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ARBITRAGE THEORY: A MATHEMATICAL INTRODUCTION*

DAVID P. ELLERMAN†

Abstract. This paper is an introduction to the mathematical theory of arbitrage. Examples of the graph-theoretic mathematics of arbitrage are given in economics, electrical circuit theory (Kirchhoff's voltage law), engineering, social group theory, and thermodynamics. The arbitrage framework is used to develop an economic interpretation for cofactors, determinants, Cramer's rule, and inverse matrices. This is applied to subjective probability theory and classical optimization theory. Arbitrage enforces the laws of probability on the market for contingent claims. Markets are mathematically associated with any classical optimization problem so that the first order necessary conditions are equivalent to the markets being arbitrage-free. The arbitrage-free market prices are the Lagrange multipliers.

Key words. arbitrage, Kirchhoff's voltage law, flows and tensions on graphs, cofactors, Cramér's rule, coherence, optimization, Lagrange multipliers

Introduction. Buy low, sell high; that is arbitrage. When the same commodity can be bought or sold at two different prices, then a sure profit is obtained by the arbitrage activity of buying at the low price and selling at the high price. By the laws of supply and demand, arbitrage tends to eliminate its own possibility by reducing price discrepencies.

This paper is a mathematical introduction to *arbitrage theory* with applications to graph theory, linear algebra, probability theory, and optimization theory.

The basic (multiplicative) arbitrage theorem, which dates from the first "modern" work in mathematical economics by Augustin Cournot [Chapter 3, 1838], is as follows:

There exists a system of absolute prices for commodities such that the exchange rates (or relative prices) are price ratios if and only if (iff) the exchange rates are arbitrage-free.

Consider, for example, an international currency market where any currency can be transformed into any other currency at given exchange rates with no transaction costs. With m currencies, let r_{ij} for $i, j = 1, \cdots, m$ be the exchange rate which specifies that one unit of the ith currency can be transformed into r_{ij} units of the jth currency. Exchange rates *multiply* along any path of exchange to yield composite rates or "cross rates." The exchange rates are derived from a price system if there exists prices $p_1, \cdots, p_m$ such that for any i, j: $r_{ij} = p_i/p_j$. Profitable arbitrage would be possible if one unit of currency could be transformed around some circle into more than one unit of the same currency. Given any circle or cycle of exchange (e.g., dollars to yen to lire to dollars), one unit of a currency can be transformed into the number of units of the same currency equal to the product of the exchange rates around the cycle. Hence the exchange rates are arbitrage-free if the product of the rates around any cycle equals one (a "Cournot cycle"?). The multiplicative arbitrage theorem states that the exchange rates are derived from a price system if and only if the exchange rates are arbitrage-free.

This arbitrage theorem is the multiplicative version of Kirchhoff's voltage law (KVL) which, in its additive form, could be stated as follows:

There exists a system of potentials at the nodes of a circuit such that the voltages on the wires between nodes are the potential differences iff the voltages add to zero around any cycle.

There is an enormous literature which uses KVL in electrical circuit theory (e.g., Desoer and Kuh [1969]), graph theory (e.g., Aigner [1979]), and network theory (e.g., Berge and Ghouila-Houri [1965]). Yet the multiplicative arbitrage interpretation of

*Received by the editors August 9, 1983.
†School of Management, Boston College, Chestnut Hill, Massachusetts 02167.

KVL given above does not seem to have appeared in that literature. Moreover, the arbitrage result in Cournot's work [1897 (orig. 1838)] precedes Kirchhoff's work [1847].

The multiplicative arbitrage result has also been unduly neglected in its home territory of mathematical economics. The arbitrage aspects of Cournot's work have received some attention from Leon Walras [1926 (orig. 1874–7)] and, recently, Michio Morishima [1977]. Conditions for freedom from arbitrage are used in theories of international trade and finance. Freedom from arbitrage has been explicitly applied as a necessary condition for competitive market equilibrium in corporate finance theory (e.g., Modigliani and Miller [1958], Miller and Modigliani [1961], Ellerman [1982]). Nevertheless, the connection between the multiplicative arbitrage theorem and KVL does not seem to have been noted in the economics literature.

There is, however, an *additive* arbitrage interpretation of KVL which is well known in economics (e.g., T. C. Koopman's transportation model in Koopmans and Reiter [1951]). Given m commodities, let v_{ij} be the additional cash or *boot* one must be paid to trade one unit of commodity i for one unit of commodity j. A negative boot would represent a cost to be paid to make the one-for-one exchange of commodity i in return for commodity j. If such exchanges could be arranged in a circular fashion so that the sum of the boots was positive, then such a "circular transformation" (Koopmans and Reiter [1951, p. 248]) would be profitable arbitrage. If the sum of the boots were negative, then the reverse circular transformation would be profitable arbitrage. The additive arbitrage interpretation of KVL is:

Given a system of boots for commodity swaps, there exists a set of unit prices for the commodities such that the boot necessary for an exchange of units is the price difference if and only if the system of boots is arbitrage-free (*i.e., sums to zero around any cycle*).

This additive arbitrage theorem will be generalized and related to the multiplicative arbitrage theorem (§2 below).

Section 1 develops the multiplicative arbitrage theorem for the general case where the arcs and nodes of a graph can take values in an arbitrary group. A universal arbitrage-free system of transformation rates is constructed. A number of examples of the arbitrage theorem are given in widely differing fields. KVL is obtained when the group is the additive group of the reals. An engineering example using gear assemblies is outlined where the group is the multiplicative group of nonzero reals R^*. A behavioral science example concerns the formation of cliques in social groups. An example from physics involving Carnot engines is also outlined where the group is the multiplicative group of positive reals R^+.

Section 2 generalizes the theory of flows and tensions on ordinary graphs (e.g., Berge and Ghouila-Houri [1965]) to graphs representing markets where a nonzero real is assigned to each arc to multiply the flow through the arcs (flows with gains).

Section 3 uses the arbitrage results to develop an economic approach to determinants. Cofactors emerge as arbitrage-free prices and provide an economic interpretation of determinants, Cramér's rule, and inverse matrices.

Section 4 is based on the observation that "making book" (i.e., placing bets so that one makes money regardless of the outcomes) is equivalent to performing arbitrage on the market for contingent commodities. Hence the theorem in subjective probability theory that coherence (inability to have book made against oneself) implies the basic laws of probability (e.g., Ramsey or de Finetti in Kyburg and Smokler (eds.) [1964], Shimony [1955]) can be reinterpreted as stating that arbitrage enforces the laws of probability on the market for contingent commodities.

Section 5 applies the arbitrage results to classical optimization theory. Given a classical optimization problem with equality constraints, "markets" are defined so that the usual first-order necessary conditions are equivalent to the markets being arbitrage-free. The (normalized) market prices, which emerge in the absence of arbitrage, are the Lagrange multipliers. Throughout this paper, the emphasis is on equational conditions as opposed to inequalities. The presentation of the arbitrage interpretation of more general optimization problems with inequality constraints is beyond the scope of this introductory treatment.

1. Arbitrage in graph theory with examples. A *directed graph* $G = (G_0, G_1, t, h)$ is given by a set G_0 of *nodes* numbered $0, 1, \cdots, m$, a set G_1 of *arcs* numbered $j = 1, \cdots, b$, and two functions t, h: $G_1 \Rightarrow G_0$ which indicate that the arc j is directed from its tail, the $t(j)$ node, to its head, the $h(j)$ node. Multiple arcs between two nodes and loops at a node are permitted. A *path* from a node i_1 to a node i_{q+1} is given by a sequence $i_1 j_1 \cdots j_q i_{q+1}$ where the i's are nodes, the j's are arcs, and for $k = 1, \cdots, q$ either $t(j_k) = i_k$ and $h(j_k) = i_{k+1}$ (so the j_k arc is oriented with the direction of the path) or $t(j_k) = i_{k+1}$ and $h(j_k) = i_k$ (so the j_k arc is oriented against the direction of the path). If no arc occurs more than once in a path, it is a *simple* path. A simple closed path ($i_1 = i_{q+1}$) is a *cycle*. A graph is *connected* if there is a path between any two distinct nodes. A node and all the nodes connected to it form a *connected component* of the graph. A graph is *complete* if there is an arc between any two distinct nodes.

Let T be any group (not necessarily commutative) with the group operation written multiplicatively, e.g., R^* the multiplicative group of nonzero reals or $GL(R^k)$ the group of invertible real $k \times k$ matrices. A function r: $G_1 \Rightarrow T$, where $r(j)$ will be written r_j for $j = 1, \cdots, b$, is a labelling of the arcs by group elements. Such group-labelled graphs have been called "voltage graphs" (Gross [1974] and Zaslavsky [1982]) or "group graphs" (Harary, Lindstrom, and Zetterstrom [1982]). But given the priority of the economic concept of market arbitrage in Cournot's work and the inappropriateness of the voltage terminology in the multiplicative context, we will use an economic terminology. A directed graph G labelled with a function r will be called a *market graph,* denoted (G, r), with the system of transformation or exchange rates r being called the *rate system.*

In the economic interpretation, different commodities are associated with each node, and the rates r_j indicate that one unit of the $t(j)$ commodity (or currency) can be transformed into or exchanged for r_j units of the $h(j)$ commodity (or currency). In this introductory treatment, the transformations are reversible in the sense that one unit of the $h(j)$ commodity can be transformed into r_j^{-1} units of the $t(j)$ commodity.

