The Mathematics of Real Estate Appraisal

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Introduction

Real estate appraisal is more of a practical art than a theoretical science. Appraisers use a number of time-honored formulas without great attention to the theoretical derivation of the formulas. While this "cookbook" approach may work as a matter of everyday practice, it leaves much to be desired from a pedagogical viewpoint. When valuation formulas do have a derivation from a certain set of assumptions, then it is quite inappropriate—particularly for the technically-oriented student—for the formulas to be taught as "recipes" established by some authority and simply to be memorized and used.

There are a number of reasonably complex formulas that are used in the income approach to real estate appraisal, particularly as developed in the United States. The necessary assumptions and the proofs of these formulas are usually to be found only in a few scarce journal articles in the United States or in out-of-print books. Hence we have attempted to give here, all in one place, fresh algebraic derivations of the major formulas to make them available to technically adept students and practitioners.

The topic of internal rates of return or IRR's is also covered largely because IRR's are often misunderstood and improperly applied in the real estate appraisal profession as well as in other areas of business. The point is that appraisers should rely on net present values, not IRR's, when giving advice about the selection of investment projects.

A number of new results are also presented:

(1) a general formula for the valuation of changing income streams defined by linear recurrence relations which has all the usual formulas for valuing changing income streams as special cases (e.g., straight line changing annuity, constant ratio changing annuity, and Ellwood J premise),

(2) an analysis of the straight line and Hoskold capitalization methods which shows that both methods are appropriate for certain declining income streams where the income decline can be motivated as the interest losses resulting from a hypothetical capital recovery sinking fund using a substandard rate (below the discount rate), and

(3) a general theorem about amortization tables where the principal reductions can be arbitrarily specified and an application of the theorem to give an alternative proof of the main result about the Hoskold capitalization method.

The Six Functions of One

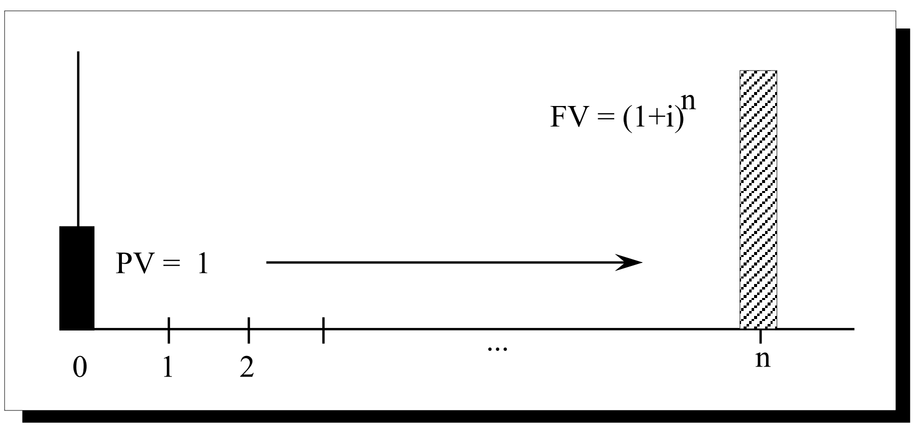
The Amount of One at Compound Interest

Throughout our discussion, we will assume that future amounts of money can be discounted back to present values or that present amounts can be compounded into future values using a **discount rate i** per period. The periods could be years, months, or any other fixed time period. Unless otherwise stated, the formulas will always assume that the interest rate (% per period) and the units of time are stated using the same period of time. The discount rate may be taken as including the risk-free interest rate and a consideration for risk and illiquidity. But it does not include any "capital recovery requirements" to be considered later.

The first basic formula

$$FV = PV(1+i)^n$$

states that given the present value of PV, that is equivalent on the market to the future value after n periods of $FV = PV(1+i)^n$. If PV = 1, then we have the **amount of one at compound interest** given in the tables. The present value is said to be "compounded" into the future value.

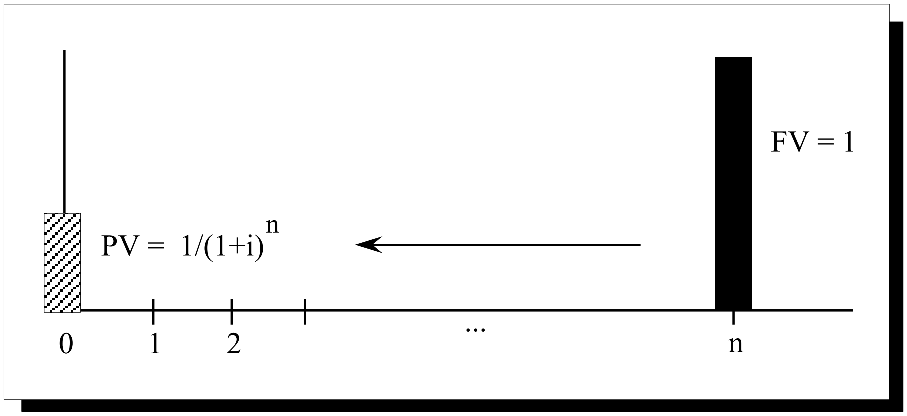


The Present Value Reversion of One

For each basic function of one, the inverse or reciprocal is also a function of one. The inverse of the amount of one at compound interest is the **present value reversion of one**.

$$PV = \frac{FV}{(1+i)^n}$$

Given a future amount FV at the end of the n^{th} period, the equivalent present value (at time zero) is obtained by dividing by the factor of $(1+i)^n$.



