fx with minimal structure so that a determination $x \Rightarrow a$ would be represented by an $\mathbf{A}$-morphism $Fx \to a$. Since $\mathbf{A}$ is the category

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of groups, the representation of the elements of a set x as elements of a group independent of any target is accomplished by the free group Fx. No extra "junk" is added other than the group elements generated by the generators x and there is no noise ("noise" in the sense of identifying distinct "signals") and no extra relations are imposed (other than those necessary to make it a group). Then the image of the generators under a group homomorphism $Fx \to a$ is a determination from the set elements of x to the group elements of a, and every such determination would generate such a group homomorphism.

The set-to-group self-determination Ga-to-a is correlated with the set map $1_{Ga} : Ga \to Ga$. For the determiners from a set to be represented as an **A**-object, the left adjoint F must be applied so, in this case, the free group functor yields the free group FGa generated by the underlying set of the group a. Thus FGa is the determinees Ga of a represented as determiners and the group homomorphism induced by the set map 1_{Ga} gives that self-determination as the counit $\varepsilon_a : FGa \to a$ (which is the adjoint transpose of 1_{Ga}). The simpler chimera counit $e_a : Ga \Rightarrow a$ just retracts the elements of the underlying set back to the group. The determinee elements of the group a are represented as the determinee elements of the set Ga but then they turn around and determine themselves by the chimera counit $e_a : Ga \Rightarrow a$ (which is the adjoint correlate of 1_{Ga}).

The set-to-group self-determination x-to-Fx is correlated with the group homomorphism $1_{Fx} : Fx \to Fx$. For the determinees of a group to be represented as an **X**-object, the right adjoint G must be applied so, in this case, the forgetful functor yields the underlying set GFx of the free group generated by the set x. Thus GFx is the determiners Fx from x represented as determinees and the set map induced by the homomorphism 1_{Fx} gives that self-determination as the unit $\eta_x : x \to GFx$ (the adjoint transpose of 1_{Fx}) The simpler chimera version $h_x : x \Rightarrow Fx$ is the injection of the generators into the free group. The determiner elements of the set x are represented as determiner elements of the free group Fx but then they are determined by themselves via the chimera unit $h_x : x \Rightarrow Fx$ (the adjoint correlate of 1_{Fx}).

In an adjunctive square, any set-to-group determination (f, f^*) expressed by a set map $f : x \to Ga$ and its associated group homomorphism $f^* : Fx \to a$ would be factored through both the sending universal $(\eta_x, 1_{Fx})$ and the receiving universal $(1_{Ga}, \varepsilon_a)$ by the indirect anti-diagonal map (Gf^*, Ff).

The chimera version of the adjunctive squares diagram always has the generic form where $x \overset{c}{\Rightarrow} a$ is the main diagonal.

$$\begin{array}{cccccc} & x & \xrightarrow{f(c)} & Ga & \\ h_x & \Downarrow & & \Downarrow & e_a \\ & Fx & \xrightarrow{g(c)} & a & \end{array}$$

In this case, the anti-diagonal map $z(c) : Fx \Rightarrow Ga$ is essentially the group homomorphism $g(c) : Fx \to a$ but where the codomain is taken as the underlying set. Intuitively, there is an inverse $s_a : a \Rightarrow Ga$ to $e_a : Ga \Rightarrow a$ such that $e_a s_a = 1_a$

and $s_a e_a = 1_{Ga}$ so $Fx \overset{z(c)}{\Rightarrow} Ga = Fx \overset{g(c)}{\to} a \overset{s_a}{\Rightarrow} Ga$. The proof of the over-and-back factorization for $f : x \to Ga$ goes around the square counter-clockwise:

$$x \overset{h_x}{\Longrightarrow} Fx \overset{z(c)}{\Rightarrow} Ga = x \overset{h_x}{\Longrightarrow} Fx \overset{g(c)}{\to} a \overset{s_a}{\Rightarrow} Ga = x \overset{c}{\Rightarrow} a \overset{s_a}{\Rightarrow} Ga = x \overset{f(c)}{\longrightarrow} Ga \overset{e_a}{\Rightarrow} a \overset{s_a}{\Rightarrow} Ga = x \overset{f(c)}{\longrightarrow} Ga$$

while the other over-and-back factorization for $g : Fx \to a$ is trivial:

$$Fx \overset{z(c)}{\Rightarrow} Ga \overset{e_a}{\Longrightarrow} a = Fx \overset{g(c)}{\to} a \overset{s_a}{\Rightarrow} Ga \overset{e_a}{\Longrightarrow} a = Fx \overset{g(c)}{\to} a.$$

The chimera isomorphism between the set-to-group main diagonals and the group-to-set anti-diagonals is: $Het(x, a) = \{x \overset{c}{\Rightarrow} a\} \cong \{Fx \overset{z(c)}{\Rightarrow} Ga\} = \mathcal{Z}(Fx, Ga)$. The zig-zag factorization is:

$$x \overset{c}{\Rightarrow} a = x \overset{h_x}{\Rightarrow} Fx \overset{z(c)}{\Rightarrow} Ga \overset{e_a}{\Rightarrow} a.$$