Let c be the path $i_1 j_1 i_2 \cdots i_k j_k i_{k+1} \cdots j_q i_{q+1}$ from node i_1 to node i_{q+1} with the length of q arcs. If the r_j's are considered as transformations to be multiplied by composition, then the *composite rate* along the path c is the product $r[c] = r_{j_q}^{a(q)} \cdots r_{j_1}^{a(1)}$, where $a(k) = 1$ or -1 depending respectively on whether the edge occurance j_k is oriented with or against the direction of the path. A function $P{:}G_0 \Rightarrow T$ labelling the nodes of G is *system of absolute prices* or, simply, a *price system,* where $P(j)$ is denoted p_j. The set of *relative prices* or *price ratios* derived from a price system P is the set of quotient transformation rates $Q(P){:}G_1 \Rightarrow T$ where $Q(P)_j = (p_{h(j)})^{-1} p_{t(j)}$.

The following result can be viewed as a finite and multiplicative version of the fundamental theorem of calculus.

LEMMA 1. *Let c be a path from i to i' and let P be a price system. Then* $Q(P)[c] = p_{i'}^{-1} p_i$.

Proof. A straightforward induction over the length of the path. QED

COROLLARY 1. *For any nodes i and i' and any paths c and c' from i to i',* $Q(P)[c] = Q(P)[c']$.

COROLLARY 2. *For any cycle or closed path c, $Q(P)[c] = 1$.*

Given a market graph (G, r), the system of transformation rates r is said to be *path independent* if for any two nodes i and i', and for any paths c and c' from i to i', $r[c] = r[c']$. The transformations r are said to be *arbitrage-free* if for any cycle c, $r[c] = 1$ ("arbitrage-free" = "balanced" in Harary, Lindstrom, and Zetterstrom [1982] or Zaslavsky [1982]). The following result can be viewed as a finite and multiplicative version of the calculus theorem which states that the following conditions are equivalent:

(1) the line integral of a vector field around any closed path is zero,

(2) the line integral of the vector field is path independent, and

(3) the vector field is the gradient of a function.

COURNOT–KIRCHHOFF ARBITRAGE THEOREM. *Let (G, r) be a market graph with r: $G_1 \Rightarrow T$ taking values in an arbitrary group T. The following conditions are equivalent:*

(1) *the system of transformation rates r is arbitrage-free,*

(2) *the system r is path independent, and*

(3) *there exists a price system P such that $Q(P) = r$.*

Proof. (3) implies (2) by Corollary 1 to Lemma 1. To show that (2) implies (1), let c be a cycle containing a node i. If c contains no other node, then c consists of one edge j which is a loop at i. Traversing the loop once and twice yields two paths from i to i so $r_j = r_j^2$ and thus $r[c] = r_j = 1$. If the cycle c contains another node i', then there are two paths c' and c'' from i to i' such that the cycle c is obtained by traversing c' from i to i' and then traversing c'' backwards from i' to i. Since $r[c'] = r[c'']$ by assumption, $r[c] = r[c'']^{-1} r[c'] = 1$. To see that (1) implies (3), on each connected component of the graph choose any maximally cycle-free subset of the set of arcs, i.e., choose a spanning tree. Assign the "numeraire" node 0 or any other fixed node the price 1. Then assign prices to the other nodes by the following rule: if j is a tree arc and $t(j)$ has a price assigned to it, then set $p_{h(j)} = p_{t(j)} r_j^{-1}$, or if j is a tree arc and the node $h(j)$ has a price assigned to it, then set $p_{t(j)} = p_{h(j)} r_j$. In this manner, a unique price is assigned to each node of the connected component since, otherwise, there would be two paths consisting of tree arcs from the given node with price 1 to another node so the tree would contain a cycle. By construction, for any tree arc j, $r_j = (p_{h(j)})^{-1} p_{t(j)} = Q(P)_j$. Consider any arc j not in the tree. Since a tree is maximally cycle-free, a cycle c is completed by adding the arc j. Hence there is a path c' inside of the tree from $h(j)$ to $t(j)$, and by condition (1), $r[c']r_j = r[c] = 1$. But $r = Q(P)$ on the tree so, by Lemma 1, $r[c'] = (p_{t(j)})^{-1} p_{h(j)}$. Then $(p_{t(j)})^{-1} p_{h(j)} r_j = 1$ so $r_j = (p_{h(j)})^{-1} p_{t(j)} = Q(P)_j$, and thus $Q(P) = r$ on all of G. QED

Let $G = (G_0, G_1, t, h)$ be a directed graph. A rate system u: $G_1 \Rightarrow F$ is a *universal arbitrage-free system* if it is arbitrage-free and if for any arbitrage-free rate system r: $G_1 \Rightarrow T$, there is a unique group homomorphism r': $F \Rightarrow T$ such that the composite map $r'u$: $G_1 \Rightarrow F \Rightarrow T$ is equal to r: $G_1 \Rightarrow T$ (i.e., $r'u = r$). A universal arbitrage-free system is unique up to isomorphism, and each arbitrage-free system of transformation rates factors through the universal system.

Let the nodes of G_0 be numbered $0, 1, \cdots, m$, and let $G_0' = G_0 - \{0\}$ be the node set without the numeraire node. $F(G_0')$ is the free group on the set G_0'. Let P_u: $G_0 \Rightarrow F(G_0')$ be the price system defined by assigning each nonzero node to itself in the free group and where $P_u(0) = 1$, the group identity (not the node with number 1). Let $u = Q(P_u)$: $G_1 \Rightarrow F(G_0')$ be the rate system derived from the price system P_u.

UNIVERSALITY THEOREM. *The rate system u: $G_1 \Rightarrow F(G_0')$ is the universal arbitrage-free rate system on G.*

Proof. Since $u = Q(P_u)$ is derived from a price system, it is arbitrage-free. Let r: $G_1 \Rightarrow T$ be any other arbitrage-free rate system on G. Since r is arbitrage-free, there is a price system P: $G_0 \Rightarrow T$ such that $Q(P) = r$. Define r': $F(G_0') \Rightarrow T$ on the generators of $F(G_0')$, the nonzero nodes, by $r'(i) = P_0^{-1} P_i$, and extend r' uniquely to all of $F(G_0')$ to make

r' a group homomorphism. Then

$$\begin{aligned} r'u(j) &= r'(h(j)^{-1} + (j)) = r'(h(j))^{-1}\, r'(t(j)) \\ &= P_{h(j)}^{-1} P_0 P_0^{-1} P_{t(j)} \\ &= P_{h(j)}^{-1} P_{t(j)} \\ &= Q(P)_j = r_j \end{aligned}$$

for any j in G_1, so $r'u = r$. QED

Example 1. *Kirchhoff's voltage law.* The arbitrage theorem is the general noncommutative version of Kirchhoff's voltage law (KVL), which is usually stated with the additive group of the reals R as the group of transformation rates. The numeraire node is then called the *ground* or *datum node.* A function U: $G_0 \Rightarrow R$ is called a *potential.* The additive version of the quotient operator $Q()$ is the difference operator $D()$ where $D(U)$: $G_1 \Rightarrow R$ is the function such that $D(U)_j = u_{t(j)} - u_{h(j)}$. Any function v: $G_1 \Rightarrow R$ such that there exists a potential U with $D(U) = v$ is called a *potential difference* or *tension* [e.g., Rockafellar, 1970, p. 205]. Given a function v: $G_1 \Rightarrow R$ and a cycle c containing the arcs $j_1, \cdots, j_q$, the sum of v around the cycle c is

$$v[c] = a(1)v_{j_1} + \cdots + a(q)v_{j_q},$$

where $a(k)$ is $+1$ or -1 depending, respectively, on whether the arc j_k is with or against the direction of the cycle. The arbitrage theorem then yields:

KVL: *For any v: $G_1 \Rightarrow R$, v is a potential difference if and only if $v[c] = 0$ for any cycle c.*

Example 2. *Gear assemblies.* A multiplicative example in engineering can be devised using bond graph theory (e.g., Thoma [1975]). When circuit graphs are used in electrical network theory, an arc represents a wire in which a two terminal circuit element, such as a resistor, can be inserted. One could, however, interpret each arc as containing a "two-port device" (in the language of bond graphs) such as a "transformer." Transformers could be electrical, translational mechanical (e.g., levers), or rotational mechanical (e.g., gears). The nonzero real transformation rates assigned to the arcs could be turns ratios, leverages, or gear ratios.