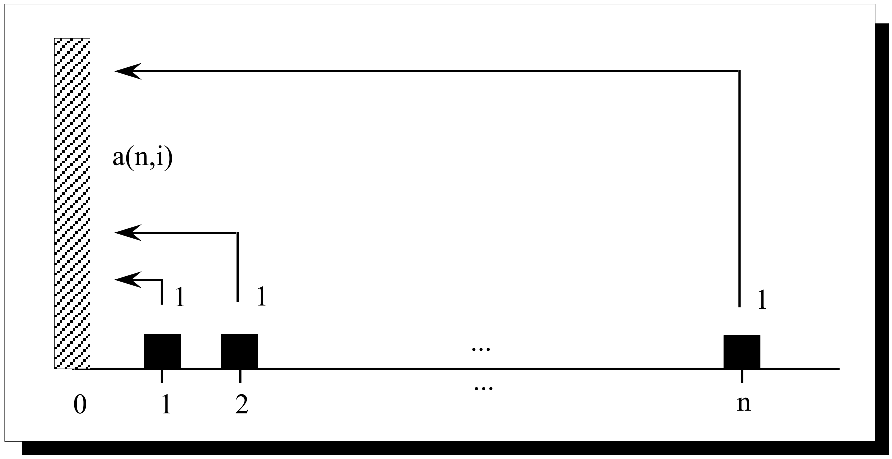
The future value is said to be "discounted" to the present value.

The Present Value of an Ordinary Annuity of One

Suppose we want to pay off a loan with a series of equal payments at the times t = 1, 2,...,n (i.e., at the end of the first period and the end of each other period up to and including the n^{th} period). We consider a series of equal payment of one. Each payment is discounted back to a present value using the present-value-of-one formula (taking care to use the right time period). Since the results are all amounts of money at the same time, they can be meaningfully added together to get the total present value of the series of equal payments. It is called the **present value of an ordinary annuity of one** and will be denoted a(n,i).

$$a(n,i) = \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + ... + \frac{1}{(1+i)^n} = \sum_{k=1}^{n} \frac{1}{(1+i)^k} = \frac{1 - \frac{1}{(1+i)^n}}{i}$$

Given a series of equal payments PMT at t = 1, 2,...,n, their present value is PMT a(n,i). Those payments would pay off a loan at time zero of that principal value of PV = PMT a(n,i).

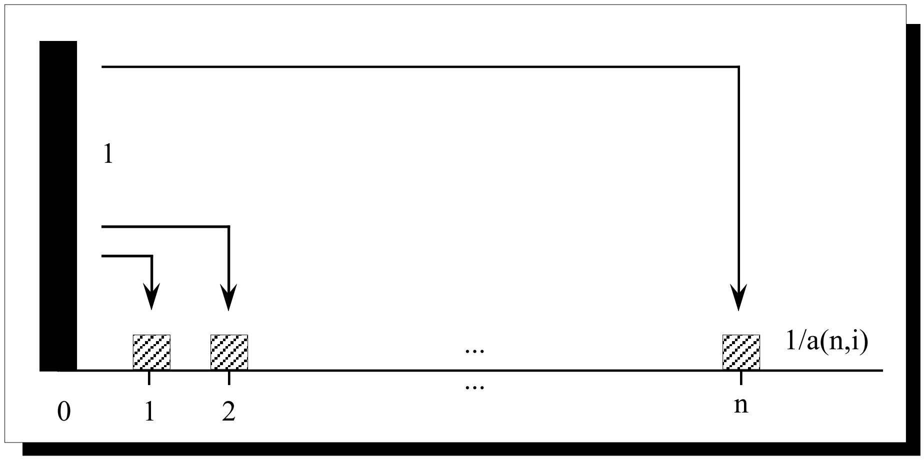


The Installment to Amortize One

If we are given the equal payments PMT, we can use the present value of an annuity of one a(n,i) to calculate the corresponding principal value PV = PMT a(n,i). But if we are given the principal PV for a loan, then we can use the reciprocal 1/a(n,i) to calculate the equal installment payments PMT = PV/a(n,i) that would pay off the loan. The equal installment payments are said to "amortize" the loan. If the loan was for PV = 1, then the reciprocal amount PMT = 1/a(n,i) is called the **installment to amortize one**.

$$\mathrm{PMT} = \frac{1}{a(n,i)} = \frac{1}{\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^n}} = \frac{i}{1 - \frac{1}{(1+i)^n}}$$

We can think of the present value PV = 1 as "growing" into the equal series of 1/a(n,i) amounts. Suppose the present amount of one is deposited in a bank account being the compound interest rate of i per period. At the end of period 1, the amount 1/a(n,i) can be withdrawn from the account leaving the remainder to accumulate interest. In a similar manner, the amount 1/a(n,i) can be withdrawn at the end of period 2 and so forth through period n. The last withdrawal of 1/a(n,i) at time n would reduce the bank account balance to zero.