The two anti-diagonal universals are $h_{x2} = z(h_x) : Fx \Rightarrow GFx$ which injects a free group into its underlying set, and $e_{a1} = z(e_a) : FGa \Rightarrow Ga$ which canonically maps the free group on the underlying set of a group a to that underlying set Ga using the group operations of a. One over-and-back identity is: $Fx \overset{h_{x2}}{\Longrightarrow} GFx \overset{e_{Fx}}{\Longrightarrow} Fx = Fx \overset{1_{Fx}}{\longrightarrow} Fx$. The map of a free group onto its underlying set and back to the group is trivially the identity group homomorphism. And the other over-and-back identity is: $Ga \overset{h_{Ga}}{\Longrightarrow} FGa \overset{e_{a1}}{\Longrightarrow} Ga = Ga \overset{1_{Ga}}{\longrightarrow} Ga$. The injection of the underlying set Ga of a group into the free group on that set followed by the projection of that free group onto Ga using the group operations of a is the identity on Ga. There are also the two over-across-and-back factorizations that might be checked:

$$x \overset{h_x}{\Longrightarrow} Fx \overset{Ff}{\longrightarrow} FGa \overset{e_{a1}}{\Rightarrow} Ga = x \overset{f}{\longrightarrow} Ga$$

$$Fx \overset{h_{x2}}{\Longrightarrow} GFx \overset{Gg}{\longrightarrow} Ga \overset{e_a}{\Longrightarrow} a = Fx \overset{g}{\longrightarrow} a.$$

The four naturally isomorphic bifunctors are:

$$Hom(Fx, a) \cong Het(x, a) \cong \mathcal{Z}(Fx, Ga) \cong Hom(x, Ga)$$

and the four associated maps in the same order are:

$$Fx \overset{g(c)}{\longrightarrow} a \leftrightsquigarrow x \overset{c}{\Longrightarrow} a \leftrightsquigarrow Fx \overset{z(c)}{\Longrightarrow} Ga \leftrightsquigarrow x \overset{f(c)}{\longrightarrow} Ga.$$

This includes all the four possibilities of group-to-group, set-to-group, group-to-set, and set-to-set morphisms. The mixed or mongrel morphisms are the heteromorphisms while the unmixed or pure morphisms are the homomorphisms in the standard theory. Note the asymmetry in that set-to-group determination is

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for arbitrary sets x and arbitrary groups a whereas the chimera morphisms in the opposite direction are only defined on the images of the two functors, i.e., are only from free groups to underlying sets of groups. This adjunction does not contemplate arbitrary determination from groups to sets. If the underlying set functor had a right adjoint, then there would be arbitrary two-way determination between sets and groups expressed by adjunctions. But it does not have a right adjoint for reasons that will now be explained.

Adjoints to underlying set functors U are particularly accessible since the underlying determinations can be expressed by easily understood chimera functions between the objects in the two categories. Given such a function $x \Rightarrow a$ (where a is from the category of objects with structure and x is a set), the existence of a left adjoint to U (i.e., a left representation of $Het(x, a) = \{x \Rightarrow \alpha\}$) will depend on whether or not there is an **A**-object Fx with the least or minimal structure so that every determination $x \to Ua$ by a set morphism will have an adjoint transpose by an **A-**morphism $Fx \to a$. But there might also be a chimera $a \Rightarrow x$ where the existence of a *right* adjoint to U will depend on whether or not there is an **A**-object Ix with the greatest or maximum structure so that any set function $Ua \to x$ can also be expressed by an **A**-morphism $a \to Ix$. The homomorphisms $a \to Ix$ would have to preserve the structure which was forgotten in $Ua \to x$ so Ix would have to carry all the possible structures that might carried by the structure-preserving morphisms $a \to Ix$. There is no 'maximal group' Ix so that $a \to Ix$ could be the adjoint transpose to $Ua \to x$; hence the functor giving the underlying set of a group has no right adjoint.

For another example, consider the underlying set functor $U : \mathbf{Pos} \to \mathbf{Set}$ from the category of partially ordered sets (an ordering that is reflexive, transitive, and anti-symmetric) with order-preserving maps to the category of sets. It has a left adjoint since each set has a least partial order on it, namely the discrete ordering. Hence any chimera function $x \Rightarrow a$ from a set x to a partially ordered set or poset a could be expressed as a set function $x \to Ua$ or as an order-preserving function $Dx \to a$ where Dx gives the discrete ordering on x. In the other direction, one can have a chimera function $a \Rightarrow x$ and an ordinary set function $Ua \to x$ but the underlying set functor U does not have a right adjoint since there is no maximal partial order Ix on x so that any determination $Ua \to x$ could be expressed as an order-preserving function $a \to Ix$. To receive all the possible orderings, the ordering relation would have to go both ways between any two points which would then be identified by the anti-symmetry condition so that Ix would collapse to a single point. Thus poset-to-set determinations expressed by $a \Rightarrow x$ cannot be represented as determination through universals, i.e., by an adjunction.

Relaxing the anti-symmetry condition, let $U : \mathbf{Ord} \to \mathbf{Set}$ be the underlying set functor from the category of preordered sets (reflexive and transitive orderings) to the category of sets. The discrete ordering again gives a left adjoint. But now there is also a maximal ordering on a set x, namely the 'indiscrete' ordering Ix on x (the 'indiscriminate' or 'chaotic' preorder on x) which has the ordering

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relation both ways between any two points. Then a preorder-to-set chimera morphism $a \Rightarrow x$ (just a set function ignoring the ordering) can be represented either as a set function $Ua \rightarrow x$ or as an order-preserving function $a \rightarrow Ix$ so that U also has a right adjoint I. Thus the determination both ways between preordered sets and sets can be given by adjunctions, i.e., represented as determination through universals.