Let us consider a rotational mechanical example using a gear assembly. There might be many gears on each shaft (so the gears are not necessarily all in the same plane). Each node represents a shaft and each arc represents a meshing of gears on the two shafts. The arcs may be oriented arbitrarily. Given a meshing or arc j, the gear on the $t(j)$ shaft is called the *driver* and the gear on the $h(j)$ shaft is the *follower.* If N_d and N_f are the number of teeth on the driver and follower respectively, then the transformation rate assigned to the arc j is the negative of the gear ratio: $r_j = -N_f/N_d$. Consider a clockwise angular velocity as positive. If the follower has an angular velocity of w_f, then the angular velocity of the driver is $w_d = r_j w_f$. The transformation rates could also be defined as the velocity ratios $r_j = w_d/w_f$. A gear assembly thus defines a market graph.

A path in the graph represents a gear train, and the composite rate along the path is the *train value.* If any two meshing gears in a train are separated and an idler gear is interposed, that changes the sign but not the absolute value of the train value. A cycle in the graph represents a circular gear train—which introduces the possibility that the gear assembly could be rigid (unable to move). A *motion* of the gear assembly is an assignment of angular velocities w: $G_0 \Rightarrow R^*$ (where R^* is the multiplicative group of nonzero reals) to the shafts such that $Q(w) = r$. The arbitrage theorem then implies that:

A gear assembly has a motion (is not rigid) if and only if the train value of any circular gear train is $+1$.

In a "plane gear assembly" all the gears are the same plane (i.e., one per shaft). In a plane gear train, each interior gear can be considered an idler gear, so the train value is determined up to sign by the size of the first and last gears. In a circular plane gear train, the first and last gears are the same, so the train value is $+1$ or -1 depending, respectively, on whether there is an even or odd number of meshings and thus gears in the circular train. Hence a plane gear assembly has a motion if and only if each circular train has an even number of gears.

Example 3. *Clique formation in social groups.* A *signed graph* (e.g., Harary [1953], Zaslavsky [1982], [1983]) is a market graph where the rate group is the multiplicative group $T = \{+1, -1\}$. The patterns of likes and dislikes in a social group can be represented by a signed graph if each node stands for a person, and a sign $+1$ or -1 on an arc j means that the person $t(j)$, respectively, "likes" or "dislikes" the person $h(j)$. In the signed graph literature, an arbitrage-free signed graph is called "balanced" due to an intuitive balance in such a pattern of likes and dislikes (e.g., Harary [1953], Flament [1963], Harary, Norman, and Cartwright [1965], Roberts [1978]). A "price system" marks each person with a $+1$ or a -1, and a pattern of likes and dislikes is derived from such a marking if a person likes others with the same marking and dislikes those with a different marking. The arbitrage theorem then yields the result:

A social group with a given pattern of likes and dislikes can be partitioned into two cliques such that all likes are intra-clique and all dislikes are inter-clique if and only if the pattern of likes and dislikes is balanced.

Example 4. *The second law of thermodynamics.* The Carnot engine approach to the second law of thermodynamics (simplified to involve only a finite number of temperatures) gives an application of the arbitrage theorem in physics. There are $m + 1$ heat reservoirs with distinct empirical (e.g., mercury thermometer) temperatures which include the freezing and boiling points of water. Given two reservoirs, one at a high temperature t_h and the other at a lower temperature t_c, a Carnot engine reversibly withdraws the heat dQ_h from the high temperature reservoir, produces the work dW, and dumps the heat dQ_c into the colder reservoir. By the first law of thermodynamics (conservation of energy), $dQ_h = dW + dQ_c$. The *efficiency* of the Carnot engine is $e = dW/dQ_h = 1 - (dQ_c/dQ_h)$, and the *efficiency debit* is $r = 1 - e = dQ_c/DQ_h$. We take each reservoir as a node, and any Carnot engine as defining an arc from the low to the high temperature reservoir with the arc rate being the efficiency debit r, so the rate group is R^+, the multiplicative group of positive reals. Carnot engines can be hooked in series and the composite efficiency debit is the product of the individual debits.

One formulation of the *second law of thermodynamics* is that between any two temperature reservoirs, all Carnot engines have the same efficiency and thus the same efficiency debit (e.g., Morse [1964b, p. 50]). Otherwise one could perform a type of "heat arbitrage" and have what is called a "perpetual motion machine of the second kind" (e.g., Castellan [1964], Chap. 8). The equality of all composite efficiency debits between any two reservoirs implies path independence. Hence, by the arbitrage theorem, the second law implies that there exist $m + 1$ positive real "prices" $T_0, T_1, \cdots, T_m$ such that for any Carnot engine operating between nodes h and c, the efficiency debit is $r = T_c/T_h$. If the "prices" are normalized so that the freezing and boiling points of water differ by 100 units, then the "prices" $T_0, \cdots, T_m$ are the *Kelvin absolute temperatures* of the reservoirs.

2. Prices and exchanges on a market graph. To relate the additive and multiplicative arbitrage theorems, we allow the rate system $r\colon G_1 \Rightarrow R$ to take its values in a field. The elements of a field form an additive group, and the nonzero elements form a (commutative) multiplicative group. All the transformation rates considered thus far have been real numbers, so little generality will be lost if we take the field to be the real numbers R. A new possibility now arises; zero can occur as a multiplicative transformation rate. In intuitive economic terms, this introduces the possibility of free goods. If $r_j = 0$, then one unit of the $t(j)$ good exchanges for zero units of the $h(j)$ good, so $t(j)$ is a "worthless" or free good. In the reverse direction, the number of units of the free $t(j)$ good that one can get in exchange for one unit of the good $h(j)$ is undefined—as is the reciprocal of $r_j = 0$.

Let $G = (G_0, G_1, t, h)$ be a directed graph. Let $r: G_1 \Rightarrow R$ be a (multiplicative) rate system possibly with zero values, so (G, r) is an extended market graph ("extended" to allow zero rates). Our purpose in this section is to generalize the theory of flows and tensions on a directed graph (e.g., Berge and Ghouila-Houri [1965]) and to provide an economic interpretation. The "classical" theory of flows and tensions is obtained as the special case where all $r_j = 1$. Flows multiplied by a rate r_j along an arc are called "flows with gains" (e.g., Jewell [1962]).

The incidence matrix treatment of flow-tension theory generalizes directly. The incidence matrix $S = [s_{ij}]$ is the $(m + 1) \times b$ matrix (nodes $\times$ arcs) where;

$$\begin{aligned}
s_{ij} &= -r_j && \text{if } t(j) \neq i = h(j),\\
s_{ij} &= 1 && \text{if } t(j) = i \neq h(j),\\
s_{ij} &= 1 - r_j && \text{if } t(j) = i = h(j), \text{ and}\\
s_{ij} &= 0 && \text{if } t(j) \neq i \neq h(j).
\end{aligned}$$

Given a vector x in R^b,

$$(Sx)_i = \sum_{t(j)=i} x_j - \sum_{h(j)=i} r_j x_j$$

for $i = 0, 1, \cdots, m$. If $Sx = 0$, there is conservation at each node. Such an x in the null space of S, Null(S), represents exchange activity which neither creates nor consumes any net amount of any good; so x is called a *circulation* (or flow). Let $y = Sx$ be in the range or column space of S, Col(S). A positive x_j indicates that x_j units of the $t(j)$ good are transformed into $r_j x_j$ units of the $h(j)$ good. Hence $y = Sx$ represents the net exchanges at the nodes. If y^+ and y^- are the positive and negative parts of y (so that $y = y^+ - y^-$), then the exchange activities x have the net effect of exchanging y^+ for y^- at the rates r. Hence the $y = Sx$ in the column space Col(S) are called *exchanges*.

The vector space dual concepts have price interpretations. Let S' be the transpose of S. Given $P = (p_0, p_1, \cdots, p_m)$ in $R^{(m+1)}$,

$$(S'P')_j = p_{t(j)} - p_{h(j)} r_j$$

for $j = 1, \cdots, b$. In the ordinary case where all $r_j = 1$, the vectors $S'P'$ in the row space Row(S) are called "tensions" (or voltages). In the economic interpretation, $p_{t(j)} - p_{h(j)}$ is the amount one must be paid "to boot" in order to exchange one unit of the $t(j)$ good for one unit of the $h(j)$ good if the unit valuations are given by P. In the general case, the swap is not "1 for 1" but "1 for r_j." Thus the general economic interpretation of $S'P'$ is a *derived boot system*, a boot system derived from the unit valuations P for the exchange rates r. At the valuation P,

$$(S'P')_j = p_{t(j)} - p_{h(j)} r_j$$

is the boot one must be paid to exchange 1 unit of $t(j)$ for r_j units of $h(j)$.

If the exchange rates r are derived from the unit valuations P, then goods are being exchanged at the price ratios so no boot is necessary. The "quid" has the same value as the "quo." Hence the vectors P in the left null space $\mathrm{Null}(S')$ are the *prices*. At such prices, one unit of $t(j)$ is equal in value to r_j units of $h(j)$.