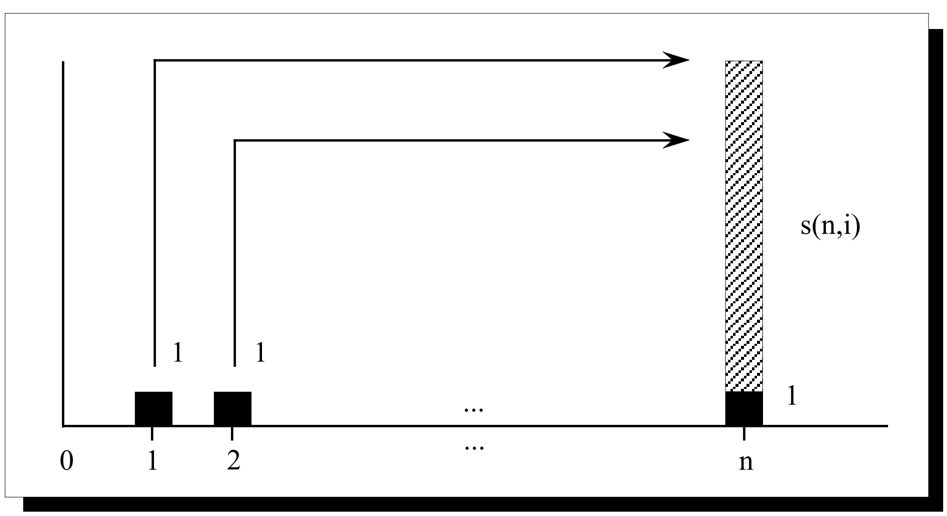
The Accumulation of One per Period

Suppose that instead of considering the present value of a series of equal payments, we consider the *future* value at time n of a series of equal amounts at time 1, 2, ..., n. This practice of depositing equal amounts over a series of time periods and letting them accumulate to a future amount is called a "sinking fund." Each deposit in the fund can be compounded to a future value at time n and the future values can be added together to get the total accumulated value of the sinking fund. If each deposit is one, then the total future amount is called the **accumulation of one per period** and is denoted s(n,i).

$$s(n,i) = (1+i)^{n-1} + (1+i)^{n-2} + ... + (1+i)^1 + 1 = \sum_{k=0}^{n-1}(1+i)^k = \frac{(1+i)^n - 1}{i}$$

Since this accumulation of one per period just restates the present value of an annuity of one as a future value at time n, we have

$$s(n,i) = (1+i)^n \, a(n,i).$$

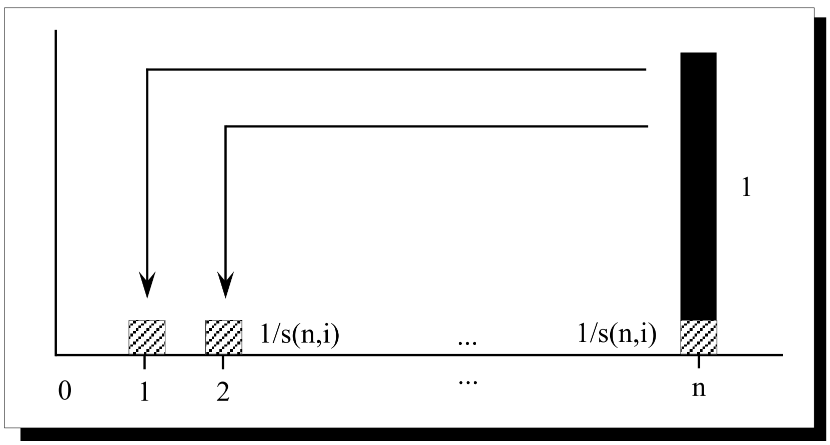


The Sinking Fund Factor

For the inverse function, we know the desired value of the accumulated fund FV at time n and we compute the sinking fund deposit (or payment PMT into the fund) at times 1,2,...,n that would accumulate to the desired amount FV. That deposit is called the **sinking fund factor** and will be denoted SFF.

$$SFF(n,i) = \frac{1}{s(n,i)} = \frac{1}{(1+i)^{n-1} + (1+i)^{n-2} + ... + (1+i)^1 + 1} = \frac{i}{(1+i)^n - 1}$$

The sinking fund factor SFF "discounts" the future value of the fund FV back into a series of equal amounts. If you had the promise to receive the future value of one at time n, then it would be equivalent for you to receive the series of equal payments SFF = 1/s(n,i) at the times 1,2,...,n.



Summary of Six Functions

The six functions can be divided into two groups: three functions and their inverses.

Function	Inverse Function
Amount of One at Compound Interest $(1+i)^n$	*Present Value Reversion of One* $(1+i)^{-n}$
Present Value of an Ordinary Annuity of One $a(n,i) = \frac{1}{(1+i)^1} + ... + \frac{1}{(1+i)^n} = \frac{1-(1+i)^{-n}}{i}$	*Installment to Amortize One* $\frac{1}{a(n,i)} = \frac{i}{1-(1+i)^{-n}}$
Accumulation of One per Period $s(n,i) = (1+i)^{n-1} + ... + (1+i)^1 + 1 = \frac{(1+i)^n - 1}{i}$	*Sinking Fund Factor* $SFF(n,i) = \frac{1}{s(n,i)} = \frac{i}{(1+i)^n - 1}$

Amortization Tables

Let us now consider a loan with the principal of PV which is to be paid off with equal payments PMT = PV/a(n,i) at times 1,2,...,n. Each payment PMT will pay some interest and pay some principal. The interest payments just service the loan; they do not reduce the principal balance. Only the remaining part of PMT can be considered as a principal payment or principal reduction. How much of each payment is considered as interest payment and how much as principal payment? The conventional way to compute interest and principal portions of loan payments is to assume that all the interest due at any time is taken out of the payment, and the remainder of the payment is principal reduction.