11 The Product-Exponential Adjunction in Sets

The product-exponential adjunction in **Set** is an interesting example of an adjoint pair of endo-functors $F : \mathbf{Set} \rightleftarrows \mathbf{Set} : G$. For any fixed (non-empty) "index" set A, the product functor $F(-) = - \times A : \mathbf{Set} \rightarrow \mathbf{Set}$ has a right adjoint $G(-) = (-)^A : \mathbf{Set} \rightarrow \mathbf{Set}$ which makes **Set** a cartesian closed category. For any sets X and Y, the adjunction has the form:

$$Hom(X \times A, Y) \cong Hom(X, Y^A).$$

Since both functors are endo-functors on **Set**, we cannot expect to find any chimera morphisms outside this category. The job of finding some 'chimera' morphisms of **Set** such that the adjunction arises out of birepresenting them is trivial; take either of the hom-sets $Hom(X \times A, Y)$ or $Hom(X, Y^A)$. But that does not show that this theory of adjoints works for the product-exponential adjunction since we don't have the chimera universals. For instance, the chimera sending universal should be a canonical morphism $h_X : X \Rightarrow FX$ but if $FX = X \times A$, there is no canonical map $X \rightarrow X \times A$ (except in the special case where A is a singleton). Similarly, the receiving universal should be a canonical map $e_Y : GY \Rightarrow Y$ but if $GY = Y^A$ then there is no canonical map $Y^A \rightarrow Y$ (unless A is a singleton). Hence there appears to be a problem in applying the theory of adjoints to the product-exponential adjunction (or to any adjunction where one category is a subcategory of the other).

The way out of this apparent problem is shown by the following general result [17, p. 83]. Consider any adjunction $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$. Let X' be the subcategory of $\mathbf{X}$ generated by the image of G where if G is one-one on objects, then its image is that subcategory. Dually, let $\mathbf{A}'$ be the subcategory of $\mathbf{A}$ generated by F (where the image is the subcategory if F is one-one on objects). If $x' = Ga$, then

$$Hom_{\mathbf{X}'}(GFx, x') \cong Hom_{\mathbf{X}'}(GFx, Ga) \cong Hom_{\mathbf{A}}(Fx, a) \cong Hom_{\mathbf{X}}(x, Ga) \cong Hom_{\mathbf{X}}(x, x').$$

Thus $\mathbf{X}'$ is a reflective subcategory of $\mathbf{X}$ where the reflector (left adjoint to the inclusion) can be taken as the functor $GF : \mathbf{X} \rightarrow \mathbf{X}'$ (where G was construed as taking values in $\mathbf{X}'$).

Dually, let $a' = Fx$ so that

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$$Hom_{\mathbf{A}}(a', a) \cong Hom_{\mathbf{A}}(Fx, a) \cong Hom_{\mathbf{X}}(x, Ga) \cong Hom_{\mathbf{A}}(Fx, FGa) \cong Hom_{\mathbf{A}'}(a', FGa).$$

Thus $\mathbf{A}'$ is a coreflective subcategory of $\mathbf{A}$ where the coreflector (right adjoint to the inclusion) can be taken as the functor $FG : \mathbf{A} \to \mathbf{A}'$ (where F is construed as taking values in $\mathbf{A}'$).

It was previously noted that in the case of a reflection, i.e., a left adjoint to the inclusion functor, heteromorphisms can be found as the morphisms with their tail in the ambient category and their heads in the subcategory. For a coreflection (right adjoint to the inclusion functor), the heteromorphisms would be turned around, i.e., would have their tail in the subcategory and their head in the ambient category. If neither $\mathbf{X}$ nor $\mathbf{A}$ were a subcategory of the other, then the above construction would not find the true chimera morphisms from objects in $\mathbf{X}$ to objects in $\mathbf{A}$. It would only find what might be viewed as "pseudo-chimera" morphisms since in the reflective case, a morphism $x \to x'$ is just a morphism $x \to Ga$ and Ga is not an $\mathbf{A}$-object at all. Or in the coreflective case, a morphism $a' \to a$ is only a pseudo-chimera morphism of the form $Fx \to a$ since Fx is not an $\mathbf{X}$-object at all. But if $\mathbf{X}$ or $\mathbf{A}$ is a subcategory of the other (including the case $\mathbf{X} = \mathbf{A}$), then there is nothing "pseudo" about the chimeras identified in the above reflective and coreflective cases. *Given* that one category is a subcategory of the other, that is as "hetero" as the chimeras can be–and that is the case at hand. In the extreme case of the ur-adjunction (the self-adjunction of the identity functor on any category), all differences between the homo- and heteromorphisms are wiped out.

In this case, $\mathbf{X} = \mathbf{Set} = \mathbf{A}$. There are dual ways of reinterpreting the adjunction—either as a reflection or coreflection. As a reflection, let **APower** be the subcategory of $G(-) = (-)^A$ images (G is one-one since A is non-empty) so that $\mathbf{APower} \hookrightarrow \mathbf{Set}$, and that inclusion functor has a left adjoint $F'(-) = (- \times A)^A : \mathbf{Set} \to \mathbf{APower}$. Then the heteromorphisms are those with their tail in **Set** and head in **APower**, i.e., the morphisms of the form $X \to Y^A$. But now we have the chimera universals. The chimera unit $h_X : X \Rightarrow F'X = (X \times A)^A$ is the canonical map that takes an x in X to the function $(x, -) : A \to X \times A$ which takes a in A to $(x, a) \in X \times A$ which is also the unit $\eta_X : X \to (X \times A)^A$ in the original product-exponential adjunction. Since the right adjoint in the reflective case is the inclusion, the chimera counit $e_{Y^A} : Y^A \to Y^A$ is the identity. The chimera adjunctive square then is the following commutative diagram.

$$\begin{array}{cccc} & X & \xrightarrow{f} & Y^A & \\ h_X & \Downarrow & & \Downarrow & e_{Y^A} \\ & (X \times A)^A & \xrightarrow{(f^*)^A} & Y^A & \end{array}$$

As a coreflection, let **AProd** be the subcategory of $F(-) = - \times A$ images (F is one-one since A is non-empty) so that $\mathbf{AProd} \hookrightarrow \mathbf{Set}$, and that inclusion functor has a right adjoint $G'(-) = (-)^A \times A : \mathbf{Set} \to \mathbf{AProd}$. Then the heteromorphisms are those with their tail in **AProd** and their head in **Set**, i.e., the morphisms of the form $X \times A \to Y$. And now we again have the chimera universals. The chimera counit $e_Y : Y^A \times A = G'Y \Rightarrow Y$ is the evaluation map which is also the counit $\varepsilon_Y : Y^A \times A \to Y$ in the original product-exponential adjunction. Since the left adjoint in the coreflective case is the inclusion, the chimera unit $h_{X\times A} : X \times A \Rightarrow X \times A$ is the identity. The chimera adjunctive square is then the following commutative diagram.

