Sci QA/A

annals of mathematical logic

AMLOAD 7 (2, 3) 113–326 (1974)

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Volume 7
Numbers 2, 3 December 1974

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM

ANNALS OF MATHEMATICAL LOGIC 7 (1974) 163–195.

SHEAVES OF STRUCTURES AND GENERALIZED ULTRAPRODUCTS*

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Received 1 May 1974

Introduction

Classical ultraproducts are constructed from indexed sets of structures, i.e., from sheaves of structures on discrete spaces. We generalize the construction so that the initial datum can be an arbitrary sheaf of structures. Boolean ultrapowers are obtained in the special case where the initial sheaf is a constant sheaf.

In Part I, we review the relevant information about sheaves on topological spaces. In Part II, we define notions of forcing and weak forcing in the stalks of any sheaf. From any sheaf of structures P on a space I, we construct a sheaf P^0 on the spectrum of prime filters of the pseudo-Boolean algebra (pBa) of opens of I. The "prime stalk theorem" is a Łoš-type theorem that characterizes the formulas weakly forced in any stalk of P^0 at a prime filter.

In Part III, we construct a sheaf P^* on the Stone space of the complete Boolean algebra of regular opens of I. The stalks of P^* are called the *ultrastalks* of P, and they are the generalized ultraproducts. With each ultrafilter in the cBa (i.e., each point in the Stone space) there corresponds a maximal filter in the pBa, and the corresponding stalks of P^* and P^0 are isomorphic, thereby yielding two constructions of the generalized ultraproducts. The "ultrastalk theorem" generalizes the Łoš ultraproduct theorem by characterizing truth in the stalks of P^* (or, equivalently, in the stalks of P^0 at maximal filters). In Part IV, a few extensions and applications are outlined.

The primary purpose of this paper is to familarize the working model

* This is an updated and revised version of our dissertation [3] written at Boston University with Professor Rohit Parikh as advisor.

theorist with the concrete machinery of sheaves of relational structures (i.e., generalized relational structures) and with the "ultrastalk" construction. The "ultrastalk" construction was originally obtained by directly generalizing the sheaf-theoretic construction of ultraproducts, but similar results could probably be obtained by concretely rendering the "generalized ultraproducts" used by Lawvere and Tierney in their theory of topoi [11].

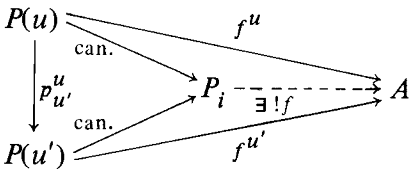
Part I. SHEAVES

Let I be a topological space, let u, u', etc. be open subsets of I, and let $O(I)$ be the inclusion partial order of the open sets of I. A *presheaf of sets* (resp., a *presheaf of relational structures of type* μ) is a functor

$$P : O(I) \xrightarrow{\text{op}} \text{Ens} \qquad (P : O(I) \xrightarrow{\text{op}} M_\mu)$$

from the opposite of the inclusion ordering to the category Ens of sets and functions (to the category M_μ of relational structures of type μ and homomorphisms). For opens $u' \subset u$, the map $p^u_{u'} : P(u) \to P(u')$ is called a *restriction map.*

If (I, P) is a presheaf of sets and $i \in I$, then the sets $P(u)$ for $u \ni i$ and the restriction maps between them form a direct system of sets. The direct limit $\varinjlim_{u \ni i} P(u) = P_i$ is called the *stalk of* P *at* i. The direct limit is formed by first taking the disjoint union $\biguplus_{u \ni i} P(u)$ and then taking equivalence classes according to the following equivalence relation: if $i \in u_1 \cap u_2$, $a_1 \in P(u_1)$, and $a_2 \in P(u_2)$, then $a_1 \sim a_2$ iff there is an open $u \ni i$ s.t. $u \subset u_1 \cap u_2$ and $p^{u_1}_u(a_1) = p^{u_2}_u(a_2)$. The canonical maps $p^u : P(u) \to P_i$, which take an element $a \in P(u)$ to its equivalence class $\underline{a} \in P_i$, commute with the restriction maps of the direct system in the sense that if $i \in u' \subset u$, then $p^u = p^{u'} \circ p^u_{u'}$. Furthermore, the canonical maps p^u enjoy the *universality property* that given any set of maps $\{P(u) \xrightarrow{f^u} A : u \ni i\}$ which also commute with the restriction maps, there is a unique map $f : P_i \to A$ s.t. $f^u = f \circ p^u$ for all $u \ni i$. This universality property is represented by the following diagram (where all triangles commute).



If (I, P) is a presheaf of structures, then the stalk P_i is constructed by taking the direct limit in the category M_μ of structures and homomorphisms. The underlying set of P_i is the same as above, and if $R(x_1, ..., x_n)$ is an atomic relation and $b_1, ..., b_n$ are equivalence classes in P_i, then $P_i \models R(b_1, ..., b_n)$ iff $\exists\, u \ni i$ and $\exists\, a_1, ..., a_n \in P(u)$ s.t. $p^u(a_k) = b_k$ for

$k = 1, ..., n$ and $P(u) \models R(a_1, ..., a_n)$. Intuitively, an atomic relation holds in the direct limit P_i iff it must hold in order for the canonical maps $p^u : P(u) \to P_i$ to be homomorphisms. The direct limit P_i enjoys the same universality property as above except that it is formulated in the category of structures (of type μ) and homomorphisms.

A presheaf of sets (I, P) is a *sheaf of sets* if the following two conditions hold (where r and s are in the customarily unmentioned index set of an open cover):

Condition (1): for any open u, any open cover $\{u_r\}$ of u, and any $a, b \in P(u)$, if $p^u_{u_r}(a) = p^u_{u_r}(b)$ for all u_r in the cover, then $a = b$;

Condition (2): for any open u and any open cover $\{u_r\}$ of u, if $\{a_r\}$ is a set of elements s.t. $a_r \in P(u_r)$ and, for any pair u_r and u_s in the cover, $p^{u_r}_{u_r \cap u_s}(a_r) = p^{u_s}_{u_r \cap u_s}(a_s)$, then there is an element $a \in P(u)$ s.t. a restricts to each a_r (i.e., $p^u_{u_r}(a) = a_r$ for every u_r in the cover). By condition (1), the element a which exists in condition (2) is unique. Since the empty set $\emptyset$ is covered by the empty cover, $P(\emptyset)$ is always a singleton.

If (I, P) is a presheaf of relational structures (of the given type μ hereafter fixed and unmentioned), then it is a *sheaf of relational structures* if conditions (1) and (2) hold as well as:

Condition (3): for any open u, any open cover $\{u_r\}$ of u, and any atomic relation $R(x_1, ..., x_n)$, if $a_1, ..., a_n \in P(u)$ are such that for every u_r in the cover $P(u_r) \models R(p^u_{u_r}(a_1), ..., p^u_{u_r}(a_n))$, then $P(u) \models R(a_1, ..., a_n)$.

If (I, P) and (J, Q) are presheaves of sets, then a *morphism* $(f, \Theta) : (I, P) \Rightarrow (J, Q)$ consists of a continuous function $f : I \to J$ and a natural transformation $\Theta : Q \Rightarrow P(f^{-1}(\cdot))$ between the two functors on $O(J)^{\mathrm{op}}$. Thus Θ is a set of functions

$$\{Q(v) \xrightarrow{\Theta_v} P(f^{-1}(v)) : v \in O(J)^{\mathrm{op}}\}$$

s.t. if $v' \subset v$ are opens in J, then the following diagram commutes:

$$\begin{array}{ccc} Q(v) & \xrightarrow{\quad \Theta_v \quad} & P(f^{-1}(v)) \\ \Big\downarrow q^v_{v'} & & \Big\downarrow p^{f^{-1}(v)}_{f^{-1}(v')} \\ Q(v') & \xrightarrow{\quad \Theta_v \quad} & P(f^{-1}(v')) \,. \end{array}$$

If (I, P) and (J, Q) are presheaves of structures, then the above notion only defines a morphism between them as presheaves of sets. A morphism between them as presheaves of structures would be the same except that the Θ_v functions would be homomorphisms.

A morphism $(f, \Theta) : (I, P) \Rightarrow (J, Q)$ induces maps between the stalks of Q and the stalks of P. For any $i \in I$, the set of maps

$$\{Q(v) \xrightarrow{\Theta_v} P(f^{-1}(v)) \xrightarrow{\text{can.}} P_i :\ v \ni f(i)\}$$

commute with the restriction maps in the direct system for $Q_{f(i)}$ so, by the universality property, there is a unique map $\Theta_i^\# : Q_{f(i)} \to P_i$ s.t. for all $v \ni f(i)$ the following diagram commutes:

$$\begin{array}{ccc}
Q(v) & \xrightarrow{\Theta_v} & P(f^{-1}(v)) \\
\big\downarrow \text{can.} & & \big\downarrow \text{can.} \\
Q_{f(i)} & \xrightarrow{\Theta_i^\#} & P_i
\end{array}$$

Given a sheaf of sets (I, P), a *subpresheaf* R of P is a subfunctor $R : Q(I) \xrightarrow{\text{op}} \text{Ens}$, i.e., for all $u \in O(I)^{\text{op}}$, $R(u) \subset P(u)$ and if $u' \subset u$, then p^u_u, restricted to the subset $R(u)$ is the restriction map $r^u_{u'}$. A *subsheaf* R of P is a subpresheaf which is a sheaf. Thus a subpresheaf R is a subsheaf if for any u, the property on $P(u)$ of belonging to the subset $R(u)$ is a *property of local character* in the sense that if for any $i \in u$, there is an open u_i s.t. $i \in u_i \subset u$ and $p^u_{u_i}(a) \in R(u_i)$, then $a \in R(u)$ (i.e., if R holds of $a \in P(u)$ locally, then R holds of a).

The notion of a sheaf of structures can be expressed using only the notions of sheaves and subsheaves of sets. If (I, P) is a presheaf of structures, then each atomic n-ary relation R defines a subpresheaf (called its *graph*) of the presheaf of sets (I, P^n), i.e.,

$$R(u) = \{\langle a_1, ..., a_n\rangle \in P^n(u) = P(u)^n : P(u) \vDash R(a_1, ..., a_n)\} .$$

If (I, P) satisfies conditions (1) and (2) (i.e., is at least a sheaf of sets), then condition (3) simply says that the graph presheaves of all the atomic relations are sheaves. Thus we may say that a *sheaf of relational structures* of the given similarity type is a sheaf of sets (I, P) together with a subsheaf $R \subset P^n$ for every n-ary atomic relation (i.e., the usual definition of relational structure with "sheaf of sets" and "subsheaf of sets" substituted respectively for "set" and "subset"). When $I = 1$, the one point space, this reduces to the usual notion of a relational structure, so a sheaf of structures can be viewed as a generalized relational structure.