Let Bal(k) be the principal balance due on the loan after the payment is made at the end of the k^{th} period. The loan begins with Bal(0) = PV. At the end of the first period, the interest due is iPV = iBal(0). Subtracting from the payment PMT gives the principal portion of the payment PMT-iBal(0). The new balance is the old balance reduced by the principal payment: Bal(1) = Bal(0) - (PMT - iBal(0)). In general, the interest due at the end of the k^{th} period is iBal(k-1) so the principal reduction by the k^{th} payment is:

$$PR(k) = PMT - iBal(k-1) = (1+i)PR(k-1).$$

The new balance at the end of the k^{th} period is:

$$Bal(k) = Bal(k-1) - PR(k) = Bal(k-1) - (PMT - iBal(k-1)).$$

The final payment at time n pays off the remaining balance of the loan so PR(n) = Bal(n-1) and Bal(n) = 0.

The computation of these interest and principal portions is usually presented in an:

Amortization Table.

Period	Beg. Balance	Payment	Interest	Prin. Reduction	End Bal.
1	PV	PMT	iPV	PMT-iPV	Bal(1)
2	Bal(1)	PMT	iBal(1)	PMT-iBal(1)	Bal(2)
...	...	...	...	...	...
n-1	Bal(n-2)	PMT	iBal(n-2)	PMT-iBal(n-2)	Bal(n-1)
n	Bal(n-1)	PMT	iBal(n-1)	PMT-iBal(n-1)	0

Some Formulas of Financial Mathematics

To derive a formula for Bal(k) the balance due at the end of the k^{th} period for a loan of principal PV, we first derive the formula for bal(k), the balance due at time k for a loan of principal 1. Then we will have: Bal(k) = PV bal(k).

We know that a(n,i) is the present value of payments of 1 at the end of each period 1,...,n. This sum can be divided into two parts, the present value of the first k payments which is a(k,i), and the value of last n-k payments at time k, namely a(n-k,i) discounted backed to time 0 by dividing by $(1+i)^k$:

$$a(n,i) = a(k,i) + \frac{a(n-k,i)}{(1+i)^k}.$$

Multiplying both sides by $(1+i)^k/a(n,i)$ and rearranging yields the formula for bal(k):

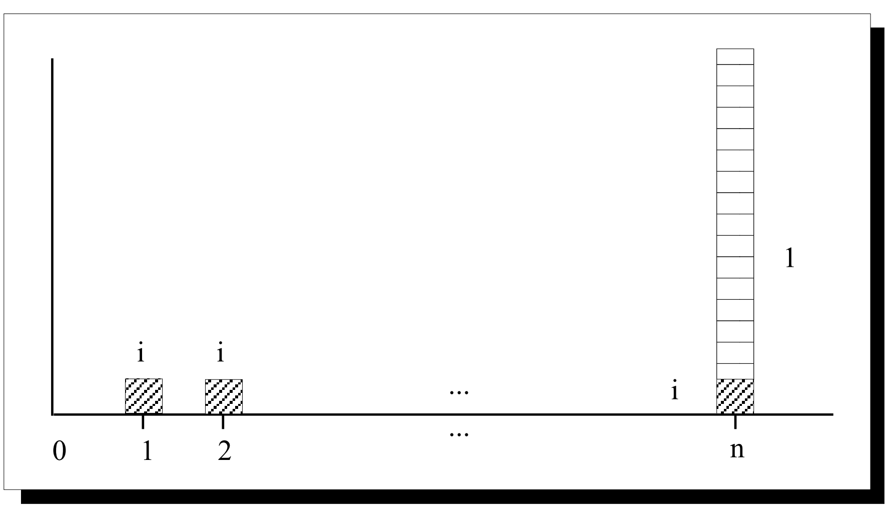
$$bal(k) = \frac{a(n-k,i)}{a(n,i)} = (1+i)^k \left[1 - \frac{a(k,i)}{a(n,i)}\right].$$

If the principal of the loan is 1, then each payment is 1/a(n,i). The balance at time k, bal(k), is the present value at that time of the last n-k payments so we have the above formula.

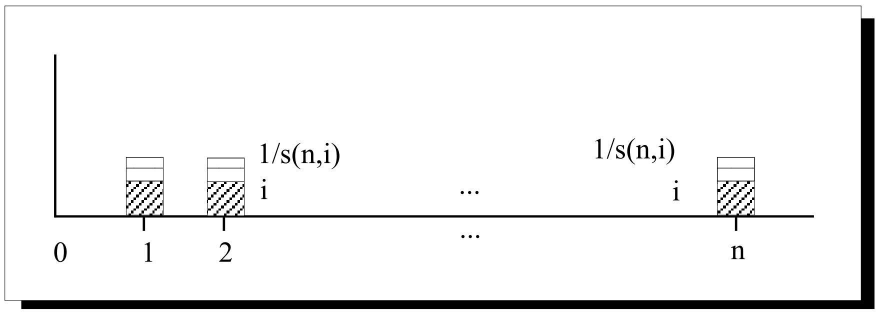
We will later have occasion to use the portion paid P = P(k) of a loan at time k which is simply one minus the balance of the loan of one at that time:

$$P = P(k) = 1 - bal(k) = 1 - \frac{a(n-k,i)}{a(n,i)} = 1 - (1+i)^k \left[1 - \frac{a(k,i)}{a(n,i)}\right].$$