$$\begin{array}{cccccc} & X \times A & \stackrel{g^* \times A}{\longrightarrow} & Y^A \times A & \\ h_{X\times A} & \Downarrow & & \Downarrow & e_Y \\ & X \times A & \stackrel{g}{\to} & Y & \end{array}$$

Hence there are in fact *two* ways of choosing the heteromorphisms and each way determines chimera universals which have all the usual factorizations and identities holding. A similar treatment would work for any other case of an adjunction $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$ where one of the categories was a subcategory of the other.

12 Galois Connections

A partially ordered set or poset is construed as a category where each hom-set either has one map (the relation $\leqslant$ holds) or no maps (the relation $\leqslant$ does not hold). A functor between posets is an order-preserving map. An adjunction between posets is usually known as a *Galois connection* [28, p.93].

A standard Galois connection is the direct image and inverse image adjunction induced by any function $f : \mathcal{X} \to \mathcal{A}$ between sets $\mathcal{X}$ and $\mathcal{A}$. Let $\mathbf{X}$ be the power set of a set $\mathcal{X}$ and let $\mathbf{A}$ be the power set of $\mathcal{A}$ both with the inclusion order. Then $F = f() : \mathbf{X} \to \mathbf{A}$, the direct image map, and $G = f^{-1} : \mathbf{A} \to \mathbf{X}$, the inverse image map, are both order-preserving functions. The adjunction, $Fx \leqslant a$ iff $x \leqslant Ga$, will be written as: $f(x) \subseteq a$ iff $x \subseteq f^{-1}(a)$ for any subsets x and a.

This adjunction is about the determination a subset x of $\mathcal{X}$ to a subset a of $\mathcal{A}$ by using the function $f : \mathcal{X} \to \mathcal{A}$ where the relation $x \Rightarrow a$ holds if for all $\zeta \in x$, $f(\zeta) \in a$ (which is just a point-wise way of saying that the direct image $f(x)$ is a subset of a). A value $Het(x, a) = \{x \Rightarrow a\}$ is a singleton if $x \Rightarrow a$ holds and is empty otherwise.

As always, the right adjoint applied to an $\mathbf{A}$-object a gives the $\mathbf{X}$-object Ga that represents all the possible determinees of a. The determinees of a are its subsets of the form $f(x)$ for some x in $\mathbf{X}$ so the representation of all the possible determinees *as an* $\mathbf{X}$*-object* would obtained as the union $\cup\{x \mid f(x) \subseteq a\} = f^{-1}(a)$ or in general for Galois connections, $\sup\{x \mid Fx \leqslant a\} = Ga$.

As always, the left adjoint applied to an $\mathbf{X}$-object x gives the $\mathbf{A}$-object Fx that represents all the possible determiners from x. A set x determines a subset of an $\mathbf{A}$-object a whenever $x \subseteq f^{-1}(a)$ so the representation of all the possible

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determinations from x as an **A**-object could be obtained as $\cap\{a \mid x \subseteq f^{-1}(a)\} = f(x)$ or in general for Galois connections, $\inf\{a \mid x \leqslant Ga\} = Fx$.

As indicated by the inclusion $f^{-1}(a) \subseteq f^{-1}(a)$, the **X**-object that represents the determinees of a can also be a source of a subset of a. The representation of $f^{-1}(a)$ as an **A**-object determiner is obtained by applying the left adjoint so we have the canonical inclusion: $f\left(f^{-1}(a)\right) \subseteq a$ which also gives the true chimera relation $e_a : f^{-1}(a) \Rightarrow a$ between subsets of different sets.

As indicated by the inclusion $f(x) \subseteq f(x)$, the **A**-object that represents the determiners from x can also be a target of x's determination. The application of the right adjoint gives the **X**-object $f^{-1}(f(x))$ so that the determination from x takes the form of the canonical inclusion: $x \subseteq f^{-1}(f(x))$ which also gives the true chimera relation $h_x : x \Rightarrow f(x)$ between subsets of different sets.

The chimera version of the adjunctive square then has the following form of an "if and only if" (iff) statement.

$$\begin{array}{ccccc} & x & \subseteq & f^{-1}(a) & \\ h_x & \Downarrow & & \Downarrow & e_a \\ & f(x) & \subseteq & a & \end{array}$$

Since the vertical relations always hold, the statement is: $f(x) \subseteq a$ iff $x \Rightarrow a$ iff $x \subseteq f^{-1}(a)$ (the representation isomorphisms in this case). The anti-diagonal relation could be defined by: $f(x) \Rightarrow f^{-1}(a)$ if $f^{-1}(f(x)) \subseteq f^{-1}(a)$ or equivalently as: $f(x) \subseteq f(f^{-1}(a))$. One over-and-back factorization is $x \subseteq f^{-1}(a)$ iff $x \subseteq f^{-1}(f(x) \subseteq f^{-1}(a)$, and the other one is: $f(x) \subseteq a$ iff $f(x) \subseteq f(f^{-1}(a)) \subseteq a$. Then the zig-zag factorization is the statement: $x \Rightarrow a$ iff $x \Rightarrow f(x) \Rightarrow f^{-1}(a) \Rightarrow a$, or in conventional terms: $f(x) \subseteq a$ iff $f(x) \subseteq f(x) \subseteq f(f^{-1}(a)) \subseteq a$. The chimera 'isomorphism' is: $x \Rightarrow a$ iff $f(x) \Rightarrow f^{-1}(a)$, or in usual terms, $f(x) \subseteq a$ iff $f^{-1}(f(x)) \subseteq f^{-1}(a)$.