The associativity of multiplication yields the adjointness result: $(PS)x = P(Sx)$. Given the unit valuations P and the exchange activity x, the total boot necessary for the exchange activity x, i.e., $(PS)x$, is equal to the value lost by the net exchange Sx, i.e., $P(Sx)$. In the next section, an economic interpretation for Cramer's rule will be provided by deriving the rule from a variation on this adjointness result.

The null space is the orthogonal complement of the row space. For the incidence matrix S, this is the real-valued market graph generalization of

KIRCHHOFF'S CURRENT LAW. *The following conditions are equivalent:*

(1) *x is a circulation (flow), and*

(2) *for any derived boot system (tension) $B = PS$, $Bx = 0$.*

The row space is the orthogonal complement of the null space. For the incidence matrix S, this is the real-valued market graph generalization of

KIRCHHOFF'S VOLTAGE LAW. *The following conditions are equivalent:*

(1) *B is a derived boot system (tension), and*

(2) *for any circulation (flow) x, $Bx = 0$.*

The ordinary KVL or additive arbitrage theorem [see the Introduction] is obtained in the special case of all $r_j = 1$. The subspace of circulations is spanned by the cycles, so the condition $Bx = 0$ only need be checked on the cycles x. Thus a boot system B is derived from a set of unit valuations P if and only if it sums to zero around every cycle (is arbitrage-free).

The orthogonality between circulations (flows or currents) and derived boot systems (tensions or voltages) on ordinary graphs (all $r_j = 1$) was first used by Heaviside in 1883 (see Penfield, Spence, and Duinker [1970]), was proven by Weyl [1923], and was rederived and used to prove network theorems by Tellegen [1952]. Hence, in the engineering literature (e.g., Desoer and Kuh [1969]), it is known as

TELLEGEN'S THEOREM. *Circulations and derived boot systems form complementary orthogonal subspaces.*

The corresponding results about nodes, using the orthogonality of the left null space and the column space of S, are of economic significance particularly in the case of general real-valued rate systems r (instead of all $r_j = 1$). The left null space $\mathrm{Null}(S')$ is the orthogonal complement of the column space $\mathrm{Col}(S)$. This could be expressed as the

PRICE LAW. *The following conditions are equivalent:*

(1) *P is a price vector, and*

(2) *for any exchange $y = Sx$, $Py = 0$.*

If no $r_j = 0$ and if there is a nonzero price vector P, then the exchange rates r are derived from it, i.e., $r = Q(P)$ in the notation of §1. Hence the price law can be interpreted as stating that any quantity exchanges made according to certain price ratios will conserve value if valued at those same prices. The price law will be used, for example, in the next section to provide an economic interpretation to the expansion of determinants by alien cofactors.

The column space $\mathrm{Col}(S)$ is the orthogonal complement of the left null space $\mathrm{Null}(S')$. This could be expressed as the

EXCHANGE LAW. *The following conditions are equivalent:*

(1) *y is an exchange, and*

(2) *for any price vector P, $Py = 0$.*

The orthogonality of the left null space and column space of S is a "Tellegen theorem" for the nodes of an extended real-valued market graph.

PRICE-EXCHANGE THEOREM. *Prices and exchanges form complementary orthogonal subspaces.*

If $r_j = 0$, j is a *zero arc*. If a node is the tail of a zero arc, it is a *free node*, otherwise a *scarce node*. The *scarce subgraph* G_s of G is the subgraph generated by the scarce nodes (the scarce nodes plus all arcs between them). The restriction r_s of r to G_s has only nonzero values so the arbitrage theorem of §1 can be applied to it. A *scarce component* is a connected component of the scarce subgraph G_s.

Let P be a price vector. If i is a free node, there is an arc j such that $t(j) = i$ and $r_j = 0$. Then $p_{t(j)} - p_{h(j)}r_j = 0$ implies that $p_i = 0$. Any price vector is zero at free nodes. If an arc j is between scarce nodes, then r_j is nonzero so the condition $p_{t(j)} - p_{h(j)}r_j = 0$ forces both prices to be zero or both to be nonzero. Hence a price vector is all zero or all nonzero on each scarce component. If nonzero, then the restriction of r to the scarce component is arbitrage-free (by the arbitrage theorem). There are two ways a good i can become a free good (i.e., $p_i = 0$). It can be directly forced to be free by being the tail of a zero arc, or it can be in a "scarce" component where arbitrage is possible. Intuitively, the possibility of arbitrage on a scarce component allows arbitrary amounts of those goods to be generated so they indirectly become free goods.

On a scarce component, a nonzero P is determined up to a scalar multiple. If P^* is another price vector nonzero on that component and i is a node in the component, then $P^{**} = P - (p_i/p_i^*)P^*$ is also a price vector (since prices form a vector space). But $p_i^{**} = 0$ so P^{**} is zero on that scarce component. Thus P and P^*, restricted to the scarce component, are scalar multiples of one another.

The structure of a price vector can now be specified; it is nonzero only on arbitrage-free scarce components where it is determined up to a scalar multiple. Hence the dimension of the space of prices, $\mathrm{Null}(S')$, is the number c of arbitrage-free scarce components. The dimension of the complementary subspace of exchanges is the number of nodes, $m + 1$, minus c. Since the row rank equals the column rank of S, the space of derived boot systems also has dimension $m + 1 - c$. Its complementary subspace of circulations has the dimension $b - m - 1 + c$, where b is the number of arcs. In the classical case where all $r_j = 1$, all nodes are scarce, r is trivially arbitrage-free, and an arbitrage-free scarce component is simply a connected component. Hence we have the result in graph theory that the dimension of the space of tensions (derived boot systems) is the number of nodes minus the number of connected components $m + 1 - c$, and the dimension of the space of flows (circulations) is $b - m - 1 + c$ (e.g., Berge and Ghouila-Houri [1965, p. 124]).

3. Determinants and market graphs. Given a square $(m + 1) \times (m + 1)$ real matrix A^*, let A be the $(m + 1) \times m$ matrix obtained by deleting the kth column, and let $A(i, \cdots, i')$ be the matrix obtained from A by deleting the rows $i, \cdots, i'$. In this section, we will show how real-valued extended market graphs can be defined from A. The main theorem is that the cofactors of the deleted kth column form a price vector for any such market graph defined from A.

The mathematics will be developed together with the economic interpretation. A commodity (node) is associated with each row. Each of the columns represents the unit level of a linear activity or process which transforms the $m + 1$ goods (e.g., Koopmans [1951]). In the jth column (all numbering of rows and columns is as in the original matrix A^*), a positive (resp. negative) a_{ij}^* indicates that the activity j at unit level of operation produces (resp. consumes) $|a_{ij}^*|$ units of the ith commodity.

Given an $m \times 1$ column vector x of activity levels for the activities in A, the net transformation or net product is the $(m + 1) \times 1$ column vector $Ax = b$. To construct an arc j with some rate r_j from node i' to node i, we need some activity levels x such that Ax is -1 in the i' row (one unit of the i' good is consumed) and is zero in every other row except row i. The entry in row i is the transformation rate r_j. The net effect of the activity levels x is to transform one unit of the i' commodity into r_j units of the i commodity. It is notationally convenient to assume, unless otherwise stated, that $i' < i$ so that row i' has the same number after row i has been deleted.

Let $A(i)$ be A after deleting row i, and let $e_{i'} = (0, \cdots, 1, \cdots, 0)'$ be the $m \times 1$ column vector with a 1 in the i' place and 0's elsewhere. Hence to define r_j, we need an x such that $A(i)x = -e_{i'}$. Let A_i be the ith row of A. Given such an x, $r_j = A_i x$ is the transformation rate on the arc being defined from node i' to node i. If such an x exists, an arc is said to be *definable* from node i' to node i. The following lemmas give the conditions for an arc to be definable and for the arc rate to be uniquely determined.