We have seen that for the case of PV = 1, the n payments PMT = 1/a(n,i) will pay off the loan. That is, the present value of those equal payments is the principal amount 1 of the loan. But there are many other future series of payments--unequal payments--which would also have that present value. For instance, we could pay the same interest on one of i at the end of each period and pay no principal until the end of the n^{th} period when we pay all the principal in one "balloon payment" of one.



That unequal series of payments has the present value of one. But how will we make the balloon payment? Suppose we make a sinking fund deposit of SFF = 1/s(n,i) at times 1,2,...,n. Those deposits will accumulate to 1 at time n to give precisely the balloon payment. But that means that the equal payments at times 1,2,...,n of the interest i plus the sinking fund factor will also have the present value of one (since that pays off that loan).



But we have another series of equal payments at t = 1,2,...,n with the present value of one, namely the installments of amortize one 1/a(n,i). Hence the two payments must be equal, and we have the important formula:

$$\frac{1}{a(n,i)} = \frac{1}{s(n,i)} + i$$

In words, the installment to amortize one is the sum of sinking fund factor plus the discount rate.

Direct Capitalization Formulas

The IRV Formula

Another useful formula can be derived by considering an infinite series of equal payments called a "perpetuity." We know that the present value of a finite series of n payments PMT at t = 1,2,...,n is

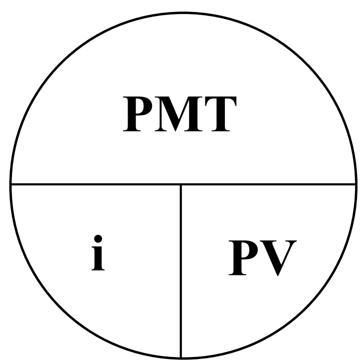
$$PV = PMT\frac{1-\frac{1}{(1+i)^n}}{i}.$$

If the series of payments goes on to infinity then we simply take $n\rightarrow\infty$ in the formula with takes the present value of one $1/(1+i)^n$ to zero. Thus we have the

$$PV = \frac{PMT}{i}.$$

Perpetuity Capitalization Formula

This is a very simple and convenient formula which can be presented in a "pie diagram."



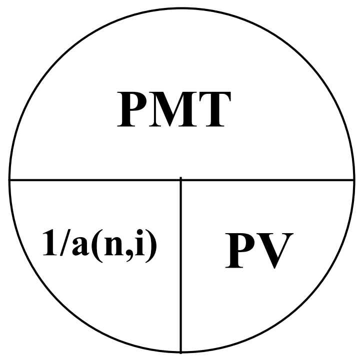
For a perpetuity payment of PMT per period, one can cover up a symbol in the pie diagram to find the formula for that amount. Cover up PV, and you see the PV = PMT/i. Cover up PMT, and you see that the perpetual payment with the present value PV is PMT = iPV.

Because of the simplicity of this type of formula, many practitioners would like to put the more complicated formulas encountered before into the same format. That is usually possible, and the results are called "direct capitalization formulas."

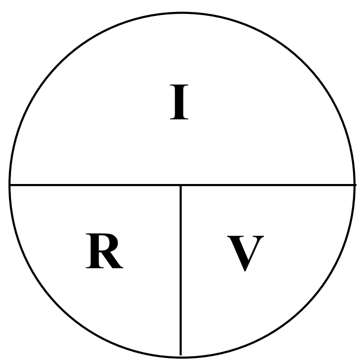
Consider, for example, the finite series of payments PMT at t = 1,2,...,n with the present value PV = PMTa(n,i). We can rewrite a(n,i) as the reciprocal of its reciprocal so that the formula is:

$$PV = PMT a(n,i) = \frac{PMT}{1/a(n,i)}.$$

Thus we see that 1/a(n,i) can be thought of as rate used to transform or "capitalize" the amount of the equal payments PMT into the present value PV. It is then called a "capitalization rate" to distinguish it from the discount rate i.



In the real estate valuation literature, the amount PMT is the income I (e.g., the net operating income NOI of an income-producing property), the capitalization rate is denoted as R, and the present value is just called the value V. Thus we have the famous I=RV formula.