The anti-diagonal universal $h_{x2} : f(x) \Rightarrow f^{-1}(f(x))$ is just the truism $f^{-1}(f(x)) \subseteq f^{-1}(f(x))$ or, equivalently, $f(x) \subseteq f(f^{-1}(f(x)))$. The other universal $e_{a1} : f(f^{-1}(a)) \Rightarrow f^{-1}(a)$ is $f^{-1}(f(f^{-1}(a))) \subseteq f^{-1}(a)$ or, equivalently, the other truism $f(f^{-1}(a)) \subseteq f(f^{-1}(a))$. One over-and-back identity is:

$$f(x) \overset{h_{x2}}{\Rightarrow} f^{-1}(f(x)) \overset{e_{f(x)}}{\Longrightarrow} f(x) \text{ iff } f(x) \subseteq f(f^{-1}(f(x))) \subseteq f(x) \text{ iff } f(x) \subseteq f(x),$$

i.e., $f(f^{-1}(f(x))) = f(x)$. The other over-and-back identity is:

$$f^{-1}(a) \overset{h_{f^{-1}(a)}}{\Longrightarrow} f(f^{-1}(a)) \overset{e_{a1}}{\Longrightarrow} f^{-1}(a) \text{ iff } f^{-1}(a) \subseteq f^{-1}(f(f^{-1}(a))) \subseteq f^{-1}(a) \text{ iff } f^{-1}(a) \subseteq f^{-1}(a),$$

i.e., $f^{-1}(f(f^{-1}(a))) = f^{-1}(a)$.

In the case of the limit and colimit adjunctions, the constant functor Δ from **Set** to $\mathbf{Set}^D$ had both a right adjoint (Lim) and a left adjoint ($Colim$) so there was two-way determination between sets and diagram functors. In the present case, the inverse image functor $f^{-1}()$ has a left adjoint so the question arises of it having a right adjoint. If it had a right adjoint, then that adjunction would be about the determination from a subset a in $\mathbf{A}$ to a subset x in $\mathbf{X}$ where the determinative relation $a \Rightarrow x$ holds if for all $\alpha \in a$, $f^{-1}(\{\alpha\}) \subseteq x$. A right adjoint applied to an $\mathbf{X}$-object x would give the $\mathbf{A}$-object that represents all the possible determinees of x. The determinees of x are its subsets of the form $f^{-1}(a)$ for some a in $\mathbf{A}$ so the representation of all the possible determinees as an $\mathbf{A}$-object would be obtained as the union $\cup\{a \mid f^{-1}(a) \subseteq x\}$ which might be denoted as $f_*(x)$ and which can also be defined directly as: $f_*(x) = \{\alpha \in \mathcal{A} \mid f^{-1}(\{\alpha\}) \subseteq x\}$. This yields the adjunction or Galois connection: $f^{-1}(a) \subseteq x$ iff $a \subseteq f_*(x)$. The two Galois connections give two determinations through universals in both directions between $\mathbf{X}$ and $\mathbf{A}$.

13 Conclusions

13.1 Summary of Theory of Adjoints

We have approached the question of "What is category theory?" by focusing on universal mapping properties and adjoint functors which seem to capture much of what is important and universal in mathematics. We conclude with a brief summary of the theory and with some philosophical speculations.

There is perhaps some irony in the theory presented here to explain one of the central concepts in category theory, the notion of an adjunction. We were required to reach outside the conventional ontology and to acknowledge the object-to-object chimera morphisms or heteromorphisms between categories. The adjunction representation theorem, which shows that all adjunctions arise from birepresentations of het-bifunctors of chimera morphisms was based on the adjunctive square construction. The adjunctive square is a very convenient and, indeed, natural diagram to represent the properties of an adjunction. An adjunction couples two categories together so that there is a form of determination that goes from the x-pair $\widehat{x} = (x, Fx)$ to an a-pair $\widehat{a} = (Ga, a)$. In a commutative adjunctive square, the main diagonal (f, g) gives the pair of determinations $f = g^* : x \to Ga$ and $g = f^* : Fx \to a$. All adjunctions can be obtained as the birepresentation of the cross-category determinations expressed by a het-bifunctor $Het : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$. For many of the adjunctions encountered by working mathematicians, the determinations can be specified concretely by some chimera morphisms or heteromorphisms $x \Rightarrow a$. But the abstract het-bifunctor, where $Het(\widehat{x}, \widehat{a}) = \{\widehat{x} = (x, Fx) \xrightarrow{(f,f^*)} (Ga, a) = \widehat{a}\}$, reproduces (up to isomorphism) any given adjunction as its birepresentation. In the adjunctive square format (abstract or concrete), each determination factors uniquely through sending and receiving universals at each end of the determination (the zig-zag factorization).

A powerful philosophical theme, which connects self-determination and uni-

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versality, emerged in the conceptual analysis of the role of certain identity morphisms in each universal of an adjunction. Each universal is self-participating but, in addition, a type of 'self-determination' is involved in the construction of each universal itself. The self-determination was represented by the identity maps 1_{Fx} and 1_{Ga}, and by their associated universal maps in the isomorphisms of bifunctors:

$$Hom(Fx, a) \cong Het(x, a) \cong \mathcal{Z}(Fx, Ga) \cong Hom(x, Ga).$$

In the zig-zag factorization, any determination given by an adjunction can be factored indirectly using the anti-diagonal map to be compatible with the self-determination represented by the universals on the sending and receiving ends.