Let Sh(I) be the category whose objects are sheaves of sets on I with natural transformations as morphisms. Note that if $P_1, P_2 \in \mathrm{Sh}(I)$, then a morphism $\Theta : P_1 \to P_2$ could be construed as a sheaf morphism (as previously defined) $(f, \Theta) : (I, P_2) \Rightarrow (I, P_1)$ in the opposite direction, where f is the identity map on I. Lawvere [11] and Tierney have shown that the sheaf Ω of germs of open subsets of I functions as a "subobject classifier" in $\mathrm{Sh}(I)$ just as Condition (2) does in the (special) case of Ens ($\cong \mathrm{Sh}(1)$). That is, if $P \in \mathrm{Sh}(I)$ then the subsheaves of P are in one-to-one correspondence with the morphisms $P \to \Omega$. For u open in I, $\Omega(u)$ is the set of open subsets of u, and if $u_1 \subset u_2$ are opens, then the restriction map $\Omega(u_2) \to \Omega(u_1)$ takes $u \subset u_2$ to $u \cap u_1 \subset u_1$. If R is a subsheaf of P, then its *characteristic morphism* (also denoted by) $R : P \to \Omega$ is defined as

$$R = \{P(u) \xrightarrow{R_u} \Omega(u) : u \in O(I)^{\mathrm{op}}\},$$

where for $a \in P(u)$,

$$R_u(a) = \bigcup \{u' \subset u : p^u_{u'}(a) \in R(u')\} .$$

Conversely, given a morphism $R : P \to \Omega$, the corresponding subsheaf R is defined by

$$R(u) = \{a \in P(u) : R_u(a) = u\} .$$

If (I, P) is a sheaf of structures, then each n-ary atomic relation R determines a subsheaf of P^n and thus a characteristic morphism denoted $\models R : P^n \to \Omega$. The values of this morphism are the "truth values"

$$\models R_u(a_1, ..., a_n) = \bigcup \{u' \subset u : P(u') \models R(p^u_{u'}(a_1), ..., p^u_{u'}(a_n))\}$$

$$= \{i \in u : P_i \models R(\underline{a_1}, ..., \underline{a_n})\} ,$$

where $a_1, ..., a_n \in P(u)$.

Before turning to examples of sheaves, we shall mention a condition sometimes used in constructing sheaves. Let P' be a "presheaf" of sets defined only on a basis B for the topology on I. P' can be canonically extended to a presheaf P on all opens of I by defining $P(u_0) = \varprojlim P'(u)$, where the (generalized inverse) limit is over all $u \subset u_0$ with $u \in B$ (the restriction maps are defined using the universality property of limits). This canonical extension P is a sheaf iff P' satisfies the following *basis condition*: for every $u \in B$ and every cover $\{u_r\}$ of u, where each $u_r \in B$, if there is a set $\{a_r\}$ where $a_r \in P'(u_r)$ s.t. any a_r and a_s have the same restrictions to $P'(u')$ for any u', where $u' \in B$ and $u' \subset u_r \cap u_s$, then there is a unique $a \in P'(u)$ which restricts to all the a_r [12].

A sheaf of structures on a discrete space will be called a *discrete* sheaf. If $\{\underline{A}_i\}_{i \in I}$ is an indexed set of structures, then we have the discrete sheaf (I, P) where $P(u) = \Pi_{i \in u} \underline{A}_i$. Then $P_i \cong \underline{A}_i$ so the indexed set of structures can be recovered from the corresponding discrete sheaf. Also if (I, P) is a discrete sheaf, then $\{P_i\}_{i \in I}$ is, of course, an indexed set of structures, and for any $u \subset I$, $P(u) \cong \Pi_{i \in u} P_i$. Thus an indexed set of structures (part of the data in the ultraproduct construction) 'is' a discrete sheaf of structures.

Another simple but important type of sheaf can be constructed from a structure $\underline{A}$ and a space I (not necessarily discrete). Then we define a sheaf A on I by $A(u) = \{u \xrightarrow{f} A : f \text{ is continuous}\}$ for u open in I (where the discrete topology is on A the underlying set of $\underline{A}$). If R is an n-ary atomic relation and $f_1, ..., f_n \in A(u)$, then $A(u) \models R(f_1, ..., f_n)$ if for all $i \in u$, $\underline{A} \models R(f_1(i), ..., f_n(i))$. Such sheaves are called *constant sheaves* because all the stalks A_i are isomorphic copies of $\underline{A}$.

This construction can be generalized to the case where $\underline{A}$ is a pseudo-Boolean-valued or Boolean-valued structure. Let I be a topological space, let $O(I)$ be the complete pseudo-Boolean algebra (cpBa) of opens in I (see [15]), and let $\text{Reg}(I)$ be the complete Boolean algebra (cBa) of regular opens in I. A *pseudo-Boolean-valued structure* (resp., *Boolean-valued structure*) $\underline{A}$ *over I* consists of: (1) a set A, (2) for each n-ary atomic relation R in the similarity type other than equality, a map $R : A^n \to O(I)$ ($R : A^n \to \text{Reg}(I)$), and (3) a binary relation $E : A^2 \to O(I)$ ($E : A^2 \to \text{Reg}(I)$) s.t. the pseudo-Boolean (Boolean) truth-values of the statements which say that E is an equivalence relation and that the atomic relations are substitutive w.r.t. E are all dense subsets of I (are all equal to I). The "interior of the closure" map $\text{IC} : O(I) \to \text{Reg}(I)$ canonically associates a Boolean-valued structure over I with each pseudo-Boolean-valued structure over I (its "Booleanization"). The maps $\models R_u : P(u)^n \to O(u) = \Omega(u)$ which constitute the characteristic morphisms for the graph subsheaves associated with the atomic relations show that any value $P(u)$ of a sheaf of structures can be viewed as a pseudo-Boolean-valued (and thus a Boolean-valued) structure (over u). Given a pseudo-Boolean-valued or Boolean-valued structure $\underline{A}$ over I, one can obtain a sheaf of structures in the following manner. Let (I, A) be the same sheaf of underlying sets as was defined before, and if R is any atomic n-ary relation (including E) and $f_1, ..., f_n \in A(u)$, then $A(u) \models R(f_1, ..., f_n)$ if for all $i \in u$, $i \in R(f_1(i), ..., f_n(i))$. The stalks A_i are all set isomorphic to the underlying set of $\underline{A}$ but the structure varies from stalk to stalk.

A *Kripke structure* is a functor $P : \langle I, \leqslant\rangle \to M_\mu$, where $\langle I, \leqslant\rangle$ is a preorder (reflexive and transitive) and where all the homomorphisms $P(i \leqslant i')$ are inclusions. It is convenient to work with the broader notion of a functor $P : \langle I, \leqslant\rangle \to M_\mu$ where the homomorphisms $P(i \leqslant i')$ are not necessarily inclusions. We topologize I by taking as opens the sets which are order-closed upwards (i.e., u is open iff if $i \in u$ and $i \leqslant i'$, then $i' \in u$). The sets $u_i = \{i' \in I\colon i \leqslant i'\}$ form a basis for this topology and the functor P defines a 'presheaf' on this basis ($P(u_i) = P(i)$) which canonically extends to a presheaf of structures P on the space I. Furthermore, every set u_i in this basis is supercompact (i.e., every open cover contains a singleton subcover [15]). By the basis condition, any 'presheaf' defined on a supercompact basis canonically extends to form a sheaf. The original values of the functor $P(i)$ are now recovered as the stalks P_i (indeed, the direct system for P_i is trivial since u_i is the smallest open containing i so $P(i) = P(u_i) \cong P_i$). It is interesting to note that, by this construction, one can obtain a sheaf of structures from an arbitrary presheaf of structures in two different ways. Firstly, a presheaf of structures on I is a functor $P : O(I)^{\mathrm{op}} \to M_\mu$ so the construction yields a sheaf of structures on the space $O(I)^{\mathrm{op}}$ (with the above defined order-closed topology). Secondly, there is a natural preorder defined on any space I, i.e., $i \leqslant i'$ if i is in the closure of the singleton $\{i'\}$. If P is a presheaf of structures on I and $i \leqslant i'$, then since any open containing i also contains i' there is a canonically induced homomorphism $P_i \to P_{i'}$ (by the universality property of direct limits). That yields an appropriate functor on the preorder and then the above construction yields a sheaf of structures on the new space I retopologized with the (richer) order-closed topology. The standard "sheaf of sections" construction allows one to construct a sheaf of structures on the original space I from a presheaf of structures on I, but we will not discuss it here.

The above examples indicate that the notion of a sheaf of relational structures is a sufficiently rich concept of a generalized relational structure to accomodate many other generalized structures used by model theorists such as indexed sets of structures, Boolean-valued structures, and Kripke structures. For more information about sheaves see [7, 8, 12 or 17].

Part II. FORCING IN SHEAVES

If (I, P) is a sheaf of structures, then each atomic n-ary relation $R(x_1, ..., x_n)$ defines a subsheaf $\models R$ of P^n. However, an arbitrary n-ary formula $\varphi(x_1, ..., x_n)$ does *not* similarly define a subsheaf or even a subpresheaf of P^n. The concept of forcing is naturally motivated in the sheaf-theoretical context by this need to appropriately modify the notion of truth or satisfaction so that a subsheaf of P^n can be associated with an arbitrary n-ary formula.

Let (I, P) be a sheaf of structures, let u_0 be an open subset of I, and let u, u', etc. denote (non-empty) open subsets of u_0. We will assume (like Abraham Robinson [16]) that formulas are built up from atomic formulas by using only the connectives of negation, conjunction, and disjunction as well as existential quantification. Let $\varphi(x_1, ..., x_n)$ be an n-ary formula and let $a_1, ..., a_n \in P(u_0)$. The notion of (strong) *forcing* (notation: $\Vdash$) will be specified by defining the characteristic morphism $\Vdash \varphi : P^n \to \Omega$ for the subsheaf $\Vdash \varphi$ of P^n to be associated with the formula. The value $\Vdash \varphi_{u_0}(a_1, ..., a_n)$ of the map

$$\Vdash \varphi_{u_0} : P(u_0)^n \to \Omega(u_0 = O(u_0))$$

will be called the *forcing-value* of the sentence $\varphi(a_1, ..., a_n)$. The intuitive idea is to define the forcing-values of atomic sentences as their truth-values, and then compute the forcing-values of non-atomic sentences in the complete pseudo-Boolean algebra of opens $O(u_0) = \Omega(u_0)$. Let "int" be the interior operator.