LEMMA 1. *An arc from node i' to node i is definable if and only if the ith row $A_{i'}$ is not in the row space of $A(i, i')$, Row $(A(i, i'))$.*

Proof. If an appropriate x exists, then $A_{i'}$ cannot be in the row space of $A(i, i')$ because x annihilates the rows generating the row space, but $A_{i'}x = -1$. Conversely, assume $A_{i'}$ is not in the row space. Since the row space and the null space of $A(i, i')$ are orthogonal, $A_{i'}$ can be uniquely decomposed into the sum of $A_{i'}^r$ in the row space and $A_{i'}^n$ in the null space. Since $A_{i'}$ is not in the row space, its null space component is nonzero so $(A_{i'}^n)^2 = \|A_{i'}^n\|^2 > 0$. Then

$$x^* = -A_{i'}^n / \|A_{i'}^n\|^2$$

is an x such that $A(i)x = -e_{i'}$. QED

LEMMA 2. *Assume an arc j is definable from i' to i. Then the arc rate r_j is uniquely determined if and only if A_i is in the row space of $A(i)$.*

Proof. If the rate r_j is uniquely determined, then for all x such that $A(i)x = -e_{i'}$, $A_i x = r_j$. The x^* from the proof of Lemma 1 is such an x. If y is any vector in the null space of $A(i)$, then $A(i)(x^* + y) = -e_{i'}$ so $A_i(x^* + y) = r_j$ and thus $A_i y = 0$. Hence A_i annihilates the null space of $A(i)$, so A_i is in the row space of $A(i)$. Conversely, if A_i is in that row space, there is a vector $W = (w_1, \cdots, w_m)$ such that $A_i = WA(i)$. Then for any x such that $A(i)x = -e_{i'}$, $A_i x = WA(i)x = -We_{i'} = -w_{i'}$. If W^* were another vector with $A_i = W^*A(i)$ but $w_{i'}^* \neq w_{i'}'$, then $(W - W^*)A(i) = 0$ where $W - W^*$ is nonzero in its i' component. But then the i' row $A_{i'}$ is a linear combination of the rows of $A(i, i')$ contrary to the assumption that an arc was definable from i' to i. QED

COFACTORS = PRICES THEOREM. *Let A^* be any square real matrix and let A be A^* with a column deleted. Let G be any real-valued (extended) market graph defined from A. The cofactors of the deleted column form a price vector for the market graph G.*

Proof. Let the kth column be the deleted column and let $p_i = A_{ik}^* = (-1)^{(i+k)}|A(i)|$ be the cofactor of a_{ik}^* in A^*. If G is a real-valued extended market graph defined from A, the theorem states that for any arc j from node i' to node i with an arc rate r_j, the price equation

$$p_{i'} - p_i r_j = 0$$

holds. The proof is divided into three cases: case (1) $r_j = 0$, case (2) $r_j \neq 0$ and $p_i = 0$ or $p_{i'} = 0$, and the main case (3) $r_j \neq 0, p_i \neq 0$, and $p_{i'} \neq 0$.

Case (1). $r_j = 0$. If $p_{i'} = 0$, then the price equation holds trivially so assume $p_{i'} \neq 0$, i.e., $|A(i')| \neq 0$. Then the price equation cannot be satisfied so we derive a contradiction from $p_{i'} \neq 0$. Since $r_j = 0$, there is an x such that $A(i)x = -e_{i'}$ and $A_i x = 0$. Hence $A(i')x = 0$ contrary to $|A(i')| \neq 0$.

Case (2). $r_j \neq 0$ *and* $p_i = 0$ or $p_{i'} = 0$. If $p_i = 0$, $|A(i)| = 0$ but $A_{i'}$ is not in the row space of $A(i, i')$ since the arc is definable. Hence there is a linear dependency among the rows of $A(i, i')$ so $|A(i')| = 0 = p_{i'}$ and the price equation holds. If $p_{i'} = 0$, then $|A(i')| = 0$. If A_i is in the row space of $A(i, i')$, then $r_j = 0$, contrary to assumption. Hence A_i is not in the row space of $A(i, i')$ so $|A(i')| = 0$ implies a linear dependency between the rows of $A(i, i')$—which implies $|A(i)| = 0 = p_i$.

Case (3). $r_j \neq 0$, $p_i \neq 0$, *and* $p_{i'} \neq 0$. Since $p_i \neq 0$, $A(i)$ can be inverted to solve the equation $A(i)x = -e_{i'}$ for $x = -A(i)^{-1}e_{i'}$. Then $r_j = A_i x = -A_i A(i)^{-1}e_{i'}$ and the price equation is

$$p_{i'} - r_j p_i = (-1)^{(i'+k)}|A(i')| + A_i A(i)^{-1}e_{i'}(-1)^{(i+k)}|A(i)| = 0.$$

The product $A(i)^{-1}e_{i'}$ extracts the i' column of $A(i)^{-1}$ which consists of the cofactors of the i' row of $A(i)$ each divided by $|A(i)|$. Canceling the $|A(i)|$, the equation to be proven reduces to

$$(-1)^{(i'+k)}|A(i')| + (-1)^{(i+k)}A_i\{\text{cofactors of } i' \text{ row of } A(i)\} = 0.$$

Beginning with $A(i')$ (where $i' < i$), $i - i' - 1$ row transpositions will move the row A_i into the position of the deleted row $A_{i'}$. The dot product

$$A_i\,\{\text{cofactors of } i' \text{ row of } A(i)\}$$

is the cofactor expansion by the i' row of the matrix obtained from $A(i')$ after the $i - i' - 1$ row transpositions. Each transposition reverses the sign of the determinant, so

$$A_i\,\{\text{cofactors of } i' \text{ row of } A(i)\} = (-1)^{(i-i'-1)}|A(i')|.$$

Hence

$$\begin{aligned}
&(-1)^{(i'+k)}|A(i)| + (-1)^{(i+k)}(-1)^{(i-i'-1)}|A(i')| \\
&= \{(-1)^{i'} - (-1)^{(2i-i')}\}|A(i')| \\
&= (-1)^{-i'}\{(-1)^{2i'} - (-1)^{2i}\}|A(i')| \\
&= 0.
\end{aligned}$$

If $i < i'$, there are $i' - i - 1$ row transpositions so the final calculation is

$$\begin{aligned}
&(-1)^{(i'+k)}|A(i')| + (-1)^{(i+k)}(-1)^{(i'-i-1)}|A(i')| \\
&= \{(-1)^{i'} - (-1)^{i'}\}|A(i')| \\
&= 0.
\end{aligned}$$

QED

If rank$(A) < m$, then all the cofactors $p_i = A^*_{ik}$ of the kth column are zero. To avoid such degeneracy, suppose that rank$(A) = m$. Then a market graph G is easily defined so that the Exchanges vector space (defined by G) is the column space of A, and the Prices space is the null space of A. If rank$(A) = m$, then there is a nonzero cofactor p_i so for any market graph definable from G, dim(Prices) > 0 and thus dim(Exchanges) $< m + 1$. Since p_i is nonzero, the rows of $A(i)$ are linearly independent so for any i', $A_{i'}$ is not in the row space of $A(i, i')$. Hence an arc is definable from any other node i' to the given node i with $p_i \neq 0$. For any such i', let x be such that $A(i)x = -e_{i'}$ and $A_i x = r_j$. Then $R_{i'i} = Ax$ is the vector with -1 in the i' place, r_j in the i place, and 0's elsewhere. The m arcs from the other i' to i constitute a market graph G sufficient for the next corollary. The m vectors

$R_{i'i}$ for $i' \neq i$ are linearly independent since each introduces a new node i'. Each is an exchange, and each is in $\mathrm{Col}(A)$ since $Ax = R_{i'i}$. Hence dim(Exchanges) $= m = \mathrm{rank}(A) = \mathrm{rank}(\mathrm{Col}(A))$. This completes the proof of the following

COROLLARY. *If* $\mathrm{rank}(A) = m$, *then a market graph* G *is definable such that Exchanges* $= \mathrm{Col}(A)$ *and Prices* $= \mathrm{Null}(A)$.

The theorem and corollary provide an economic interpretation for determinants (a dual quantity interpretation is also available). Given a square matrix, delete any one column and use the remaining m columns as activities or processes to define transformation rates for market graphs on the rows as nodes. For any such graph, the cofactors of the deleted column form a price vector. Under the nondegeneracy assumption that some of those cofactors are nonzero, all the columns of A can be obtained as exchanges. Then the orthogonality between prices and exchanges provides an economic interpretation to the expansion of A^* by alien cofactors.

The economic interpretation of the *determinant* $|A^*|$ is given by its cofactor expansion by the kth column:

$|A^*| = p_0 a^*_{0k} + \cdots + p_m a^*_{mk}$ *is the value of the unit level of the kth activity* (*kth column*) *when evaluated at the prices* (*cofactors*) *defined by the other m activities in* A^*.