13.2 Determination through universals: the main features

In the adjunctions of category theory, we have seen that on the receiving end, a determination might be expressed by a direct map $Fx \xrightarrow{g} a$ or by an indirect map $Fx \xrightarrow{Fg^*} FGa$ factoring through the receiving universal $FGa \xrightarrow{\varepsilon_a} a$. In chimera terms, the same direct map $Fx \xrightarrow{g(c)} a$ has the over-and-back factorization through the indirect anti-diagonal map $Fx \overset{z(c)}{\Longrightarrow} Ga$ and the chimera counit or receiving universal $Ga \overset{e_a}{\Longrightarrow} a$. Mathematically the direct and the indirect-through-the-universal determinations are equal but in the empirical sciences there might be a question of whether a determinative mechanism or process was of the first direct type or the second indirect type factoring through the universal. With direct determination, the receiver has the passive role of receiving the determination. In the second type of mechanism, the receiver of the determination plays a more active or self-determining role of generating a wide ('universal') range of possibilities and then the determination takes place indirectly through the selection of certain of those possibilities to be actively implemented.

Several main features of this determination through universals might be singled out for the receiving case (the sending case is dual).

Universality: While an external direct determination specifies or determines a particular set of possibilities, the determination through a universal constructs the object representing all the possibilities that might be directly determined—as indicated by its universal mapping property.

Autonomy: The universal is constructed in a manner independent of any external determiners (e.g., neither any x nor any f or g were involved in constructing the receiving universal in the conventional form $FGa \xrightarrow{\varepsilon_a} a$ or in the chimera form $Ga \overset{e_a}{\Longrightarrow} a$).

Self-determination: The morphism associated with the universal potentially determines all the possibilities (e.g., the receiving universal in either form was the adjoint correlate of the identity 1_{Ga}).

Indirectness: The particularization comes only with the indirect factor map that picks out or selects certain possibilities.

Composite Effect: The composition of the specific factor map followed by the universal morphism then implements the possibilities to agree with the given direct determination.

In grand philosophical terms, the factorization through universals of an adjunction gives an approach, albeit in rather abstract mathematical terms, to resolving what is perhaps the central conundrum of philosophy, the reconciliation of external determination ("necessity" or "heteronomy") and self-determination ("freedom" or "autonomy"). We turn to what this determination through universals might mean in the life sciences and in social philosophy.[18]

13.3 Determination through universals in the life sciences

The debate between selectionist and instructionist mechanisms can be seen in this light. In the original debate about evolution, the Lamarckian position was that the environment directly "instructed" the organism about adaptive features which were then inherited by the organism's offspring. In the Darwinian selectionist theory, the species generated a wide range of possibilities (e.g., through mutations and sexual reproduction) and then the environment has only the indirect role of selecting which features have a survival advantage and which will thus be differentially propagated to the offspring.

The mathematics provides a highly abstract, atemporal, and idealized model so one does not expect a perfect fit to any processes in the life or human sciences. With that caveat, the main features of determination through universals seem to be present in the selectionist account of biological evolution.

Universality: The selectionist theory is an example of population thinking because it is the population, not the individual organism, that explores the universe of possibilities by variation through mutation and sexual reproduction.[19]

Autonomy: In the modern treatment of genetic evolution, there is some emphasis on the "fundamental dogma" that there is no information flow from

[18]In the physical sciences, there is the obvious possibility of viewing the expansion of the wave packet in quantum mechanics and then its collapse to realize a specific actuality as an application of determination through universals. However, I am not prepared to investigate that possibility here so the focus is on the life and human sciences.

[19]Some versions of Darwinian evolution have taken the problem of the generation of variety more seriously than others. In particular, Sewall Wright's shifting balance theory [36] has emphasized the advantages of having the population split up into various subpopulations or 'demes' that will encourage wider variation (a practice also followed by artificial breeders). Separation of subpopulations allows more variation to be tested and fixed but there also has to be migration between the subpopulations so that 'improvements' or 'discoveries' will be transmitted to the whole population. The proper mix is a question of shifting balances.

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the environment to 'direct' the process of generating genetic variety. This is the aspect of autonomy. The possibilities are generated independent of any external determiners.

Self-determination: In the biological context, this is the self-reproduction of organisms. In the mathematical example of the limit adjunction, we saw how the determinees became determiners in the construction of the limit $LimD$ so the determinees determined themselves via the projection maps $e_D : LimD \Rightarrow D$. In a temporal context, the switching of determinee and determiner roles would be sequential. This switching of roles is involved in all biological reproduction, the offspring becomes the parent. At the molecular level of reproduction, a DNA sequence is first the determinee when it is formed on a given template, and then after the splitting of the double helix, it becomes itself a template or determiner to 'determine' or reproduce itself.

Indirectness: The selective effect of the environment on the variety of possibilities that have been generated is modelled mathematically by the indirect factor map.

Composite Effect: The composite effect of this external selection and potential self-reproduction of the variants is to 'implement' or differentially reproduce the selected variants. That is mathematically modelled by the composition of the indirect factor map (which selects certain possibilities) with the universal morphism—which 'by itself' would self-determine or reproduce all possibilities—so that the *composite* effect is to differentially amplify or 'implement' the selected possibilities.

Today the idea of an instructive versus a selective process has been generalized to a number of other processes. A common theme is that learning processes are originally thought to be instructive (i.e., direct $Fx \xrightarrow{g(c)} a$) but are then found to be selective (i.e., indirect through the universal $Fx \overset{Fg^*}{\to} FGa \overset{\varepsilon_a}{\to} a$ or $Fx \overset{z(c)}{\Longrightarrow} Ga \overset{e_a}{\Rightarrow} a$). The key component of any selectionist mechanism is the generator of diversity that generates the 'universal' range of possibilities (e.g., the construction in the mathematics of going from a to FGa or simply to Ga) so that certain possibilities can then be selected (e.g., by the indirect morphism $Fx \overset{Fg^*}{\to} FGa$ or $Fx \overset{z(c)}{\Longrightarrow} Ga$) to determine the eventual outcome by a self-determinative process (e.g., by the canonical morphism $FGa \overset{\varepsilon_a}{\to} a$ or $Ga \overset{e_a}{\Longrightarrow} a$).