(1) $\varphi = R$ atomic:

$$\Vdash \varphi_{u_0}(a_1, ..., a_n) = \models R_{u_0}(a_1, ..., a_n) .$$

(2) $\varphi = \psi \vee \chi$:

$$\Vdash \varphi_{u_0}(a_1, ..., a_n) = \Vdash \psi_{u_0}(a_1, ..., a_n) \cup \Vdash \chi_{u_0}(a_1, ..., a_n) .$$

(3) $\varphi = \psi \wedge \chi$:

$$\Vdash \varphi_{u_0}(a_1, ..., a_n) = \Vdash \psi_{u_0}(a_1, ..., a_n) \cap \Vdash \chi_{u_0}(a_1, ..., a_n) .$$

(4) $\varphi = \neg \psi$:

$$\Vdash \varphi_{u_0}(a_1, ..., a_n) = \text{int}\,(u_0 - \Vdash \psi_{u_0}(a_1, ..., a_n)) .$$

(5) $\varphi = (\exists x)\, \psi$:

$$\Vdash \varphi_{u_0}(a_1, ..., a_n) = \bigcup_{u \subset u_0} \ \bigcup_{a \in P(u)} \Vdash \psi_u(a, p_u^{u_0}(a_1), ...) .$$

For any n-ary formula $\varphi(x_1, \ldots, x_n)$, the above definitions determine the characteristic morphism $\Vdash \varphi : P^n \to \Omega$. Forcing by the structures $P(u_0)$ is defined by

$$P(u_0) \Vdash \varphi(a_1, \ldots, a_n) \quad \text{iff} \quad \Vdash \varphi_{u_0}(a_1, \ldots, a_n) = u_0 .$$

The subsheaf associated with a formula is the subsheaf determined by the forcing-graph of the formula, i.e.,

$$\Vdash \varphi(u_0) = \{\langle a_1, \ldots, a_n\rangle \in P^n(u_0) \colon P(u_0) \Vdash \varphi(a_1, \ldots, a_n)\} .$$

Our results below will focus on the following notion of forcing in stalks: for $i \in u_0$, $P_i \Vdash \varphi(\underline{a_1}, \ldots, \underline{a_n})$ iff $i \in \Vdash \varphi_{u_0}(a_1, \ldots, a_n)$. Then we have

$$\Vdash \varphi_{u_0}(a_1, \ldots, a_n) = \{i \in u_0 \colon P_i \Vdash \varphi(\underline{a_1}, \ldots, \underline{a_n})\} .$$

Abraham Robinson's concept of (infinite) forcing [16], defined in terms of a *class* of structures Σ, generalizes the concept of forcing in Kripke structures [6] which only involves a *set* of structures. When Σ is a *set* $\{P_i\}_{i \in I}$ indexed by I, then a partial ordering can be defined on I (i.e., $i \leqslant i'$ iff P_i is a substructure of $P_{i'}$) which immediately yields a Kripke structure. Let (I, P) be the sheaf of structures constructed (as above) from the Kripke structure. Then, for sentences expressed using our basic notation, Robinson-forcing by the structure P_i, Kripke structure forcing by the point i, sheaf forcing by the stalk P_i, and sheaf forcing by the value $P(u_i)$ all *agree.*

For $u \in O(I)$, the unary operation of pseudo-complementation is defined by $-u = \operatorname{int}(I - u)$ (where the binary operation also denoted by "$-$" is set difference). The notion of *weak-forcing* or *wforcing* (notation: $\Vdash^*$) is determined by the following definition of the wforcing-values:

$$\Vdash^* \varphi_{u_0}(a_1, \ldots, a_n) = --(\Vdash \varphi_{u_0}(a_1, \ldots, a_n)) \cap u_0 ,$$

where $a_1, \ldots, a_n \in P(u_0)$. The wforcing-value $\Vdash^* \varphi_{u_0}(a_1, \ldots, a_n)$ is a regular open subset of the subspace u_0. For $\varphi = R$ atomic, the wforcing-value is the 'regularization' (relative to u_0) of the truth-value $\vDash R_{u_0}(a_1, \ldots, a_n)$, and for non-atomic sentences the wforcing-values are obtained by computing in the complete Boolean algebra $\operatorname{Reg}(u_0)$ of regular open subsets of the subspace u_0. In explicit terms, the following relationships hold (and they are an alternative means of defining the wforcing-values):

(1) $\varphi = R$:

$$\Vdash^* \varphi_{u_0}(a_1, \ldots, a_n) = --[\vDash R_{u_0}(a_1, \ldots, a_n)] \cap u_0 .$$

(2) $\varphi = \psi \vee \chi$:

$$\Vdash^* \varphi_{u_0}(\ldots) = --[\Vdash^* \psi_{u_0}(\ldots) \cup \Vdash^* \chi_{u_0}(\ldots)] \cap u_0 \, .$$

(3) $\varphi = \psi \wedge \chi$:

$$\Vdash^* \varphi_{u_0}(\ldots) = \Vdash^* \psi_{u_0}(\ldots) \cap \Vdash^* \chi_{u_0}(\ldots) \, .$$

(4) $\varphi = \neg \psi$:

$$\Vdash^* \varphi_{u_0}(\ldots) = \operatorname{int}(u_0 - \Vdash^* \psi_{u_0}(\ldots)) \, .$$

(5) $\varphi = (\exists x)\, \psi$:

$$\Vdash^* \varphi_{u_0}(a_1, \ldots, a_n) =$$

$$= --\left[\bigcup_{u \subset u_0} \bigcup_{a \in P(u)} \Vdash^* \psi_u(a, p_u^{u_0}(a_1), \ldots, p_u^{u_0}(a_n))\right] \cap u_0 \, .$$

As with (strong) forcing, the wforcing-values define the characteristic morphism $\Vdash^* \varphi : P^n \to \Omega$ which in turn determines the subsheaf:

$$\Vdash^* \varphi(u_0) = \{\langle a_1, \ldots, a_n \rangle \in P^n(u_0) \colon \Vdash^* \varphi_{u_0}(a_1, \ldots, a_n) = u_0\} \, .$$

Then wforcing by a value of the sheaf $P(u_0)$ is defined by:

$$P(u_0) \Vdash^* \varphi(a_1, \ldots, a_n) \quad \text{iff} \quad \Vdash^* \varphi_{u_0}(a_1, \ldots, a_n) = u_0 \, .$$

And wforcing by a stalk P_i is defined by:

$$P_i \Vdash^* \varphi(\underline{a_1}, \ldots, \underline{a_n}) \quad \text{iff} \quad i \in \Vdash^* \varphi_{u_0}(a_1, \ldots, a_n) \, .$$

Then

$$\Vdash^* \varphi_{u_0}(a_1, \ldots, a_n) = \{i \in u_0 \colon P_i \Vdash^* \varphi(\underline{a_1}, \ldots, \underline{a_n})\} \, .$$

As usual, a sentence is wforced iff its double negation is forced.

Let $O(I)$ be the complete pseudo-Boolean algebra (cpBa) of open subsets of I. A filter F (i.e., a proper filter $F \neq O(I)$) is *prime* if $u \cup u' \in F$ implies $u \in F$ or $u' \in F$ for any $u, u' \in O(I)$. Let $\Pr(I)$ be the set of (proper) prime filters of $O(I)$ with the following topology: the basic opens are the sets $X(u) = \{F \in \Pr(I) \colon u \in F\}$ for $u \in O(I)$. $\Pr(I)$ is the *prime spectrum* of $O(I)$ (see [15]). The map $\eta : I \to \Pr(I)$ which takes i to $F_i = \{u \in O(I) \colon i \in u\}$ (the principal prime filter generated by i), is a continuous map since $\eta^{-1}(X(u)) = u$. Given a sheaf of structures P on I, we define the (direct image) sheaf of structures P^0 on $\Pr(I)$ by the condition: $P^0(X) = P(\eta^{-1}(X))$ for X open in $\Pr(I)$. The sheaf of structures $(\Pr(I), P^0)$ will be called the *prime sheaf of* (I, P), and its stalks will be called the *prime stalks of* (I, P). A filter $F \in \Pr(I)$ is *maximal* if for any open u, $u \in F$ or $-u \in F$. The stalks of P^0 at maximal filters will be called the *max-stalks of* (I, P). The operation which takes a sheaf of structures (I, P) to its prime sheaf $(\Pr(I), P^0)$ easily extends to an endofunctor on the category of sheaves of structures (of a given type) and their morphisms. This functor will be called the *prime functor.*

The following Łoš-type theorem establishes the relationship between wforcing in a given sheaf and in its prime sheaf. Assume that (I, P) is a sheaf of structures, $\varphi(x_1, ..., x_n)$ is an n-ary formula, $a_1, ..., a_n$ are in $P(u_0)$, and $F \in X(u_0)$ (i.e., $u_0 \in F$).

Prime Stalk Theorem

$$P_F^0 \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n}) \quad \text{iff} \quad \{i \in u_0 : P_i \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\} \in F.$$

Proof. The proof is by induction over the complexity of φ and we will be using the following facts about basic opens in $\Pr(I)$ (see [15]);

(1) $--X(u) = X(--u)$,

(2) $X(u_1) \cup X(u_2) = X(u_1 \cup u_2)$,

(3) $X(u_1) \cap X(u_2) = X(u_1 \cap u_2)$,

(4) $-X(u) = X(-u)$,

(5) $--\bigcup_r X(u_r) = X(--\bigcup_r u_r)$,

where $\{u_r\}$ is any set of open subsets of I. We prove the theorem by showing that

$$\{F \in X(u_0) : P_F^0 \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\} = X(\{i \in u_0 : P_i \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\}).$$

That is, we show that the wforcing-value of the sentence in the prime sheaf is the basic open determined by the wforcing-value of the sentence in the original sheaf.

(1) $\varphi = R$ atomic:

$$\{i \in u_0 : P_i \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\} = --[\{i \in u_0 : P_i \vDash \varphi(...)\}] \cap u_0$$

and, similarly in $\Pr(I)$,

$$\{F \in X(u_0) : P_F^0 \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\} = --[\{F \in X(u_0) : P_F^0 \vDash \varphi(...)\}] \cap X(u_0)$$

(and since the truth-value of an atomic sentence in the prime sheaf is the basic open defined by the truth-value in the original sheaf, we may continue)

$$= - -X(\{i \in u_0: P_i \vDash \varphi(...)\}) \cap X(u_0)$$

$$= X(- -\{i \in u_0: P_i \vDash \varphi(...)\} \cap u_0)$$

$$= X(\{i \in u_0: P_i \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\}) .$$

(2) $\varphi = \psi \vee \chi$:

$$\{F \in X(u_0): P_F^0 \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\} =$$

$$= - - [\{F \in X(u_0): P_F^0 \Vdash^* \psi(...)\}$$

$$\cup \{F \in X(u_0): P_F^0 \Vdash^* \chi(...)\}] \cap X(u_0)$$

(so by induction hypothesis)

$$= - -[X(\{i \in u_0: P_i \Vdash^* \psi(...)\}) \cup X(\{i \in u_0: P_i \Vdash^* \chi(...)\})] \cap X(u_0)$$

$$= X(- -[\{i \in u_0: P_i \Vdash^* \psi(...)\} \cup \{i \in u_0: P_i \Vdash^* \chi(...)\}] \cap u_0)$$

$$= X(\{i \in u_0: P_i \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\}) .$$

(3) $\varphi = \psi \wedge \chi$: similar to (2).