I=RV Formula

We previously saw that the capitalization rate R = 1/a(n,i) could be expressed as the sum of the sinking fund factor and the discount rate so we have:

$$V = \frac{I}{\frac{1}{a(n,i)}} = \frac{I}{i + \frac{1}{s(n,i)}}.$$

Cross-multiplying shows that each income I is the sum of the amount V/s(n,i) and iV. The latter is simply the interest on the value V and it is called the "return ON investment." Since the amount V/s(n,i) is the sinking fund deposit which would accumulate to V at time n, it is called the "capital recovery" part of the income or the "return OF investment."

$$I = iV + V/s(n,i) = \text{Return on investment} + \text{Return of investment}.$$

The Cap-Rate Style of Reasoning

There are various ways to express the formulas of financial mathematics. The income approach to real estate appraisal, particularly in the USA, has developed a strong tendency to express formulas in a certain way which, in turn, promotes a certain "style" of reasoning. In making the following remarks about this "cap-rate style" our purpose is not to criticize it but only to point out that it is a free choice and that other choices would also be quite possible. Let us begin with the basic formula to capitalize a perpetual income stream of one dollar payments or incomes:

$$V = \frac{1}{i}.$$

How should the formula be changed to value a truncated income stream stopping at time n? The "cap-rate style" is to change the formula by modifying the capitalization rate to account for the truncation of the income stream at t = n to obtain:

$$a(n,i) = \frac{1}{i + \frac{1}{s(n,i)}}.$$

The new formula is explained using the reasoning about "return on investment" and "return of investment." Since the income stream terminates, the underlying asset has wasted away so the capitalization rate must be "loaded" with the sinking fund factor SFF(n,i) = 1/s(n,i) to account for the return *of* investment.

There is, however, another perfectly equivalent way to modify the perpetuity formula to account for the truncation of the income stream. Instead of changing the denominator (the capitalization rate), change the numerator (the income). Instead of loading the cap rate, we can make a deduction from the income (1 per year) to turn it into a perpetual income stream which can then be capitalized by the same denominator of i. What is the deduction to perpetualize the income--to replace the truncated stream with a perpetual stream with the same value? From the first income of 1 at time 1, set aside $1/(1+i)^n$ which is equivalent to another 1 at time n+1 (i.e., which would accumulate to 1 at time n+1 in a sinking fund). From the second income of 1 at t = 2, set aside another $1/(1+i)^n$ which accumulates to 1 at time n+2, and so forth. By making the $1/(1+i)^n$ deduction from each of the 1's in the truncated income stream, one generates another stream of 1's at the times n+1, n+2, ..., n+n. The same deductions are made from those 1's, and so forth. Thus the perpetual version of the truncated income stream of n 1's at times 1, 2,..., n is $1-1/(1+i)^n$ which can then be capitalized by dividing by the interest rate:

$$a(n,i) = \frac{1 - \frac{1}{(1+i)^n}}{i}.$$

This formula is also in the IRV format but it reflects the opposite "income style" of reasoning, i.e., modify the income instead of modifying the capitalization rate. Instead of using cap rate reasoning about loading the cap rate to account for the return of investment, we can use the familiar reasoning about charging depreciation against income so that an asset can be replaced when it wastes away. The amount $1/(1+i)^n$ is the depreciation charge against each "1" so that it can be replaced n years later to perpetuate the income stream.

We will see again and again that formulas are developed in real estate mathematics so that the changes are made to the cap rates, not the incomes. That in turn determines the style of reasoning and explanation, e.g., loading cap rates to recover capital instead of charging depreciation against income to replace capital. It is not a question of right or wrong. Both the formulas for a(n,i) are correct and equivalent. Some formulas might be more elegantly expressed by modifying cap rates, while other formulas will find simpler forms by changing the income terms. The mathematics of real estate valuation has chosen the cap-rate road, not the income road. With the increasing use of electronic computers to value uneven cash flows, the form of the formulas will become less important but the cap-rate style of reasoning will probably have a longer lasting influence.

Adjusting Capitalization Rates for Appreciation and Depreciation

We are considering a series of payments or income I that terminates at $t = n$. There is no further value after that time so this corresponds in real estate valuation to an asset or property that wastes completely away at $t = n$. Clearly there are other possibilities so we should see how the formulas in the capitalization rate format could be adjusted.

For instance, if the asset had the same value V at time n as at time 0, then it would be equivalent to the perpetuity of incomes I and the value would be $V = I/i$. Thus when the asset does not depreciate or appreciate, the sinking fund factor disappears.

What is the general formula in the capitalization rate format when we have a series of equal incomes I at $t = 1,2,...,n$ and then a future value FV at $t = n$? The total present value would be the usual sum of all the discounted values.

$$V = \frac{I}{(1+i)^1} + \frac{I}{(1+i)^2} + \ldots + \frac{I}{(1+i)^n} + \frac{FV}{(1+i)^n}$$

$$= \frac{I}{1/a(n,i)} + \frac{FV}{(1+i)^n}$$

The sinking fund deposits at $t = 1,2,...,n$ which accumulate to FV at $t = n$ are FV/s(n,i) and the present value at $t = 0$ of those deposits is

$$\frac{FV}{(1+i)^n} = \frac{FV/s(n,i)}{1/a(n,i)}.$$

Substituting in the previous formula yields

$$V = \frac{I}{1/a(n,i)} + \frac{FV}{(1+i)^n} = \frac{I + FV/s(n,i)}{1/a(n,i)}$$
$$= \frac{I + FV/s(n,i)}{\frac{1}{s(n,i)} + i}.$$

Cross-multiplying and solving for I yields

$$I = \frac{V - FV}{s(n,i)} + iV = V\left[\frac{1 - FV/V}{s(n,i)} + i\right] = V \times R^*$$

where the modified capitalization rate

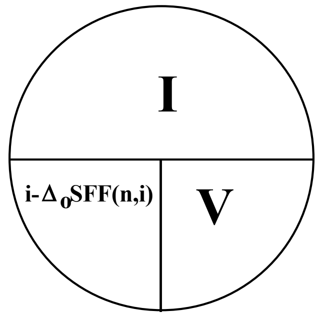
$$R^* = \frac{1 - FV/V}{s(n,i)} + i = \frac{1}{s(n,i)} - \frac{FV/V}{s(n,i)} + i$$

reflects the future value FV at t = n. When FV = V, the capitalization rate reduces to the discount rate i. When FV = 0, we have the previous formula R = 1/s(n,i) + i where the asset has wasted away at t = n.