A particularly striking application of the selectionist approach is to the immune system. The early theories were instructional; the external molecule or antigen would enter the system and instruct the immune mechanism with its template to construct antibodies that will neutralize the antigen. In 1955, Niels Jerne [20] proposed the selectionist theory of the immune system which is now

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accepted (with variations added by many other researchers). The immune system takes on the active role of generating a huge variety of antibodies and the external antigen has the passive role of simply selecting which antibody fits it like a key in a lock. Then that antibody is differentially amplified in the sense of being cloned into many copies to lock-up the other instances of the antigen. A similar example was the originally instructivist account of bacteria 'learning' to tolerate antibiotics or to consume a new substance but now these processes are recognized as selectionist. A wide variety of bacterial mutations are constantly being generated and those that can tolerate antibiotics or digest a new substrate will differentially thrive in such an environment.

Peter Medawar (who in addition to Jerne received a Nobel Prize for work related to the immune system) illustrated the difference between an instructive and selective (or "elective") mechanism using as an analogy the difference between a phonograph (or "gramophone") and a jukebox. With a phonograph, one has to externally supply the specific musical instructions (a record) to the machine which then plays the record. But a jukebox has a wide repertoire of musical instructions (records) inside it; externally there is only the selection of the record to be played.

The analogy with the determination through universals can be illustrated using the conventional exponential adjunction. Taking the set $X = 1$, a single record played on the phonograph might be compared to the function $g : 1 \times A \rightarrow Y$ where A is a set of parameters—going over the set of parameters A 'plays' the record to produce the music Y. But the jukebox contains within it a large 'universal' repertoire of records Y^A which might be played. The adjoint transpose $g^* : 1 \rightarrow Y^A$ picks out the same record that played as $g : 1 \times A \rightarrow Y$ out of the universal repertoire Y^A. The operation of the jukebox playing a record from its internal repertoire is represented by the counit $\varepsilon_Y : Y^A \times A \rightarrow Y$. To get the same effect with the jukebox as with the phonograph, the functor $- \times A$ is applied to the selection $g^* : 1 \rightarrow Y^A$ of the record to be played:

$$\text{"}\mathit{Phonograph}\text{"} \quad 1 \times A \xrightarrow{g} Y = 1 \times A \xrightarrow{g^* \times A} Y^A \times A \xrightarrow{\varepsilon_Y} Y \quad \text{"}\mathit{Jukebox}\text{"}$$

> During the past ten years [1950s], biologists have come to realize that, by and large, organisms are very much more like juke-boxes than gramophones. Most of the reactions of organisms which we were formerly content to regard as instructive are in fact elective.[32, p. 90]

In the mathematics, the morphisms are equal, but empirically the point is that a selective mechanism is represented by the factorization through the universal, not by the direct morphism that mathematically would give the same end results (e.g., the playing of the record).

In a selectional mechanism, there is also instruction or determination but it comes from within—the autonomy and self-determination that emerges in the mathematics with the counit $\varepsilon_Y : Y^A \times A \to Y$ being the adjoint transpose of the identity map $1_{Y^A} : Y^A \to Y^A$ (the external X plays no role). The point about the jukebox was not that it had no musical instructions (records) but that they were embodied within the entity rather than externally supplied. "The instructions an organism contains are not musical instructions inscribed in the grooves of a gramophone record, but *genetical* instructions embodied in chromosomes and nucleic acids." [32, p. 90] Heinz Pagels makes the same point connecting evolution and the immune system.

> Like evolution, the immune response is also a selective system in which the system is instructed from within—the genetic instructions plus random variation—but the selection depends on the external environment—the specific invading antigens. [33, p. 265]

Language learning by a child is another example of a process that was originally thought to be instructive. But Noam Chomsky's theory of generative grammar postulated an innate universal grammar (the instructions from within) that would unfold according to the linguistic experience of the child. The child did not 'learn' the rules of grammar; the linguistic experience of the child would select how the universal mechanism would develop or unfold to implement one rule rather than another. This connection is not new. Niels Jerne's Nobel Lecture was entitled *The Generative Grammar of the Immune System*.

An everyday example of indirect determination is a person's understanding of spoken language. The naive viewpoint is that somehow the meaning of the spoken sentences is transmitted from the speaker to the listener. But, in fact, it is only the physical sounds that are transmitted. The syntactic analysis and the semantic component have to be generated by the listener so the heard sounds only have the role of selecting which generative processes will be triggered. Here again, Chomsky has emphasized the universality of the internal mechanism to generate an understanding of a potential infinity of sentences which have never been heard before. Descartes emphasized this universality of language and reason: "reason is a universal instrument which can serve for all contingencies" [7, p. 116] so Chomsky has referred to the generative grammar approach as "Cartesian linguistics."

> In summary, one fundamental contribution of what we have been calling "Cartesian linguistics" is the observation that human language, in its normal use, is free from the control of independently identifiable external stimuli or internal states and is not restricted to any practical communicative function, in contrast, for example, to the pseudo language of animals. It is thus free to serve as an instrument of free thought and self-expression. The limitless possibilities of thought and imagination are reflected in the creative aspect of language use. The language provides finite means but infinite possibilities of expression constrained only by rules of concept formation and sentence formation, these being in part particular and idiosyncratic but in part universal, a common human endowment. [6, p. 29]

The general features of universality, autonomy (independence from external stimulus control), and self-determination are clear.

After Gerald Edelman received the Nobel prize for his work on the selectionist approach to the immune system, he switched to neurophysiology and developed the theory of neuronal group selection or neural Darwinism.