(4) $\varphi = \neg \psi$:

$$\{F \in X(u_0): P_F^0 \Vdash^* \varphi(...)\} = \text{int } [X(u_0) - \{F: P_F^0 \Vdash^* \psi(...)\}]$$

$$= - [\{F \in X(u_0): P_F^0 \Vdash^* \psi(...)\}] \cap X(u_0)$$

(so by induction hypothesis)

$$= - [X(\{i \in u_0: P_i \Vdash^* \psi(...)\})] \cap X(u_0)$$

$$= X(-[\{i: P_i \Vdash^* \psi(...)\}] \cap u_0)$$

$$= X(\{i \in u_0: P_i \Vdash^* \varphi(...)\}) .$$

(5) $\varphi = (\exists x) \psi$: Recall that

$$\{i \in u_0: P_i \Vdash^* (\exists x) \psi(x, \underline{a_1}, ..., \underline{a_n})\} =$$

$$= - - \left[\bigcup_{u \subset u_0} \bigcup_{a \in P(u)} \{i \in u: P_i \Vdash^* \psi(\underline{a}, \underline{a_1}, ..., \underline{a_n})\} \right] \cap u_0$$

and similarly in $\Pr(I)$, where the first union may be taken over basic opens $X(u) \subset X(u_0)$. Thus

$$\{F \in X(u_0)\colon P_F^0 \Vdash^* (\exists x)\, \psi(x, \underline{a_1}, \ldots, \underline{a_n})\} =$$

$$= --\Big[\bigcup_{X(u) \subset X(u_0)} \ \bigcup_{a \in P^0(X(u))} \{F \in X(u)\colon P_F^0 \Vdash^* \psi(\underline{a}, \underline{a_1}, \ldots, a_n)\}\Big] \cap X(u_0)$$

$$= --\Big[\bigcup_{u \subset u_0} \ \bigcup_{a \in P(u)} X(\{i \in u\colon P_i \Vdash^* \psi(\underline{a}, \underline{a_1}, \ldots, \underline{a_n})\})\Big] \cap X(u_0)$$

$$= X(--\Big[\bigcup_{u \subset u_0} \ \bigcup_{a \in P(u)} \{i \in u\colon P_i \Vdash^* \psi(\underline{a}, \underline{a_1}, \ldots, \underline{a_n})\}\Big] \cap u_0)$$

$$= X(\{i \in u_0\colon P_i \Vdash^* (\exists x)\, \psi(x, \underline{a_1}, \ldots, \underline{a_n})\})\,.$$

This result would not hold if weak-forcing (wforcing) was replaced by forcing because, in general, $\bigcup_r X(u_r)$ is only a dense subset of $X(\bigcup_r u_r)$.

Part III. ULTRASTALKS

Our main theorem, which shows that max-stalks are generalized ultraproducts, is a sharpening of the prime stalk theorem that replaces weak-forcing by truth on the left hand side when F is a maximal filter. We will first define a new sheaf on a homeomorphic copy of the subspace of maximal filters which will have stalks isomorphic to the max-stalks. Let $\mathrm{Ult}(I)$ be the Stone space of the complete Boolean algebra $\mathrm{Reg}(I)$ of regular open (regopen) subsets of the space I. $\mathrm{Ult}(I)$ is homeomorphic with the max spectrum of the cpBa $O(I)$, i.e., with the subspace of maximal filters in $\mathrm{Pr}(I)$. If F is an ultrafilter of $\mathrm{Reg}(I)$, the corresponding maximal filter is $F^0 = \{u \in O(I)\colon -u \notin F\}$. If F is a maximal filter of $O(I)$, the corresponding ultrafilter is $F^* = \{--u \in \mathrm{Reg}(I)\colon u \in F\}$.

Given a sheaf of structures (I, P), we will construct another sheaf of structures $(\mathrm{Ult}(I), P^*)$ called the *ultrasheaf of* (I, P). The stalks of the ultrasheaf of (I, P) will be called the *ultrastalks of* (I, P). If u_0 is a regopen subset of I and $X(u_0)$ is the basic open of $\mathrm{Ult}(I)$ determined by u_0 (i.e., $X(u_0) = \{F \in \mathrm{Ult}(I)\colon u_0 \in F\}$), then we define P^* on the basic opens in the following manner:

$$P^*(X(u_0)) = \varinjlim_{u \subset_{\mathrm{d}} u_0} P(u)\,,$$

where the direct limit is over the structures $P(u)$ for u an open dense subset of u_0 (notation: $u \subset_{\mathrm{d}} u_0$). If u_0' is another regopen s.t. $X(u_0') \subset X(u_0)$, then $u_0' \subset u_0$, and if $u \subset_{\mathrm{d}} u_0$, then $u \cap u_0' \subset_{\mathrm{d}} u_0'$. The homomorphisms

$$\{P(u) \xrightarrow{\text{rest.}} P(u \cap u_0') \xrightarrow{\text{can.}} P^*(X(u_0'))\colon u \subset_{\mathrm{d}} u_0\}$$

commute with the homomorphisms in the direct system for $P^*(X(u_0))$, so by the universality property for direct limits there is a unique homomorphism $P^*(X(u_0)) \to P^*(X(u_0'))$ s.t. for any $u \subset_{\mathrm{d}} u_0$ the following diagram commutes:

$$\begin{array}{ccc} P(u) & \xrightarrow{\text{rest.}} & P(u \cap u_0') \\ \Big\downarrow{\scriptstyle\text{can.}} & & \Big\downarrow{\scriptstyle\text{can.}} \\ P^*(X(u_0)) & \longrightarrow & P^*(X(u_0'))\,. \end{array}$$

By taking these as the restriction homomorphisms, we have a 'presheaf' of structures on the basic opens of $\mathrm{Ult}(I)$. As indicated in Part I, such a 'presheaf' on basic opens can be canonically extended to a presheaf of structures defined on all the opens of $\mathrm{Ult}(I)$. The long and tedious proof that this presheaf is, in fact, a sheaf [the ultrasheaf of (I, P)] is relegated to an appendix.

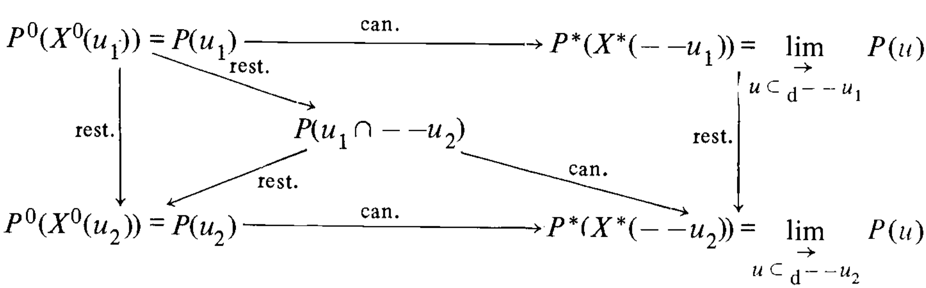
The operation which takes a sheaf of structures (I, P) to its ultrasheaf $(\mathrm{Ult}(I), P^*)$ is functorial only on sheaf of structures morphisms (f, Θ), where f is an open continuous map. Given an arbitrary sheaf of structures (I, P), we can now obtain the prime sheaf $(\mathrm{Pr}(I), P^0)$ and the ultrasheaf $(\mathrm{Ult}(I), P^*)$. There is a morphism $(f, \Theta) : (\mathrm{Ult}(I), P^*) \Rightarrow (\mathrm{Pr}(I), P^0)$, where for any ultrafilter F, $f(F) = F^0$ the corresponding maximal filter. Let us distinguish the opens of the prime spectrum and the Stone space with the superscripts of "0" and "*" respectively. Then $f^{-1}(X^0(u)) = X^*(--u)$, so $f : \mathrm{Ult}(I) \to \mathrm{Pr}(I)$ is a continuous function. For basic opens, let

$$\Theta_{X^0(u)} : P^0(X^0(u)) \to P^*(f^{-1}(X^0(u)))$$

be the canonical homomorphism

$$P^0(X^0(u)) = P(u) \xrightarrow{\text{can.}} \varinjlim_{u' \subset_{\mathrm{d}} --u} P(u')$$
$$= P^*(X^*(--u)) = P^*(f^{-1}(X^0(u))) ,$$

which we have since u is an open dense subset of $--u$. For other opens Y^0 in $\mathrm{Pr}(I)$, $P^0(Y^0) = P(\eta^{-1}(Y^0)) = P^0(X^0(\eta^{-1}(Y^0)))$ and $f^{-1}(Y^0) = f^{-1}(X^0(\eta^{-1}(Y^0)))$, so let $\Theta_{Y^0} = \Theta_{X^0(\eta^{-1}(Y^0))}$ (where $\eta : I \to \mathrm{Pr}(I)$ is the continuous map taking i to the principal prime filter F_i generated by i). To check that (f, Θ) is a morphism, we must check that the following square commutes for any $u_2 \subset u_1$. The square commutes by the commutativity of the constituent triangles.



For each ultrafilter F, the morphism induces a homomorphism $\Theta_F^\# : P^0_{f(F)} \to P_F^*$ which is an isomorphism. Thus, a generalized ultraproduct may be constructed as a max-stalk, i.e., as a 'one-stage' direct limit $P^0_{F^0} = \varinjlim_{u \in F^0} P(u)$, or as an ultrastalk, i.e., as a 'two-stage' direct limit $P_F^* = \varinjlim_{u_0 \in F} \varinjlim_{u \subset_d u_0} P(u)$. Henceforth we will use ultrastalks and max-stalks interchangeably.

Let (I, P) be an arbitrary sheaf of structures, let $F \in \mathrm{Ult}(I)$, and let $F^0 = \{u \in O(I)\colon --u \in F\}$ be the corresponding maximal filter. Let $\varphi(x_1, ..., x_n)$ be an n-ary formula and let $a_1, ..., a_n \in P(u_0)$, where $u_0 \in F^0$ (i.e., $--u_0 \in F$).

Ultrastalk Theorem

$$P^0_{F^0} \vDash \varphi(\underline{a_1}, ..., \underline{a_n}) \quad \text{iff} \quad \{i \in u_0\colon P_i \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\} \in F^0,$$

i.e.,

$$P_F^* \vDash \varphi(\underline{a_1}, ..., \underline{a_n}) \quad \text{iff} \quad --\{i \in u_0\colon P_i \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\} \in F.$$

Proof. By the prime stalk theorem, it suffices to prove that $P^0_{F^0} \vDash \varphi(a_1, ..., a_n)$ iff $P^0_{F^0} \Vdash^* \varphi(a_1, ..., a_n)$ (i.e., that truth = wforcing in max-stalks) which we will do by induction over the complexity of φ.