Adding the second subscript to indicate the column, the prices used to evaluate the kth activity are $p_{ik} = A^*_{ik}$, the cofactors of the kth column. They will be called the *commodity k-prices*. If $|A^*| \neq 0$, the value of the kth activity can be taken as "numeraire" (price $= 1$) so the *normalized commodity k-prices* are $p^*_{ik} = p_{ik}/|A^*| = A^*_{ik}/|A^*|$ for $i = 0, \cdots, m$. Let $P^*_k = (p^*_{0k}, \cdots, p^*_{mk})$ be the row vector of normalized commodity k-prices. At these prices, the (imputed) *activity k-prices* are $P^*_k A^* = (0, \cdots, 1, \cdots, 0) = I_k$ (the kth row of the $m + 1$ dimensional identity matrix I). Only the kth activity has k-value (i.e., value at k-prices) and it is the numeraire. Let P^* be the matrix of normalized commodity prices, the kth row being P^*_k, for $k = 0, 1, \cdots, m$. Then $P^*A^* = \mathrm{I}$ so the matrix of normalized commodity prices is the *inverse matrix* of A^*, i.e.,

$$P^* = A^{*-1}.$$

Given an $(m + 1) \times 1$ column vector x of activity levels for all the $m + 1$ activities, the net product of the transformations is $A^*x = b$. Assuming $|A^*| \neq 0$, the k-value of the activities is $P^*_k A^* x = (0, \cdots, 1, \cdots, 0)x = x_k$. The k-value of the net product b is

$$P^*_k b = (p^*_{0k} b_0 + \cdots + p^*_{mk} b_m)/|A^*|.$$

In economic terms,

$$\begin{aligned} &\text{Value of activities } x \text{ at } P^*_k A^* \text{ the activity } k\text{-prices} \\ &= P^*_k A^* x = P^*_k b \\ &= \text{Value of the net product } b \text{ at commodity } k\text{-prices } P^*_k. \end{aligned}$$

This is the economic interpretation of

CRAMER'S RULE. *If* $|A^*| \neq 0$ *and* $A^*x = b$,

$$x_k = (A^*_{0k} b_0 + \cdots + A^*_{mk} b_m)/|A^*|.$$

Cofactors encode transformations. The cofactors = prices theorem shows that the cofactors of a column in a square matrix A^* are prices which encode information about the transformations defined by the remaining columns of the matrix. Under the nondegeneracy assumption for A (some nonzero cofactors of the deleted column), any market graph definable from A will have some arbitrage-free components. Let us (for the first time) allow the kth column to be used to define transformation rates on a market graph. Will the transformations added by the kth column make arbitrage possible? That information is supplied by evaluating the kth column by the k-prices (i.e., by evaluating the determinant). If the kth column has zero k-value, then it adds no new transformations; it is definable from the other activities. But if the kth activity has a nonzero k-value (i.e., the determinant is nonzero), then it is not contained in the other activities and arbitrage is possible. Any arc rate r_j can be defined from node i' to node i by inverting A^* to solve $A^*x = R_{i'i}$ where $R_{i'i}$ is the vector $(0, \cdots, -1, \cdots, r_j, \cdots, 0)$ with -1 in the i' place, r_j in the i place and zeros elsewhere. Then a market graph can be expanded so that arbitrage allows any vector to be obtained as a net product, i.e., the equation $A^*x = b$ always has a solution, and the only price vector is the zero vector.

The arbitrage possibilities allowed with all the columns of A^* can be summarized as the following

DETERMINANT = ARBITRAGE-PROFIT THEOREM. $|A^*| = 0$ *implies freedom from arbitrage, and* $|A^*| \neq 0$ *implies profitable arbitrage is possible.*

The value of the determinant $|A^*|$ can be interpreted as a measure of the "profit" obtainable from arbitrage on market graphs definable from A^* (the "profit" being the k-value of the kth activity at unit level). This result about the zero or nonzero value of a determinant opens up the possibility of arbitrage interpretations of the vanishing determinant conditions which occur in mathematics and science. One such interpretation is developed in the next section.

More columns could be added to the square matrix A^* to form a matrix A^{**}. More arcs could be defined on the same node set by using m or $m + 1$ columns at a time in the manner indicated above. It is useful to apply the "arbitrage" terminology directly to any $(m + 1) \times n$ matrices A^{**} where $n > m$ and the row rank of A^{**} is at least m. Such a matrix is said to be *arbitrage-free* if any market graph definable from A^{**} has some arbitrage-free components, i.e., if the $m + 1$ rows of A^{**} are linearly dependent. This will be applied to classical optimization theory in the last section.

4. Arbitrage in probability theory. The zero or nonzero value of a determinant indicates, under the economic interpretation, the absence or presence of arbitrage. This theme will be illustrated using the derivation of the laws of probability from the requirement of coherence in betting behavior.

Making book means making a series of bets so that one has positive net winnings regardless of the events which occur. This "getting-something-for-nothing" suggests arbitrage and properly so; making book is performing arbitrage on the market for contingent commodities. A *contingent commodity* (or "contingent claim" or "lottery ticket") is a commodity which one receives conditioned on the occurance of some event, e.g., three oranges if event E occurs. We will only need to consider contingent money, e.g., \$5 if E occurs. We assume small enough stakes so that it is reasonable to suppose that bettors maximize expected monetary values (thus avoiding the complications of utility theory).

A person subjectively assigns a probability $p(E)$ to an event E if the person is just willing to pay $p(E)S$ in order to receive the stake S if E occurs. In terms of contingent money, the probability $p(E)$ is the price the person is just willing to pay for the contingent dollar "\$1 if E."

Suppose a bettor places two bets with a bookie; the bettor pays \$.50 to have \$1 if E, and also pay \$.55 to have \$1 if not-E. By taking both bets, a bookie would "make book" against the bettor. No matter what happens the bettor gets \$1 by paying \$1.05, so \$.05 is lost. The bettor's probability judgments are said to be *incoherent* if book can be made against the bettor. Ramsey [1960 (orig. 1926)] and Finetti [1964 (orig. 1937)] showed that the laws of probability theory, e.g., $p(E) + p(\text{not-}E) = 1$, could be derived from the requirement of coherence. Bookies thrive on the incoherence of different people's probability judgments.

By slightly reconceptualizing the bets, we can see that by making book, the bookie is performing arbitrage on the market for contingent commodities. When the bookie sold (and the bettor bought) the contingent commodity \$1 if not-E for \$.55, that could be reconceptualized as the bookie buying (and the bettor selling) the contingent commodity \$1 if E for \$.45 = \$1 − \$.55. To see the equivalence between buying \$1 if E for \$.45 and selling \$1 if not-E for \$.55, consider the payoffs in each eventuality. If E occurs, having bought \$1 if E for \$.45 implies a net gain of \$.55. But selling \$1 if not-E for \$.55 implies the exact same consequence when E occurs, a gain of \$.55 (over the zero return if the person held onto the \$1 if not-E claim). If not-E occurs, having bought \$1 if E for \$.45 implies a loss of \$.45. Having sold \$1 if not-E for \$.55 implies the same because one gave up the \$1 (which one would have had when not-E occurred) in return for \$.55, a net loss of \$.45.

In general, selling the contingent claim \$1 if not-E for \$$p(\text{not-}E)$ is equivalent to buying the contingent claim \$1 if E for \$1 − $p(\text{not-}E)$. The bookie's two bets can now be reformulated. By taking the \$.55 bet on not-E, the bookie bought from the bettor the contingent commodity \$1 if E for \$.45. Then by taking the \$.50 bet on E, the bookie sold the \$1 if E back to the bettor for \$.50. This is the circular arbitrage or buy-low-sell-high operation on the market for contingent claims which netted the bookie \$.05.

This arbitrage operation can be expressed in the determinantal framework of the last section. There are two commodities, \$1 if E and \$1 if not-E. There are two activities, betting on E and betting on not-E. Betting \$.50 to get \$1 if E is the activity represented by the column vector $(1 - .50, -.50)'$. As before, the negative components represent the inputs and the positive components are the outputs of the activity. By performing the activity of betting \$.50 to get \$1 if E, the bettor loses the input \$.50 if not-E and gains the output \$1 − \$.50 if E. Betting \$.55 to get \$1 if not-E is the activity represented by the vector $(-.55, 1 - .55)'$. The column vector $s = (s_1, s_2)'$ of activity levels are the stakes for the two bets. The net product vector $w = (w_1, w_2)'$ gives the winnings in terms of the contingent claims.

$$\text{If } A^* = \begin{pmatrix} I - .50 & -.55 \\ -.50 & 1 - .55 \end{pmatrix}, \quad \text{then } A^*s = w.$$

Taking the cofactors of (say) the first column, the prices defined by the second column are $P = (.45, .55)$. According to the bet on not-E, those are the prices (= probabilities) assigned to the contingent claims, \$1 if E and \$1 if not-E. If one does undertake the first activity (at the activity level or stake of 1), then at the prices established by the second activity the net value is

$$|A^*| = (.45, .55)(.50, -.50)' = -.05,$$

the arbitrage losses of the bettor undertaking both activities at level 1.

The bettor is assumed to be just willing to take each individual bet, so the bettor would also be willing to reverse each individual bet. If the determinant were positive, then the bookie could still make book (perform arbitrage) by reversing each individual bet.

Hence coherence, the inability to make book by performing arbitrage on the market for contingent claims, implies that the determinant must be zero.