It is convenient to restate the modified capitalization rate in terms of an appreciation ratio Δ_o so that $100\Delta_o$ is the percentage of appreciation (and where depreciation would be treated as a negative percent). The future value is $FV = (1+\Delta_o)V$. Then the capitalization rate can be expressed as

$$R^* = \frac{1}{s(n,i)} - \frac{FV/V}{s(n,i)} + i$$
$$= \frac{1 - (1 + \Delta_o)}{s(n,i)} + i$$
$$= i - \frac{\Delta_o}{s(n,i)} = i - \Delta_o SFF(n,i).$$

In the real estate literature, the subtraction of the appreciation term to find the capitalization rate R* is called "unloading" for the appreciation and "loading" for the depreciation (negative appreciation).



Direct Capitalization Formula with Appreciation or Depreciation

For no appreciation or depreciation, $\Delta_o = 0$, and for the fully depreciated asset, $\Delta_o = -1$.

Band of Investment Formulas

We have an income property which yields the net operating income NOI at the end of each year. A portion M of the value V is financed by a mortgage at the interest rate i (M is also called "loan to value ratio") so MV is the principal of the mortgage. After subtracting the debt service from the NOI, the remainder is the cash return to the equity holder which is to be discounted at the equity yield rate of Y.

A "band of investment" formula is a way to derive a direct capitalization rate R so that the value V is obtained by capitalizing the NOI, i.e., V = NOI/R. We will derive the formulas for R under a range of assumptions.

In all cases, the value of the property V is the sum of the value of the equity interest in the property plus the face value of the mortgage:

Value = Equity + Mortgage Value.

Interest-only Loan, No Change in Asset Value, and No Sale of Asset

It is assumed that the asset yields an infinite stream of annual net operating incomes NOI and that the mortgage is an interest-only loan so the debt service is MVi. Thus the equity stream capitalizes to the value [NOI - MVi]/Y and the mortgage value is MV so the total value equation is:

$$V = \frac{[\mathbf{NOI - MVi}]}{Y} + MV.$$

Collecting the V-terms to the left side we have:

$$V\left[1 + \frac{Mi}{Y} - M\right] = \frac{NOI}{Y}$$

so dividing and rearranging yields:

$$V = \frac{NOI}{Y\left[1 + \frac{Mi}{Y} - M\right]} = \frac{NOI}{\left[Mi + (1-M)Y\right]}.$$

Thus the value V could be obtained by capitalizing the NOI at the direct capitalization rate R where

$$R = Mi + (1-M)Y$$

is the weighted average of the interest rate i and the equity yield rate Y with the weights being the mortgage and equity portions of the value.

Interest-only Loan, No Change in Asset Value, and Resale of Asset after H Years.

The conditions are as above except that the asset is sold for the value V (no change in asset value) after the holding period of H years. Then the value of the equity is the present value of equity cash return over the holding period plus the present value of the sales proceeds net of paying off the mortgage:

$$\text{Equity} = a(H,Y)\left[NOI - MVi\right] + \left(1+Y\right)^{-H}\left[V - MV\right].$$

Adding in the mortgage face value yields the value equation:

$$V = a(H,Y)\left[NOI - MVi\right] + (1+Y)^{-H}\left[V - MV\right] + MV.$$

Collecting the V-terms to the left yields:

$$V\left[1 + a(H,Y)\,Mi - (1+Y)^{-H}[1-M] - M\right] = a(H,Y)\,NOI.$$

Solving for V and rearranging yields:

$$V = \frac{NOI}{\left[\frac{(1-M)}{a(H,Y)} + Mi - \frac{(1+Y)^{-H}\left[1-M\right]}{a(H,Y)}\right]}$$

where the denominator can be written as:

$$R = Mi + \frac{(1-M)\left[1-(1+Y)^{-H}\right]}{a(H,Y)}.$$

But $a(H,Y) = [1 - (1+Y)^{-H}]/Y$ so we have the previous formula:

$$R = Mi + (1-M)Y.$$

Mortgage Amortization over Holding Period, Asset Depreciation Equal to Mortgage, and Asset Resale after H Years