(1) $\varphi = R$ atomic: Let

$$u_1 = \{i \in u_0\colon P_i \vDash R(\underline{a_1}, ..., \underline{a_n})\} = \bigcup \{u \subset u_0\colon P(u) \vDash R(p_u^{u_0}(a_1), ..., p_u^{u_0}(a_n))\}$$

so by condition (3) in the definition of a sheaf of structures,

$$P(u_1) \vDash R(p_{u_1}^{u_0}(a_1), ..., p_{u_1}^{u_0}(a_n)).$$

Hence

$$\begin{aligned} P^0_{F^0} \Vdash^* R(\underline{a_1}, ..., \underline{a_n}) \quad &\text{iff} \quad \{i \in u_0\colon P_i \Vdash^* R(\underline{a_1}, ..., \underline{a_n})\} \in F^0 \\ &\text{iff} \; --u_1 \cap u_0 \in F^0 \quad \text{iff} \quad u_1 \in F^0 \\ &\text{iff} \quad P^0_{F^0} \vDash R(\underline{a_1}, ..., \underline{a_n}). \end{aligned}$$

(2) $\varphi = \psi \vee \chi$:

$$\begin{aligned} P^0_{F^0} \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n}) \quad &\text{iff} \quad \{i \in u_0\colon P_i \Vdash^* \psi \vee \chi(...)\} \in F^0 \\ &\text{iff} \quad \{i \in u_0\colon P_i \Vdash^* \psi(...)\} \in F^0 \\ &\quad \text{or } \{i \in u_0\colon P_i \Vdash^* \chi(...)\} \in F^0 \end{aligned}$$

$$\text{iff} \quad P^0_{F^0} \Vdash^* \psi(...) \text{ or } P^0_{F^0} \Vdash^* \chi(...)$$

$$\text{iff} \quad P^0_{F^0} \models \psi(...) \text{ or } P^0_{F^0} \models \chi(...)$$

$$\text{iff} \quad P^0_{F^0} \models \varphi(\underline{a_1}, ..., \underline{a_n}) .$$

(3) $\varphi = \psi \wedge \chi$: Similar to 2.

(4) $\varphi = \neg \psi$:

$$P^0_{F^0} \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n}) \quad \text{iff} \quad \{i \in u_0 \colon P_i \Vdash^* \neg \psi(...)\} \in F^0$$

$$\text{iff} \quad \{i \in u_0 \colon P_i \Vdash^* \psi(...)\} \notin F^0$$

$$\text{iff} \quad \text{not } P^0_{F^0} \Vdash^* \psi(...)$$

$$\text{iff} \quad \text{not } P^0_{F^0} \models \psi(...)$$

$$\text{iff} \quad P^0_{F^0} \models \varphi(\underline{a_1}, ..., \underline{a_n}) .$$

(5) $\varphi = (\exists x)\, \psi$:

$$P^0_{F^0} \Vdash^* (\exists x)\, \psi(x, \underline{a_1}, ..., \underline{a_n})$$

$$\text{iff} \quad \{i \in u_0 \colon P_i \Vdash^* (\exists x)\, \psi(x, ...)\} \in F^0$$

$$\text{iff} \quad \bigcup_{u \subset u_0} \bigcup_{a \in P(u)} \{i \in u \colon P_i \Vdash^* \psi(\underline{a}, ...)\} = u_1 \in F^0 .$$

The wforcing-values in this union form an open cover of u_1. We claim that given any open cover $\{u_r\}_{r \in A}$ of an open u, there is a *disjoint* open cover $\{u_s\}_{s \in B}$ which covers a dense subset of u s.t. each u_s is contained in some u_r. Consider the class of all disjoint open covers $\{u_s\}_{s \in B}$ of some subset of u s.t. for each s there is an r s.t. $u_s \subset u_r$. Order the class by inclusion (not refinement). The class is non-empty and the union of each chain is in the class so, by Zorn's lemma, there is a maximal element which clearly must cover a dense subset of u (thus proving the claim). Applying this topological fact to our problem, there is a disjoint open cover $\{u_s\}_{s \in B}$ s.t. $\bigcup_{s \in B} u_s = u_2 \subset_{\mathrm{d}} u_1$, and for each s, there is an a in some $P(u)$ s.t.

$$u_s \subset \{i \in u \colon P_i \Vdash^* \psi(\underline{a}, \underline{a_1}, ..., \underline{a_n})\} .$$

Let $a_s \in P(u_s)$ be the restriction of such an element a so that for any

$i \in u_s$, $P_i \Vdash^* \psi(a_s, a_1, ..., a_n)$. Since the $\{u_s\}$ are disjoint, the individuals $\{a_s\}$ agree on intersections, so they patch together to yield an element $a_0 \in P(u_2)$ (here is where we made crucial use of condition (2), the patching property of sheaves) such that for all $i \in u_2$, $P_i \Vdash^* \psi(\underline{a_0}, \underline{a_1}, ..., \underline{a_n})$. Continuing the proof:

$$P^0_{F^0} \Vdash^* (\exists x)\, \psi(x, \underline{a_1}, ..., \underline{a_n}) \quad \text{iff} \quad u_1 \in F^0$$

iff $\exists u_2$ and $\exists a_0 \in P(u_2)$ s.t.

$$\{i \in u_2 \colon P_i \Vdash^* \psi(\underline{a_0}, \underline{a_1}, ..., a_n)\} \in F^0$$

iff $\exists u_2$ and $\exists a_0 \in P(u_2)$ s.t.

$$P^0_{F^0} \Vdash^* \psi(\underline{a_0}, \underline{a_1}, ..., \underline{a_n})$$

iff $\exists u_2$ and $\exists a_0 \in P(u_2)$ s.t.

$$P^0_{F^0} \models \psi(\underline{a_0}, \underline{a_1}, ..., \underline{a_n}) \quad \text{iff} \quad P^0_{F^0} \models (\exists x)\, \psi(x, \underline{a_1}, ..., \underline{a_n})\,.$$

Corollary on Forcing

$$P^0_{F^0} \models \varphi(\underline{a_1}, ..., \underline{a_n}) \quad \textit{iff} \quad \{i \in u_0 \colon P_i \Vdash \varphi(\underline{a_1}, ..., \underline{a_n})\} \in F^0\,.$$

Proof. By definition,

$$\{i \in u_0 \colon P_i \Vdash^* \varphi(\underline{a_1}, ..., \underline{a_n})\} = --\{i \in u_0 \colon P_i \Vdash \varphi(\underline{a_1}, ..., \underline{a_n})\} \cap u_0.$$

Now $u_0 \in F^0$, and since F^0 is maximal, for any open u, $u \in F^0$ iff $--u \in F^0$. Thus the forcing-value is in F^0 iff the wforcing-value is in F^0.

The ultrastalk (or max-stalk) construction generalizes the classical ultraproduct construction and the more recent Boolean ultrapower construction (see [18] and [14] – one might note that what Vopěnka calls a "sheaf" is not a sheaf but a Boolean-valued relation). Classical ultraproducts are the ultrastalks of discrete sheaves and Boolean ultrapowers are the ultrastalks of constant sheaves. In the latter case, let (I, A) be the constant sheaf of structures constructed from the structure $\underline{A}$ and the space I (i.e.,

$$A(u) = \{u \xrightarrow{f} A \colon f \text{ locally constant}\})\,.$$

Then the elements of A^* (Ult(I)) are in one-to-one correspondence with the functions $f \colon A \to \text{Reg}(I)$ which partition the cBa. The ultrastalk A^*_F is the Boolean ultrapower of $\underline{A}$ w.r.t. the ultrafilter F in the cBa Reg(I).

Part IV. EXTENSIONS AND APPLICATIONS

A *presheaf of (universal) algebras* of a given type on the space I is a functor from $O(I)^{\text{op}}$ into the category of algebras of that type and homomorphisms. A *sheaf of algebras* is a presheaf of algebras which satisfies conditions (1) and (2) of Part I. Thus a sheaf of algebras of a given type is a sheaf of sets (I, P) together with a morphism $P^n \to P$ for each n-ary atomic operation in the type. In a sheaf of algebras (I, P), if an n-ary atomic operation f is construed as an $(n + 1)$-ary atomic relation $f(x_1, ..., x_n) = x_{n+1}$, then that atomic relation automatically satisfies condition (3) of Part I, i.e., the graph of the operation constitutes a subsheaf of the $(n + 1)^{\text{st}}$ power sheaf P^{n+1}. Hence a sheaf of algebras 'is' a sheaf of relational structures so our results apply to all the sheaves of algebras (e.g., groups, rings, fields, etc.) which are so ubiquitous in modern mathematics. At the end of this section, we will outline an application involving sheaves of rings.

It is clear (from the last paragraph) that we could have begun with the (seemingly) broader notion of a relational structure as a set with atomic relations as well as atomic operations defined on it (where individual constants are 0-ary operations). Then a *presheaf of relational structures* of a given type on I is a functor from $O(I)^{\text{op}}$ into the category of structures of that type and homomorphisms (where the latter preserve both atomic relations and operations). A *sheaf of relational structures* of a given type is a presheaf of structures of the given type such that conditions (1) and (2) of Part I are satisfied and such that the atomic relations satisfy condition (3). Thus a sheaf of structures of a given type is a sheaf of sets (I, P) together with a morphism $P^n \to P$ for each n-ary atomic operation in the type and together with a subsheaf of P^n for each n-ary atomic relation in the type. Since the graph of an n-ary operation automatically defines a subsheaf of P^{n+1} (i.e., satisfies condition (3)), this definition of a sheaf of structures is equivalent to the one given in Part I.

It should be noted that we have *not* excluded the empty relational structures, even though that is customary in model theory. If the similarity type μ is without individual constants, then the empty structure 'of type μ' is a member of M_μ, the category of structures of that type and homomorphisms. The terminal object in the category Sh(I) of sheaves of sets on I and morphisms (natural transformations) is the empty product or 0^{th} power $(I, 1)$ where $1(u) = 1 = \{0\}$ for any open $u \subset I$.