> [T]he theoretical principle I shall elaborate here is that the origin of categories in higher brain function is somatic selection among huge numbers of variants of neural circuits contained in networks created epigenetically in each individual during its development; this selection results in differential amplification of populations of synapses in the selected variants. In other words, I shall take the view that the brain is a selective system more akin in its workings to evolution than to computation or information processing.[8, p. 25]

In simpler terms, the brain generates an immense variety of groups of neural circuits (like the variety of antibodies) and then external stimuli only select which neural groups will be differentially amplified. What at first looks like the external environment instructing the brain is seen instead as a selectionist process.

From antiquity, some schools of thought (e.g., Neo-Platonism) have emphasized the general point that understanding and learning are mistakenly seen as a direct instructive process rather than an indirect composite effect of catalyzing a universal internal generative process. In the early fifth century, Augustine in *De Magistro* (The Teacher) made the point.

> But men are mistaken, so that they call those teachers who are not, merely because for the most part there is no delay between the time of speaking and the time of cognition. And since after the speaker has reminded them, the pupils quickly learn within, they think that they have been taught outwardly by him who prompts them.(Chapter XIV)

Wilheim von Humboldt made the same point even recognizing the symmetry between speaker and listener.

> Nothing can be present in the mind (Seele) that has not originated from one's own activity. Moreover understanding and speaking are but different effects of the selfsame power of speech. Speaking is never comparable to the transmission of mere matter (Stoff). In the person comprehending as well as in the speaker, the subject matter must be developed by the individual's own innate power. What the listener receives is merely the harmonious vocal stimulus. [19, p. 102]

In the mathematics of adjunctions we have seen the symmetry between the self-determination involved in both the sending and receiving universals (e.g., the zig-zag factorization).

In all these cases—from the beginnings of evolutionary selection up through the highest human functions—one of the central points is that being on the receiving end of a determination does not imply passivity. There is an indirect mode

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of determination through a receiving universal where the receiver in some form actively generates or has the capacity to actively generate a well-nigh 'universal' set of possible receptions (received determinations or determinees). Then the determination (e.g., learning) through the receiving universal takes the form of selecting which 'message' is actively generated so that receiving the determination is quite consistent with the active self-determination of the receiver. Such a mechanism seems key to understanding how an organism can perceive and learn from its environment without being under the direct stimulus control of the environment—thus resolving the ancient conundrum of receiving an external determination while exercising self-determination.

In the adjunctions of category theory, we have seen an abstract version of this conceptual structure of indirect determination through universals that express self-determination. The importance of these conceptual structures *in mathematics* was the original reason why we have focused on adjunctions. We have given a few hints at how these adjunctive structures of determination through universals might also model processes of central importance in the life sciences that serve, in varying degrees, to make external determination more indirect and thus more consistent with self-determination.

13.4 Determination through universals in social philosophy

The conceptual structure of determination through universals might also be used as a normative model. Here the *locus classicus* is Immanuel Kant. The mathematics of adjunctions sustains Kant's philosophical intuition that universality (the first version of the categorical imperative) is closely related to autonomy and self-determination (the second and third versions of the categorical imperative). However, it is not clear that he worked out the correct notion of universality that would correspond to, say, the autonomy principle of always treating persons as ends-in-themselves rather than just as means ([22]; see Chapter 4 in [13]). In any case, there may be something to Michael Arbib and Ernest Manes' linking of "The Categorical Imperative" (subtitle of their book) to universal constructions in category theory [1, p. vii]. And it might be noted that Kant also saw an active role for the mind in perception and cognition somewhat along the lines described in the previous section.

Perhaps the most important applications come in political and economic theory. In political theory, all heteronomous governance relations can be restructured to form a political democracy so that people are, at least in theory, jointly self-governing. In economics, the problematic determinative relation is that of the employer and employee. But it is also always possible to restructure a firm as a workplace democracy so that everyone working in it is jointly 'self-employed' or self-managing.[20]

[20] The relevant political theory is outlined in [15] and the concepts of workplace democracy are dealt with at book length in [12] (which can be downloaded from www.ellerman.org).

More generally, normative questions of heteronomy versus autonomy arise in social philosophy where determinative relations between persons are the central topic. The setting is where one person (or group of persons) has the sender or 'determiner' role and another person (or group of persons) is in the receiver or 'determinee' role. For the person in the sending role trying to influence, instruct, counsel, control, or determine others, the message is that this can structured as firstly being self-determination—leading by example, practicing what one preaches, and teaching something by doing it oneself. But the greater problem is the self-determination or autonomy of the person or persons in the receiving position. The mathematical analogy shows that (in an adjunctive context) there is always a way to rearrange matters so that any external determination becomes indirect by factoring through the receiving universal that realizes the self-determination of the receiver.

Across human affairs, there are relationships of teacher to student, manager to subordinate, counselor to client, psychologist to patient, and helper to doer where the first party tries to influence, control, or otherwise determine the actions and beliefs of the second party. The perennial conundrum is that most 'help' or educational instruction occurs in a manner that overrides or undercuts the self-determination and autonomy of the persons in the receiving position. It is a most subtle matter to see how such heteronomous external determination might be 'factored' to take an indirect form that would respect the self-determination and autonomy of the people in the receiver role.[21]

This theory of adjoints gives an account of an adjunction based on determination through universals expressing a type of self-determination at both the sending and receiving end of a determination. The salient features were summarized in adjunctive squares which show that—by the zig-zag factorization—it is always possible to factor any determination expressed by an adjunction indirectly through the sending and receiving universals. It is heartening to see the "unreasonable effectiveness of mathematics" (to use Eugene Wigner's expression) to capture this basic conceptual theme of structuring an external determinative relationship to be indirect and compatible with self-determination on each end of the determination.

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[21] These matters have been dealt with at book length in [14] (with some emphasis on development assistance).

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