Let $E_1, \cdots, E_n$ be n mutually exclusive and jointly exhaustive events. One and only one of the events must occur. There are n contingent commodities: \$1 if $E_1, \cdots$, \$1 if E_n. For $k = 1, \cdots, n$, a bettor is just willing to pay $\$p_k$ to receive \$1 if E_k. That activity is expressed by the column vector $(-p_k, \cdots, 1 - p_k, \cdots, -p_k)'$ where $1 - p_k$ is in the kth place with $-p_k$ elsewhere. The bettor gives up the inputs $\$p_k$ if E_j, for any $j \neq k$, in order to obtain the output $\$1 - p_k$ if E_k. Taking the cofactors of the first column, the prices of the n contingent claims (= probabilities of the n events) defined by the other $n - 1$ activities are

$$P = (1 - p_2 - \cdots - p_n, p_2, \cdots, p_n).$$

Evaluating the first activity at these prices, its value, the value of the determinant, is $1 - p_1 - p_2 - \cdots - p_n$. Arbitrage is not possible if and only if the determinant is zero. Hence freedom from arbitrage (coherence) implies

$$p_1 + p_2 + \cdots + p_n = 1.$$

As another example, we give the arbitrage/coherence derivation of the conditional probability formula: $p(E \mid C) = p(E)/p(C)$, where the event E is contained in the conditioning event C. There are three contingent commodities: \$1 if $E \& C$, \$1 if not-$E \& C$, and \$1 if not-C. If a bettor pays $\$p(E)$ for \$1 if E, the transformation of the three contingent claims is $(1 - p(E), -p(E), -p(E))'$. If the bettor pays $\$p(C)$ for \$1 if C, then the transformation of the three contingent claims is expressed by the vector $(1 - p(C), 1 - p(C), -p(C))'$, where the first two contingent claims, \$1 if $E \& C$ and \$1 if not-$E \& C$, both involve C occurring. The third bet is a more complicated conditional bet. The bettor pays $\$p(E \mid C)$ to receive \$1 if $E \& C$ occurs and thus to lose $\$p(E \mid C)$ if not-E/C occurs. But if the conditioning event C does not occur, the $\$p(E \mid C)$ is refunded. This betting activity is expressed by the vector $(1 - p(E \mid C), -p(E \mid C), 0)'$.

The matrix of activities is

$$A^* = \begin{pmatrix} 1 - p(\mathrm{E}) & 1 - p(C) & 1 - p(E \mid C) \\ -p(E) & 1 - p(C) & -p(E \mid C) \\ -p(E) & -p(C) & 0 \end{pmatrix}.$$

Taking the cofactors of (say) the third column, the prices of the contingent claims defined by the first two betting activities are

$$P = (p(E), p(C) - p(E), 1 - p(C)).$$

The event E is contained in C so $E \& C = E$, and thus the price of \$1 if $E \& C$ is $\$p(E)$. The events $E \& C$ and not-$E \& C$ partition C, so the bets assign the price $\$p(C) - p(E)$ to \$1 if not-$E \& C$. And $\$1 - p(C)$ is the price assigned to \$1 if not-C. Evaluating the third activity at these prices,

$$|A^*| = p(E) - p(C)p(E \mid C).$$

Hence freedom from arbitrage implies the conditional probability formula when E is contained in C: $p(E) = p(C)p(E \mid C)$.

5. Arbitrage in classical optimization theory. In a classical optimization problem, an objective function is being maximized subject to m equational constraints. We will consider market graphs with $m + 1$ nodes $0, 1, \cdots, m$. The commodity associated with node 0 is the objective function. At the other nodes, the commodities are the scarce resources associated with the m constraints. The intuitive idea is to let the arc rates be defined by the linearization at the margin of the constraints and the objective function. If a market graph defined by this marginal data permitted arbitrage, then there is an exchange that is positive in the zero place and zero elsewhere. Thus the objective function can be marginally increased with no net change in the scarce resources, so the original point could not be optimal. Hence a necessary condition for optimality is that any market graph defined from the marginal data must be arbitrage-free.

Before turning to a more exact mathematical formulation, we will investigate the use of market graphs, illustrated as "arbitrage diagrams," as a pedagogical tool. Consider the case of one constraint so there are two commodities, the objective function and the scarce resource. Suppose the problem is to maximize the area of a rectangular field of width W and length L with a given amount of fencing F where one length of the field is provided by a river. Hence the problem is to maximize WL subject to $L + 2W = F$. The two commodities in the market are area and fencing. Given an extra 1 yard of fencing, there are two ways it could be transformed into area: add it to the length or add it to the widths. Adding it to the length increases the area by $W(L + 1) - WL = W$. If the extra yard is added to the widths, it must be split equally between the two widths to maintain the rectangular shape. Thus the other transformation rate of fencing into area is $(W + 1/2)L - WL = L/2$. The arc rates can be illustrated by what we will call an *arbitrage diagram:*

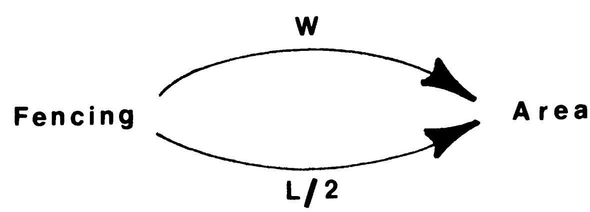

DIAGRAM 1

If the two rates were unequal, arbitrage would be possible. Each rate is the marginal value, in terms of area, of the fencing in the two uses (width and length). If the rates are unequal, then some of the scarce resource should be shifted from the low-value use to the high-value use, and that is arbitrage. Shifting some marginal fencing out of the low-valued use is sacrificing some area to buy some fencing at the low price. Then that freed-up fencing is sold at the high price in return for more area. This buy-low-sell-high operation secures more area with no net change in the amount of fencing used so the previous width and length could not be optimal. A necessary condition for optimality is that the fencing-area market illustrated in the arbitrage diagram be arbitrage-free: $W = L/2$. That common transformation rate $W = L/2$ of the resource into the objective function is the Lagrange multiplier which emerges in the usual Lagrangian treatment of the problem.

Arbitrage diagrams are particularly useful in teaching economic theory. Consider the consumer utility maximization problem. The first case treated is typically with two goods X and Y with unit prices P_x and P_y. What are the necessary conditions for the consumer to maximize utility $U(X, Y)$ subject to the budget constraint of just spending the income I, i.e., subject to the budget line $P_xX + P_yY = I$? The conventional approach is to draw the following diagram with an indifference or iso-utility curve tangent to the budget line.

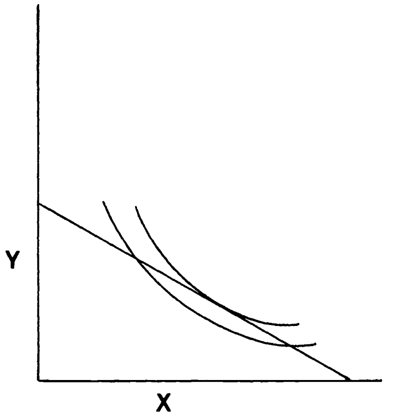

DIAGRAM 2

The slope of the budget line is $-P_x/P_y$. If MU_x and MU_y are the partial derivatives of the utility function (the marginal utilities), then the slope of the indifference curve is $-MU_x/MU_y$. The highest indifference curve attainable by the consumer is the one tangent to the budget line. At that point the two slopes are equal, so the utility maximization condition is (canceling the minus signs)

$$P_x/P_y = MU_x/MU_y.$$

In the arbitrage diagram approach to the problem, a new market is defined where the commodities are income and utility. There are two ways of transforming income, at the margin, into utility: buy more X or buy more Y. If an extra \$1 is expended on X, it will purchase $1/P_x$ units of X, each of which yields MU_x extra units of utility. Hence MU_x/P_x is the rate of transforming income into utility by purchasing X, and similarly MU_y/P_y is the rate obtained by spending the marginal dollar on Y. This market is illustrated by the following arbitrage diagram.

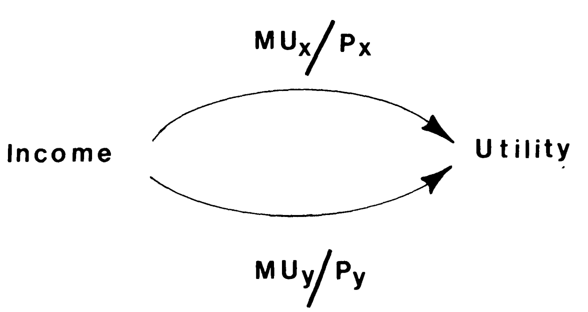

DIAGRAM 3

Utility maximization requires the income-utility market to be arbitrage-free:

$$MU_x/P_x = MU_y/P_y.$$

The arbitrage diagram approach has a distinctive pedagogical advantage in the n goods case where $P_1, \cdots, P_n$ are the unit prices and $MU_1, \cdots, MU_n$ are the marginal utilities. The conventional indifference curve diagram does not generalize to n goods, and hand-waving about the tangency of hypersurfaces to hyperplanes is of little pedagogical value. Yet the arbitrage diagram generalizes immediately. There are n ways to transform income into utility: buy one of the n goods. As before, the transformation rate is MU_i/P_i for $i = 1, \cdots, n$. This market is illustrated by the following arbitrage diagram.