A sheaf of sets (I, P) is said to be *non-empty* if there exists a morphism from the terminal sheaf 1 to P (i.e., if $P(I) \neq \emptyset$). If one wishes to exclude the empty structure, then one may add to the above definition of a sheaf of structures the stipulation that the underlying sheaf of sets (I, P) be non-empty (or simply exclude the empty structure from the category of structures of that type and homomorphisms). The stipulation would always be satisfied if the type included constants (i.e., 0-ary operations) since a constant is given by a morphism $P^0 = 1 \to P$. However, it is worth noting that the above results do not depend upon that customary exclusion. As Sabah Fakir pointed out to me, the usual 'quotient of direct product' construction and the direct limit or sheaf theoretic construction of classical ultraproducts yield isomorphic results only when empty factors are excluded. When some of the structures in the indexed set of structures are empty, the usual construction collapses to the empty structure and violates the Łoš theorem, whereas the structure yielded by the direct limit construction continues to satisfy the Łoš theorem. Accordingly, the Łoš theorem specifies when a classical ultrastalk is non-empty. That is, for discrete I and an ultrafilter F,

$$P_F^0 \cong P_F^* \models (\exists x)(x = x) \quad \text{iff} \quad \{i \in I\colon P_i \models (\exists x)(x = x)\} \in F\,.$$

The Feferman–Vaught (F–V) generalized product construction takes as part of its initial data an indexed set of structures (see [4]), i.e., a discrete sheaf of structures. As with ultraproducts, the generalized product construction and theorem can be generalized to arbitrary sheaves of structures (as initial data). Stephen Comer [2] has extended the F–V theorem to the case where the initial datum is a sheaf of structures on a Boolean space with clopen truth-values – where the classical case is obtained by taking (what we would call) a classical ultrasheaf, i.e., the ultrasheaf (= prime sheaf) of a discrete sheaf. However, if we begin with an arbitrary sheaf of structures, then the ultrasheaf construction yields a sheaf of structures on a Boolean space. By the ultrastalk theorem, the truth-values are all clopen (i.e., the wforcing-values are regopen, but wforcing = truth in ultrastalks and regopen = clopen in an extremally disconnected Boolean space). In this manner, the general case (arbitrary initial sheaf) is reduced to Comer's theorem. Comer [2] and Angus Macintyre [13] have obtained decidability and model-completeness results by using certain sheaves of structures on Boolean spaces with clopen truth-values that are constructed with different techniques.

The ultrasheaf construction is functorial on morphisms (f, Θ) with f

open. If (I, P) and (J, Q) are arbitrary sheaves of structures and $(f, \Theta) : (I, P) \Rightarrow (J, Q)$ is a sheaf of *sets* morphism (i.e., the maps $\Theta_v : Q(v) \to P(f^{-1}(v))$ are not assumed to be homomorphisms) with f open (as well as continuous), then there is a canonically defined morphism

$$(\mathrm{Ult}(f), \Theta^*) : (\mathrm{Ult}(I), P^*) \Rightarrow (\mathrm{Ult}(J), Q^*)$$

between the respective ultrasheaves. This morphism induces for each $F \in \mathrm{Ult}(I)$, an *ultrastalk map*

$$(\Theta^*)^{\#}_F : Q^*_{\mathrm{Ult}(f)(F)} \to P^*_F .$$

There is an ultrastalk map theorem which is a Łoš-type theorem that characterizes the formulas preserved by an ultrastalk map $(\Theta^*)^{\#}_F$ in terms of what sets are in the ultrafilter F.

We will illustrate this Łoš-type mapping theorem by considering the classical case where I and J are discrete spaces (so the ultrastalks are ultraproducts). Let $\varphi(x_1, ..., x_n)$ be an n-ary formula, let $g_1, ..., g_n \in Q(J) \cong \Pi_{j\in J} Q_j$, and let $h_k = \Theta_J(g_k)$ so that $(\Theta^*)^{\#}_F(\underline{g_k}) = \underline{h_k}$ for $k = 1, ..., n$. Then the following equivalence holds:

if $Q^*_{\mathrm{Ult}(f)(F)} \vDash \varphi(\underline{g_1}, ..., \underline{g_n})$, then $P^*_F \vDash \varphi(\underline{h_1}, ..., \underline{h_n})$ if and only if

$$\{i \in I\text{: if } Q_{f(i)} \vDash \varphi(\underline{g_1}, ..., \underline{g_n}), \text{ then } P_i \vDash \varphi(\underline{h_1}, ..., \underline{h_n})\} \in F .$$

Let us say that a map *truth-preserves* a formula $\varphi(x_1, ..., x_n)$ if the map preserves each instance of truth. Then one can easily prove the following 'classical' result;

Ultraproduct Map Theorem

$$Q^*_{\mathrm{Ult}(f)(F)} \xrightarrow{(\Theta^*)^{\#}_F} P^*_F \textit{ truth-preserves } \varphi(x_1, ..., x_n)$$

$$\text{iff } \{i \in I\text{: } Q_{f(i)} \xrightarrow{\Theta^{\#}_i} P_i \textit{ truth-preserves } \varphi(x_1, ..., x_n) \ \in F\} .$$

The usual proof of Frayne's lemma is essentially an implicit application of this result. The ultrastalk map theorem extends this ultraproduct map theorem to the general case (arbitrary sheaves of structures) with the only important change being that the conditions on the original stalk maps $\Theta^{\#}_i : Q_{f(i)} \to P_i$ must be restated in terms of weak-forcing rather than truth.

When a notion of forcing is defined, then it is customary to say that a structure is *generic* if the notions of forcing and truth coincide in that structure. In Abraham Robinson's approach to model theoretic forcing, a notion of (infinite) forcing is defined with respect to a (usually proper) class of classical structures, and then generic structures are to be found within that class. We have defined a forcing notion within each generalized relational structure (i.e., each sheaf of structures) which yields a forcing notion on the set of classical structures that are the stalks of the sheaf. By the ultrastalk theorem, ultrastalks and max-stalks are generic stalks (and an ultrasheaf is generic in the sense that all its stalks are generic). This suggests an alternative tactic to that of searching for generic stalks among the stalks of a given sheaf of structures.

By applying the prime functor, we 'blow up' the given sheaf of structures (I, P) to obtain a new sheaf $(\mathrm{Pr}(I), P^0)$ with certain generic stalks, some of which are 'generic completions or developments' of the original stalks P_i in the following sense. Each original stalk P_i is isomorphically reproduced as the principal prime stalk $P^0_{F_i}$ (where F_i is the principal prime filter generated by i), and the notions of forcing and wforcing (as well as truth) in $P^0_{F_i}$ agree with the corresponding notions in P_i. The wforcing notion in the stalks of $(\mathrm{Pr}(I), P^0)$ is related to wforcing in the stalks of (I, P) by a Łoš-type theorem (i.e., the prime stalk theorem). The new sheaf $(\mathrm{Pr}(I), P^0)$ always has certain stalks which are generic (the max-stalks), and truth in these generic stalks is related to wforcing and forcing in the original sheaf (I, P) by Łoš-type theorems (the ultrastalk theorem and the corollary on forcing). We noted above (in Part I) that there is a preorder defined on the points of any space, and if $i \leqslant i'$, then there is an induced homomorphism $P_i \to P_{i'}$ which preserves the forcing and wforcing of all formulas. In the prime spectrum $\mathrm{Pr}(I)$, which is T_0, this topologically defined partial order on the prime filters as points coincides with the inclusion relation between the prime filters. If we begin at any prime stalk P^0_F (principal or otherwise) and move to larger and larger prime filters F' containing F, then the corresponding prime stalks $P^0_{F'}$ become 'increasing generic' (intuitively speaking) until we arrive 'in the limit' at a max-stalk which is generic. This development of structures is traced by the induced homomorphisms $P^0_F \to P^0_{F'}$ which preserve the forcing and wforcing of all formulas. Furthermore, $P^0_F \Vdash^* \varphi(a_1, ..., a_n)$ iff $P^0_{F'} \models \varphi(a_1, ..., a_n)$ for all generic developments $P^0_{F'}$ of P^0_F, i.e., for all generic $P^0_{F'}$ with $F \subset F'$ (where the data is the same as in the prime stalk theorem).

For an application to sheaves of rings, consider the sheaf (I, C) of rings of germs of real-valued continuous functions on a (completely regular) space I (i.e., $C(u) = \{f\colon u \xrightarrow{f} \mathbf{R}$ is continuous$\}$). It is well-known that when I is discrete, then the residue class fields of the ring $C(I)$ are classical ultraproducts (ultrapowers in fact). In order to generalize that result when I is not discrete, we consider $(\mathrm{Ult}(I), C^*)$, the ultrasheaf of (I, C) [N.B., C^* is not bounded functions but is our notation for the ultrasheaf of C.] The ring of global sections $C^*(\mathrm{Ult}(I)) = \varinjlim_{u \subset_d I} C(u)$ is, by the results of Fine, Gillman, and Lambek [5], the complete or *maximal ring of quotients* $Q(C(I))$ (also denoted as $Q(I)$) of $C(I)$. Then the ultrastalks of C, i.e., the stalks of the ultrasheaf C^*, are precisely the residue class fields of $Q(I) \cong C^*(\mathrm{Ult}(I))$. Hence, the generalized ultraproducts associated with any ring of real-valued continuous functions $C(I)$ are the residue class fields of its maximal ring of quotients. It was not evident that the maximal ring of quotients was involved in the classical case of discrete I because then $C(I) \cong Q(I) \cong C^*(\mathrm{Ult}(I))$.

In more detail, let $\underline{f} \in Q(I) \cong \varinjlim_{u \subset_d I} C(u)$ be the equivalence class of some $f \in C(u)$ for some open u dense in I. If $Z_0(\underline{f}) = \mathrm{int}(Z(f)) = \mathrm{int}(f^{-1}(0))$, then $Z_0(\underline{f})$ is a uniquely determined open, independently of the particular representative f. If M is a maximal ideal in $Q(I)$, then $F_M = \{Z_0(\underline{f})\colon \underline{f} \in M\}$ is a maximal filter in the cpBa of opens of I. Conversely, if F is a maximal filter, then $M_F = \{\underline{f}\colon Z_0(\underline{f}) \in F\}$ is a maximal ideal in $Q(I)$. These operations are inverse to one another, i.e., $F_{(M_F)} = F$ and $M_{(F_M)} = M$. The ring theoretic maximal spectrum of $Q(I)$ is homeomorphic with the subspace of $\mathrm{Pr}(I)$ of maximal filters, which is homeomorphic with $\mathrm{Ult}(I)$. Since $Q(I)$ is regular (in the sense of Von Neumann) all prime ideals are maximal, so in fact the ring theoretic prime spectrum $\mathrm{Spec}(Q(I))$ is homeomorphic with $\mathrm{Ult}(I)$ (see [5, Corollaries 10.17 and 11.12]).

For u an open subset of I, let $E(u)$ denote the continuous real-valued function defined on the dense open subset $u \cup -u$ which has value 0 on u and value 1 on $-u$ (so that $Z_0(\underline{E(u)}) = u$). Then the class $\underline{E(u)} \in Q(I)$ is an idempotent and all idempotents in $Q(I)$ have that form for some (regular) open u. The Boolean algebra of idempotents of $Q(I)$ is complete and isomorphic to $\mathrm{Reg}(I)$, the cBa of regopens of I. Moreover, the ideals M of $Q(I)$ are Z_0-ideals in the sense that $\underline{f} \in M$ iff $\underline{E(Z_0(\underline{f}))} \in M$.

Let M be a maximal ideal of $Q(I)$, let $F = F_M$ be the corresponding maximal filter, and let F^* be the corresponding ultrafilter. It is now easy to verify that $C^*_{F^*} \cong Q(I) \,/\, M$, i.e., that the ultrastalks of (I, C) are the residue class fields of $Q(I)$. If $\underline{f}, \underline{g} \in C^*(\mathrm{Ult}(I)) \cong Q(I)$, then

$$\begin{aligned} C^*_{F^*} \models \underline{f} = \underline{g} \quad & \text{iff} \quad --\mathrm{int}(\{i\colon f(i) = g(i)\}) \in F^* \\ & \text{iff} \quad \mathrm{int}(\{i\colon f(i) - g(i) = 0\}) \in F \\ & \text{iff} \quad Z_0(\underline{f - g}) \in F \\ & \text{iff} \quad \underline{f} - \underline{g} \in M \\ & \text{iff} \quad Q(I)\,/\,M \models \underline{f} = \underline{g}\ , \end{aligned}$$

and similarly for the ring operations. If the usual order relation is included in the structure, then we have

$$\begin{aligned} C^*_{F^*} \models \underline{f} \leqslant \underline{g} \quad & \text{iff} \quad --\mathrm{int}(\{i\colon f(i) \leqslant g(i)\}) \in F^* \\ & \text{iff} \quad \mathrm{int}(\{i\colon f(i) \leqslant g(i)\}) \in F \\ & \text{iff} \quad \mathrm{int}(\{i\colon 0 \leqslant (g-f)(i)\}) \in F \\ & \text{iff} \quad Z_0(\underline{g - f} - |\underline{g - f}|) \in F \\ & \text{iff} \quad \underline{g - f} - |\underline{g - f}| \in M \\ & \text{iff} \quad Q(I)\,/\,M \models 0 \leqslant \underline{g - f} \\ & \text{iff} \quad Q(I)\,/\,M \models \underline{f} \leqslant \underline{g}\ . \end{aligned}$$

Thus the ultrastalks are the residue class fields of $Q(I)$. Let $\varphi(x_1, ..., x_n)$ be an n-ary formula in the language of ordered rings and let $f_1, ..., f_n \in C(u)$ for some open u dense in I. Then the ultrastalk theorem implies the following theorem:

Theorem

$$Q(I)\,/\,M \models \varphi(\underline{f_1}, ..., \underline{f_n}) \quad \textit{iff} \quad E(\{i \in u\colon C_i \Vdash^* \varphi(\underline{f_1}, ..., \underline{f_n})\}) \in M.$$

If a commutative ring R (with unity) is a direct product of fields, then $R \cong Q(R)$ = the maximal ring of quotients of R [10]. Daigneault and Kochen [9] have shown that the residue class fields of R ($\cong Q(R)$) are classical ultraproducts. A commutative ring R is said to be *reduced* (or semi-prime) if it contains no non-zero nilpotents. Reduced commutative rings can be represented as the subrings of direct products of fields. The above theorem can be generalized to any reduced commutative ring R by applying the ultrasheaf construction to the affine scheme of R and then using the results of Banaschewski [1]. The residue class fields of the maximal ring of quotients $Q(R)$ of a reduced commutative ring R are generalized ultraproducts.

Appendix I. The ultrasheaf theorem

Theorem. (Ult(I), P^*) *is a sheaf of relational structures.*

Proof. Let s and s' be subscript variables which range over a fixed index set S that is hereafter unmentioned. To show that P^* is a sheaf of sets, we need to verify the basis condition of Part I. That is, given a basic open $X(u_0)$, a cover of $X(u_0)$ by basic opens $\{X(u_s)\}$, and a set of elements $\{a_s \in P^*(X(u_s))\}$ s.t. for any s and s', a_s and $a_{s'}$ have the same restriction to $P^*(X(u))$ for any basic open $X(u) \subset X(u_s) \cap X(u_{s'})$, then we must show that there exists a unique element $a_0 \in P^*(X(u_0))$ which restricts to the elements $\{a_s\}$. We first show uniqueness, so suppose $a_1, a_2 \in P^*(X(u_0))$ both restrict to all the elements $\{a_s\}$. Now a_1 and a_2 are equivalence classes in the direct limit $P^*(X(u_0)) = \varinjlim_{u \subset_d u_0} P(u)$ so there are $a_1' \in a_1$ and $a_2' \in a_2$ where both $a_1', a_2' \in P(u_0')$ for some $u_0' \subset_d u_0$. Then for each index s, a_1' and a_2' go by the restriction map $P(u_0') \to P(u_0' \cap u_s)$ to some a_{1s} and a_{2s}, respectively, where they are both in the equivalence class $a_s \in P^*(X(u_s))$. Thus there is a $u_s' \subset_d u_s$ (and we may take $u_s' \subset u_0' \cap u_s$) s.t. a_{1s} and a_{2s} restrict to the same element $a_s' \in P(u_s')$ [since a_{1s} and a_{2s} are in the same equivalence class in the direct limit $P^*(X(u_s))$]. Hence we have a set of elements $\{a_s' \in P(u_s')\}$ which agree on the intersections of the $\{u_s'\}$ (since they are all restrictions of a_1' as well as a_2'), so by the patching property of the sheaf P [i.e., condition (2)], there is an element $a' \in P(\bigcup_s u_s')$ which restricts to the $\{a_s'\}$ and by the uniqueness property of the sheaf P [i.e., condition (1)] that element is unique. The set $u_0'' = \bigcup_s u_s'$ is a subset of u_0' and the restrictions of a_1' and a_2' to $P(u_0'')$ in turn restrict to all the $\{a_s'\}$ so by uniqueness in P, a_1' and a_2' have the same restriction (i.e., a') to $P(u_0'')$. By assumption, $X(u_0) = \bigcup_s X(u_s)$ so $\bigcup_s u_s = u_0$ and thus $\bigcup_s u_s \subset_d u_0$. For each s, $u_s' \subset_d u_s$ so $u_0'' = \bigcup_s u_s' \subset_d \bigcup_s u_s \subset_d u_0$, i.e., $u_0'' \subset_d u_0$. Hence $P(u_0'')$ is in the direct system for $P^*(X(u_0))$, and a_1' and a_2' have the same restriction to $P(u_0'')$, so they are in the same equivalence class, i.e., $a_1 = a_2$. Thus we have *uniqueness.*

Given a set of elements $\{a_s \in P^*(X(u_s))\}$ such that for any indices s and s', a_s and $a_{s'}$ have the same restriction to any $P^*(X(u))$ for $X(u) \subset X(u_s) \cap X(u_{s'})$, it remains to show the existence of an element $a_0 \in P^*(X(u_0))$ which restricts to all the elements $\{a_s\}$. Since $X(u_s) \cap X(u_{s'})$ is itself a basic open, we need only consider a set of elements which agree on intersections. Moreover, $X(u_0)$ is compact so there is a finite subcover

$\{X(u_k)\colon k = 1, ..., n\}$. Let a_k be the given element of $P^*(X(u_k))$ for $k = 1, ..., n$. We first show that it is sufficient to show the existence of an $a_0 \in P^*(X(u_0))$ which restricts to the $\{a_k\colon k = 1, ..., n\}$. Given such an a_0, the restriction of a_0 in $P^*(X(u_s))$ and a_s must both have the same restriction to $P^*(X(u_s) \cap X(u_k))$ for $k = 1, ..., n$. But the set $\{X(u_s) \cap X(u_k)\colon k = 1, ..., n\}$ is a cover of the basic open $X(u_s)$ by basic opens so, by the above uniqueness result, the restriction of a_0 in $P^*(X(u_s))$ and a_s are the same element, i.e., a_0 restricts to each a_s. Thus it suffices to show the existence of an appropriate a_0 for finite covers.

We proceed by induction over n, the number of basic opens in the cover. It is trivial for $n = 1$, so suppose we have an n element cover $\{X(u_k)\colon k = 1, ..., n\}$ of $X(u_0)$ and a set $\{a_k \in P^*(X(u_k))\colon k = 1, ..., n\}$ of elements which agree on intersections. The union of the first $n - 1$ sets in the cover is a basic open and those $n - 1$ sets are a cover of it, so by induction hypothesis there is an element $a' \in P^*(\bigcup_{k=1}^{n-1} X(u_k))$, which restricts to the a_k for $k = 1, ..., n - 1$. If a' and a_n have the same restriction to the intersection, i.e., in $P^*(X(u_n) \cap \bigcup_{k=1}^{n-1} X(u_k))$, then the induction step reduces to the case of $n = 2$. But that intersection is a basic open, $\{X(u_n) \cap X(u_k)\colon k = 1, ..., n - 1\}$ is a cover of it by basic opens, and a_n and a' have the same restriction to each set in that cover, so by uniqueness a_n and a' have the same restriction to the intersection.

Thus we only need consider the case $X(u_0) = X(u_1) \cup X(u_2)$, where $a_1 \in P^*(X(u_1))$ and $a_2 \in P^*(X(u_2))$ have the same restriction to $P^*(X(u_1) \cap X(u_2))$. The cover

$$\{X(u_1) \cap X(-u_2), X(-u_1) \cap X(u_2), X(u_1) \cap X(u_2)\}$$

is a disjoint cover of $X(u_0)$ by basic opens and it refines the two element cover. Let ${}_2a_1 \in P^*(X(u_1) \cap X(-u_2))$ be the restriction of a_1, let ${}_1a_2 \in P^*(X(-u_1) \cap X(u_2))$ be the restriction of a_2, and let $a_{12} \in P^*(X(u_1) \cap X(u_2))$ be the restriction of a_1 and a_2. As $X(u_1) \cap X(-u_2) = X(u_1 \cap -u_2)$, there is an open ${}_2u_1 \subset_{\mathrm{d}} u_1 \cap -u_2$ and an element ${}_2a'_1 \in P({}_2u_1)$ such that ${}_2a'_1 \in {}_2a_1$. In a similar manner we have sets ${}_1u_2$ and u_{12} open dense in $-u_1 \cap u_2$ and $u_1 \cap u_2$, respectively, and we have elements ${}_1a'_2 \in P({}_1u_2)$ and $a'_{12} \in P(u_{12})$ such that ${}_1a'_2 \in {}_1a_2$ and $a'_{12} \in a_{12}$. Since ${}_2u_1$, ${}_1u_2$, and u_{12} are disjoint, the elements ${}_2a'_1$, ${}_1a'_2$ and a'_{12} agree on intersections, so by the patching property of P, there is an element $a' \in P(u')$, where $u' = {}_2u_1 \cup {}_1u_2 \cup u_{12}$, which restricts to them. Also

$$u' \subset_{\mathrm{d}} (u_1 \cap -u_2) \cup (-u_1 \cap u_2) \cup (u_1 \cap u_2) \subset_{\mathrm{d}} --(u_1 \cup u_2) = u_0$$

so $P(u')$ is in the direct system for $P^*(X(u_0))$, and hence $a' \in P(u')$ determines an equivalence class $a_0 \in P^*(X(u_0))$. Now a_0 restricts to ${}_2a_1$, ${}_1a_2$, and a_{12} so it remains to show that a_0 restricts to a_1 and a_2. The restriction of a_0 to $P^*(X(u_1))$ and a_1 have the same restriction to $P^*(X(u_1) \cap X(-u_2))$ and to $P^*(X(u_1) \cap X(u_2))$ (that is, ${}_2a_1$ and a_{12}), and

$$\{X(u_1) \cap X(-u_2), X(u_1) \cap X(u_2)\}$$

is a cover of the basic open $X(u_1)$ by basic opens, so by uniqueness again, a_0 restricts to a_1. Similarly, a_0 restricts to a_2. This completes the proof that $(\mathrm{Ult}(I), P^*)$ is a sheaf of sets, i.e., that it satisfies conditions (1) and (2) in the definition of a sheaf of structures (see Part I).