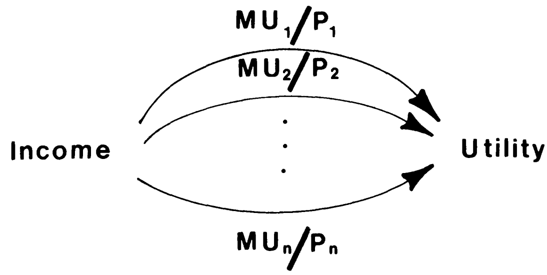

DIAGRAM 4

Utility maximization implies that the market must be arbitrage-free:

$$MU_1/P_1 = MU_2/P_2 = \cdots = MU_n/P_n.$$

That common rate is *the* transformation rate of income into utility, the marginal utility of income, and it is the "mysterious" Lagrange multiplier which students will find in more advanced treatments of the topic. Similar applications of arbitrage diagrams abound in economic theory.

In the general one constraint case, a function $y = f(x_1, \cdots, x_n)$ is to be maximized subject to a constraint $g(x_1, \cdots, x_n) = b$. Let $\nabla f(x) = (f_1, \cdots, f_n)$ and $\nabla g(x) = (g_1, \cdots, g_n)$ be the (nonzero) gradients of partial derivatives of f and g evaluated at $x = (x_1, \cdots, x_n)$. Let A^{**} be the $2 \times n$ matrix with the f gradient as the first row and minus the g gradient as the second row (the minus sign indicates the scarce resource is an input). There are n ways to marginally transform some b into y: vary x_j for $j = 1, \cdots, n$. If $g_j \neq 0$, let the jth column of A^{**} be considered as the second column in a two column activity matrix A^*, where any other column of A^{**} can be taken as the first column. The cofactors of the first column of A^* are $(-g_j, -f_j)$ so the rate of transformation of the scarce resource b into the objective function y defined by varying x_j is $r_j = f_j/g_j$. The absence of arbitrage implies the equality of those rates for $j = 1, \cdots, n$. In terms of determinants, if column k of A^{**} is taken as the first column of A^*, the determinant $|A^*| = -f_k g_j + f_j g_k$ must be zero to prevent profitable arbitrage. But any column of A^{**} could be taken as the first colunm of A^*. Hence freedom from arbitrage implies that

$$f_1/g_1 = f_2/g_2 = \cdots = f_n/g_n,$$

and that common rate is the Lagrange multiplier.

The arbitrage framework can be used to give a heuristic motivation for the Lagrangian function. Think of the process of transforming the resource into the objective via the functions g and f as "production." Suppose there was an *alternative* means to transform b into y at the rate p. Instead of requiring all the resource b to be used up in the production of y, the leftover resource $b - g(x_1, \cdots, x_n)$ could be sold at the price p.

Hence instead of enforcing the constraint $g(x_1, \cdots, x_n) = b$, one could maximize the total sum of the $y = f(x_1, \cdots, x_n)$ obtained from the resource by production plus the $y = p[b - g(x_1, \cdots, x_n)]$ obtained by the alternative transformation in exchange for the unused resource. That is, one could maximize without constraint the Lagrangian function:

$$L = f(x_1, \cdots, x_n) + p[b - g(x_1, \cdots, x_n)].$$

The arbitrage-free condition equalizes all the transformation rates:

$$f_1/g_1 = \cdots = f_n/g_n = p.$$

There is, however, no given "alternative" means to transform the resource into the objective. But arbitrage only requires the use of two rates. Hence when considering any one rate f_j/g_j, let p stand for an unknown or undetermined transformation rate which represents whatever rates are available by other means (e.g., varying the other x_k's). Freedom from arbitrage requires that each rate f_j/g_j be equal to p, and thus the rates are led ("as if by an invisible hand") to equal each other. Only when all the rates f_j/g_j for $j = 1, \cdots, n$ are equal does the indeterminancy of p disappear. No matter which rate f_j/g_j is considered, all the other alternative rates f_k/g_k are the same, so that common rate is the "Lagrange multiplier" p.

For the general classical optimization problem, let $f^i(x_1, \cdots, x_n)$ for $i = 0, 1, \cdots, m$ (where $m < n$) be continuous differentiable functions from R^n to R. The classical optimization problem has the following form:

$$\begin{aligned} &\text{Maximize } y = f^0(x_1, \cdots, x_n) \\ &\text{subject to:} \quad f^1(x_1, \cdots, x_n) = b_1, \\ &\qquad\qquad\qquad f^m(x_1, \cdots, x_n) = b_m. \end{aligned}$$

Let $F(x) = [\partial f^i(x)/\partial x_j)]$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$ be the $m \times n$ matrix of partial derivatives of the constraints. Let $\nabla f^0(x)$ be the gradient of partial derivatives of the objective function. A point $\overline{x} = (\overline{x}_1, \cdots, \overline{x}_n)$ is *feasible* if it satisfies the constraints. A point is *regular* if $F(x)$ has linearly independent rows when evaluated at that point. A basic theorem of classical optimization theory, proven using the implicit function theorem, is that if $\overline{x}$ is feasible, regular, and is a local maximum of $f^0(x)$ subject to the constraints, then the following condition holds.

Necessary condition—first version (NC1). For any $x = (x_1, \cdots, x_n)'$, if $F(\overline{x})x = 0$, then $\nabla f^0(\overline{x})x = 0$.

NC1 can be restated in an equivalent form using Lagrange multipliers by using the general linear algebra result that the row space and null space of a matrix are orthogonal complements. Since NC1 states that the gradient of f^0 at $\overline{x}$ is orthogonal to every x in the null space of $F(\overline{x})$, the gradient must be in the row space. This yields the equivalent

Necessary condition—second version (NC2). There exists $p = (p_1, \cdots, p_m)$ such that $\nabla f^0(\overline{x}) = pF(\overline{x})$, i.e., such that the Lagrangian function

$$L = f^0(x) + p_1[b_1 - f^1(x)] + \cdots + p_m[b_m - f^m(x)]$$

has zero partial derivatives at $\overline{x}$ (e.g., Varaiya [1972]).

We will show that "markets," i.e., market graphs, can be defined in terms of the marginal data $F(\overline{x})$ and $\nabla f^0(\overline{x})$ so that the necessary conditions NC1 and NC2 are equivalent to the markets being arbitrage-free. The normalized market prices which arise out of the absence of arbitrage are the Lagrange multipliers.

Let A^{**} be the $(m + 1) \times n$ matrix formed with $\nabla f^0(\bar{x})$ as the first row and with $-F(\bar{x})$ as the last m rows. Since the bottom m rows of A^{**} are linearly independent, there are m columns of A^{**} which form an $(m + 1) \times m$ matrix A such that the bottom m rows of A are linearly independent. Let A be considered as the last m columns of an $(m + 1) \times (m + 1)$ matrix A^* with any other column of A^{**} as the first column. Let $P^* = (p_0^*, p_1^*, \cdots, p_m^*)$ be the cofactors of the "unknown" first column. The cofactor p_0^* is the determinant of the m linearly independent bottom rows of A so $p_0^* \neq 0$. Thus commodity 0 can be taken as the numeraire to obtain the normalized price vector $P = (1, p_1, \cdots, p_m)$ where $p_i = p_i^*/p_0^*$ for $i = 1, \cdots, m$. Let j be an arc from node i to the numeraire node 0. Since P^* is a price vector, $p_i^* - r_j p_0^* = 0$ so the normalized price p_i is the rate of transformation $r_j = p_i^*/p_0^* = p_i$ of the ith scarce resource into the objective function, i.e., the Lagrange multiplier associated with the ith constraint.

Optimality implies that P annihilates all of A^{**}, i.e., that A^{**} is an arbitrage-free matrix. Clearly $PA = 0$ by expansion with alien cofactors. Let C be any other column of A^{**} not in A. Taking C as the first column of A^*, if $PC \neq 0$, then arbitrage would be possible (on a market graph definable from A^*). Since A^* would be nonsingular, a net product or exchange $(1, 0, \cdots, 0)' = b = A^*x^*$ can be obtained which is 1 in the zero place and 0 elsewhere. It can be interpreted as the result of an arbitrage operation which yields one more unit of the numeraire with no cost in extra resources utilized. The vector x^* of activity levels only involves the x's corresponding to the columns of A^*. Extending x^* to an $n \times 1$ column vector x with zeros for the other activities yields $A^{**}x = b$. Thus x annihilates $F(\bar{x})$ but not the gradient of f^0 in violation of NC1. Hence optimality implies $PC = 0$ for any column C, i.e., $PA^{**} = 0$. As an $(m + 1) \times n$ matrix with $n > m$ and row rank at least m, A^{**} must be arbitrage-free.

These results can be summarized in the following

ARBITAGE-FREE PRICES = LAGRANGE MULTIPLIERS THEOREM. *Let $\bar{x}$ be feasible, regular, and a local maximum. Then any market graph defined from the marginal data $\nabla f^0(\bar{x})$ and $-F(\bar{x})$ must be arbitrage-free. There is a vector $P = (1, p_1, \cdots, p_m)$ of normalized cofactors which is a price vector for any such market graph, and the normalized cofactors $p_1, \cdots, p_m$ are the Lagrange multipliers associated with the m constraints.*

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