The remaining condition (3) states that for any n-ary atomic relation $R(x_1, ..., x_n)$, the graph subpresheaf of $(P^*)^n$ determined by the relation is in fact a subsheaf. On a basic open $X(u_0)$ the graph subpresheaf has the value

$$\{\langle a_1, ..., a_n\rangle \in P^*(X(u_0))^n \colon P^*(X(u_0)) \models R(a_1, ..., a_n)\} = \varinjlim_{u \subset_{\mathrm{d}} u_0} \models R(u)\,,$$

where $\models R$ is the given graph subsheaf of P^n associated with the atomic relation. But this presheaf on $\mathrm{Ult}(I)$ is just $(\models R)^*$, so by the above proof, it is a sheaf of sets and thus a subsheaf of $(P^*)^n$. Hence, $(\mathrm{Ult}(I), P^*)$ is a sheaf of relational structures – the ultrasheaf of (I, P).

Appendix II. The stalk space approach to sheaves

We have exclusively used the presheaf approach to sheaves which has been generalized in algebraic geometry (sheaves on a Grothendieck topology) and the theory of topoi. However, the horticultural terminology is derived from the original definition of a sheaf as a special type of fiber space. This fiber space or stalk space approach is used, for example, in representation theory.

In the stalk space approach, a *sheaf of sets* is a triple (S, I, p) where S and I are topological spaces and $p : S \to I$ is a local homeomorphism, i.e., for any $s \in S$, there is an open neighborhood $v \ni s$ such that p restricted to v is a homeomorphism onto an open subset $p(v)$ of I. This implies that p is continuous and open. For any $i \in I$, the *stalk* S_i *at* i is the fiber $p^{-1}(i)$. If one wishes to exclude empty stalks, then one should additionally stipulate that p be onto. The map p is called the *projection map*, I is called the *base space*, and S is called the sheaf space or *stalk space* since its underlying set is the (disjoint) union of the stalks.

For $R \subset S$, $(R, I, p \upharpoonright R)$ is a sheaf iff R is open, and in that case $(R, I, p \upharpoonright R)$ is said to be a *subsheaf* of (S, I, p). If (S_1, I, p_1) and (S_2, I, p_2) are sheaves with the same base space I, their *product* is the sheaf $(S_1 \times_I S_2, I, p)$, where

$$S_1 \times_I S_2 = \{\langle s_1, s_2\rangle \in S_1 \times S_2 : p_1(s_1) = p_2(s_2)\}$$

with the restriction of the product topology (terminology: fibered product or pullback) and where $S_1 \times_I S_2 \xrightarrow{p} I$ is defined by $p(\langle s_1, s_2\rangle) = p_1(s_1) = p_2(s_2)$. Finite products of sheaves on I and finite powers of a sheaf on I are similarly defined. The empty product or 0^{th} power of a sheaf on I is the sheaf $(I, I, 1)$ where 1 is the identity map. For sheaves (S_1, I, p_1) and (S_2, I, p_2), a map $f : S_1 \to S_2$ which commutes with the projection maps is a *morphism*

$$(S_1, I, p_1) \xrightarrow{f} (S_2, I, p_2)$$

if f is continuous.

For a sheaf (S, I, p) and any open $u \subset I$, a *section over* u is a continuous map $f : u \to S$ such that $p \circ f$ is the identity on u [N.B., a section is here defined as a *continuous* right-inverse.] Let $\Gamma(u, S)$ be the set of sections over u. If for $u' \subset u$ we take the "restriction map" $\Gamma(u, S) \to \Gamma(u', S)$ to be the restriction map, we obtain a presheaf $\Gamma(\cdot, S) : O(I)^{\text{op}} \to \text{Ens}$. This presheaf clearly satisfies conditions (1) and (2) of Part I so it is a sheaf of sets (as defined in the presheaf approach) called the *sheaf of sections.*

Let (I, P) be a presheaf of sets on I, let $S = \bigcup_{i \in I} P_i$ be the disjoint union of the stalks, and let $p : S \to I$ take each element of P_i to i. For any open $u \subset I$ and any element $a \in P(u)$, let $v_a \subset S$ be the image of the map $u \to S$ which takes $i \in u$ to the equivalence class $\underline{a} \in P_i$. If we topologize S by taking the sets v_a, for opens $u \subset I$ and elements $a \in P(u)$, as a basis, then (S, I, p) is a sheaf of sets (as defined in the stalk space approach). The sheaf of sections $(I, \Gamma(\cdot, S))$ is called the *associated sheaf* (or, sheafification) of the presheaf (I, P). If (I, P) was a sheaf, then and only then it would be isomorphic with its associated sheaf (i.e., $P(u) \cong \Gamma(u, S)$ for all opens $u \subset I$). Conversely, if we begin with a sheaf (S, I, p), construct the sheaf of sections, and then (as above) reconstruct a stalk space sheaf, the resultant sheaf is isomorphic with (S, I, p). Hence, the presheaf approach and the stalk space approach to sheaves are equivalent.

A triple (S, I, p) is a *sheaf of (universal) algebras* of a given type if:

(1) $p : S \to I$ is a local homeomorphism,

(2) each stalk $S_i = p^{-1}(i)$ is an algebra of the given type,

(3) for each n-ary atomic operation f, the induced map from the n^{th} fibered power $S \times_I \ldots \times_I S$ (n times) to S is continuous.

Thus a sheaf of algebras of a given type is a sheaf of sets (S, I, p) together with a morphism from the n^{th} power of (S, I, p) to (S, I, p) for each n-ary atomic operation in the type.

Let us first use the broad definition of a "relational structure" as a set (possibly empty) with atomic relations and operations defined on it (where constants are 0-ary operations). A triple (S, I, p) is a *sheaf of relational structures* of a given type if:

(1) $p : S \to I$ is a local homeomorphism,

(2) each stalk $S_i = p^{-1}(i)$ is a relational structure of the given type,

(3) for each n-ary atomic relation R, its graph
$\{\langle s_1, \ldots, s_n\rangle \in S \times_I \ldots \times_I S \colon S_i \models R(s_1, \ldots, s_n), \text{ where } i = p(s_1) = \ldots = p(s_n)\}$
is an open subset of the n^{th} fibered power $S \times_I \ldots \times_I S$ (n times),

(4) for each n-ary atomic operation f, the induced map from the n^{th} fibered power $S \times_I \ldots \times_I S$ to S is continuous.

Thus a sheaf of relational structures of a given type is a sheaf of (underlying) sets (S, I, p) together with a subsheaf of the n^{th} power of (S, I, p) for each n-ary atomic relation in the type, and together with a morphism from the n^{th} power of (S, I, p) to (S, I, p) for each n-ary atomic operation in the type. This is simply the usual definition of a relational structure with "sheaf of sets", "subsheaf", and "morphism" substituted respectively for "set", "subset", and "function".

In a topological context, the appropriate topology on $2 = \{0, 1\}$ is usually the non-discrete T_0 topology $\{\emptyset, \{1\}, 2\}$. For example, with that topology on 2, the set of open subsets $\Omega(I)$ (or topological power set) of a space I is isomorphic with the set of continuous functions $I \to 2$. Requirement (3) above could be restated as:

(3′) for each n-ary atomic relation R, the characteristic function $\chi_R : S \times_I \dots \times_I S \to 2$ is continuous.

Angus Macintyre [13] has given a definition of a sheaf of relational structures which is similar to the above except that he stipulates an onto projection map and puts the discrete topology on 2 [in his version of requirement (3′)]. An important use of the stalk space approach is the representation of certain structures as structures of global sections $\Gamma(I, S)$ (with the structure inherited from the direct product of the stalks). An onto projection map p guarantees non-empty stalk structures S_i, the existence of a right-inverse to p on I, and the existence of a global section in the ultrasheaf of the sheaf of sections, but it does not guarantee the existence of a global section of p (i.e., a continuous right-inverse to p on I). That is, the structure of global sections $\Gamma(I, S)$ might be empty even with an onto projection map. A sheaf of sets on I, (S, I, p), is said to be *non-empty* if there is a morphism from the empty product or 0^{th} power $(I, I, 1)$ to (S, I, p) [i.e., if (S, I, p) has a global section]. A non-empty sheaf must have an onto projection map. If one desires to work exclusively with non-empty relational structures, then one must require that the underlying sheaf of sets be non-empty. This remark is germane only for similarity types without individual constants because a 0-ary operation is interpreted as a morphism $(I, I, 1) \to (S, I, p)$. A sheaf of structures in a type with constants would automatically have a non-empty underlying sheaf of sets.

Macintyre's use of the discrete topology on 2 [in this version of requirement (3′)] implies that the graph of an atomic relation must be clopen instead of simply open. That strong requirement seems too restrictive because when an n-ary atomic operation is construed as an $(n+1)$-ary atomic relation, the graph will not in general be clopen. Under that strong condition, a sheaf of algebras would not in general qualify as a "sheaf of structures" if operations were construed as atomic relations. However, it is immediate that the graph of an n-ary operation is open, i.e., that it defines a subsheaf of the $(n+1)^{\text{st}}$ power sheaf, since the graph of a continuous function is homeomorphic with its domain (which, in this case, is the stalk space of the n^{th} power sheaf). These definitional comments, of course, do not affect Macintyre's results.

If (I, P) is a presheaf of structures, then the corresponding sheaf (S, I, p) (constructed as above) is a sheaf of structures. If each set of sections $\Gamma(u, s) \subset \Pi_{i\in u} S_i$ inherits its structure from the direct product, then the sheaf of sections is a sheaf of structures (i.e., also satisfies condition (3) of Part I) called the *associated sheaf of structures* of the presheaf of structures (I, P). Then (I, P) is a sheaf of structures iff it is isomorphic to its associated sheaf of structures. Conversely, (S, I, p) is a sheaf of structures iff the (stalk space) sheaf of structures reconstructed from the sheaf of structures of sections is isomorphic to (S, I, p) (i.e., the stalk spaces are homeomorphic over I and the stalks $S_i \cong \Gamma_i$ are isomorphic as structures). In short, the presheaf approach and the stalk space approach to sheaves of relational structures are equivalent, because the approaches are equivalent for sheaves of sets and the graphs of atomic relations are simply subsheaves of sets of various powers of the underlying sheaf of sets.

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