Assume that the mortgage with an annual interest rate of i is amortized over the holding period of H years in 12H monthly payments. The monthly payment for a loan of 1 is 1/a(12H,i/12) so the annual debt service or *mortgage constant* R_m is 12 times the monthly payment for a loan of 1. We furthermore assume that the asset value depreciates exactly as the mortgage is paid off so the resale value at the end of the holding period is V-MV (and there is no remaining mortgage to pay off). Hence the value equation is:

$$V = a(H,Y)\left[NOI - MVR_m\right] + (1+Y)^{-H}\left[V - MV\right] + MV.$$

By comparing this value equation with the previous one, we see that the only difference is that i is replaced by R_m so the direct capitalization rate will be:

$$R = MR_m + (1-M)Y.$$

Ellwood and Akerson Formulas with Constant Income

We now consider a more general case where the mortgage is amortized over a period longer than the holding period. With monthly payments, the mortgage constant R_m is 12 times the monthly payment and the balance due on the mortgage at the end of the holding period is MVbal(12H). We further assume that the asset appreciates by the proportion Δ_o over the holding period so the resale value is $(1+\Delta_o)V$. These assumptions yield the value equation:

$$V = a(H,Y)\left[NOI - R_m MV\right] + (1+Y)^{-H}\left[(1+\Delta_o)V - MVbal(12H)\right] + MV.$$

Collecting the V terms to the left yields:

$$V\left[1 + a(H,Y)R_m M - \frac{1+\Delta_o}{(1+Y)^H} + \frac{Mbal(12H)}{(1+Y)^H} - M\right] = a(H,Y)\,NOI.$$

Dividing through and rearranging terms gives:

$$V = \frac{NOI}{\left[\frac{1}{a(H,Y)} + R_m M - \frac{1+\Delta_o}{a(H,Y)(1+Y)^H} + \frac{Mbal(12H)}{a(H,Y)(1+Y)^H} - \frac{M}{a(H,Y)}\right]}.$$

The denominator is the direct capitalization rate R. We can then use the previous equations

$$s(H,Y) = a(H,Y)(1+Y)^H \quad \text{and} \quad \frac{1}{a(H,Y)} = Y + \frac{1}{s(H,Y)}$$

to simplify the rate R to

$$R = Y + \frac{1}{s(H,Y)} + R_m M - \frac{1}{s(H,Y)} - \frac{\Delta_o}{s(H,Y)} + \frac{M bal(12H)}{s(H,Y)} - MY - \frac{M}{s(H,Y)}.$$

Canceling terms and using the equations SFF(H,Y) = 1/s(H,Y) and P = P(12H) = 1-bal (12H) we can simply the expression to:

$$R = Y - M\left[Y + PSFF(H,Y) - R_m\right] - \Delta_o SFF(H,Y).$$

The expression in the square brackets is called the *Ellwood C factor* so the direct capitalization rate can be written in the Ellwood form as:

$$R = Y - MC - \Delta_o SFF(H,Y)$$

Ellwood Formula

where C = Y + P SFF(H,Y) - R_m.

If we regroup the terms in another way reminiscent of the band of investment formula than we have the:

$$R = MR_m + (1-M)Y - MPSFF(H,Y) - \Delta_o SFF(H,Y)$$

Akerson Formula.

The Valuation of Changing Income Streams

Introduction

There is, of course, a general formula for the value V of any income stream $I_1, I_2, ..., I_n$:

$$V = \sum_{k=1}^{n} \frac{I_k}{(1+i)^k}$$

but it is in fact the definition of the present value of the income stream. We will consider changing income streams where the I_k's are defined in a regular manner by some relationship, and then we will seek a concise formula for the above defined value V (that is not just the defining summation of the present values). These concise formulas are of more theoretical than practical importance in the sense that an appraiser equipped with an electronic spreadsheet can now directly use the definition to arrive at a numerical value for the present value of a projected numerical income stream.

We will present a formula for the valuation of changing income streams defined by linear recurrence relations (linear difference equations) which seems to be new and to have all the usual formulas for valuing regular income streams as special cases (e.g., straight line changing annuity, exponential or constant ratio changing annuity, and streams changing according to the Ellwood J premise).

As a special application, we show that the straight line and Hoskold methods of capitalizing income streams can be seen as the discounted present value of declining streams where the decline in income can be conceptualized as interest losses. These losses result, as it were, from a make-believe reinvestment of a capital recovery portion of the income in a hypothetical sinking fund with an interest rate below the discount rate (0 in the straight line case and some "safe" rate i_s in the Hoskold case). The declining income stream of the straight line case can be evaluated using a known formula for the straight line changing annuity. The more general formula given here is needed for the declining income stream of the Hoskold case.

Valuing Income Streams Defined by Linear Recurrence Relations

Consider the general linear recurrence relation defined by

$y_0 = c$ and $y_k = my_{k-1} + b$ for some constants m, b, and c.

The general solution has the form

$$y_n = m^n c + m^{n-1} b + \cdots + mb + b$$

which can be expressed by the formula

$$y_n = \begin{cases} m^n c + \dfrac{b\left[m^n - 1\right]}{m-1} & \text{for} \quad m \neq 1 \\ c + nb \quad \text{for} \quad m = 1. \end{cases}$$

Taking the k^{th} year's income as y_k for k = 1,...,n, the present value of the income stream is

$$V_n = \sum_{k=1}^{n} \frac{y_k}{(1+i)^k}.$$

It will be useful to notice the recurrence relation for the V_k's:

$$V_k = \frac{m}{1+i}\left[V_{k-1} + c\right] + ba(k,i).$$

In Appendix 1, we derive the formula for V_n in the following four cases where we use the notation a_n = a(n,i). Since the y_k's are defined by general linear recurrence relations, we will call the formula the **general linear recurrence valuation formula**.

Case 1 for m ≠ 1, 1+i: $$V_n = \left[\frac{b}{m-1} + c\right] \frac{m\left[1 - \left(\frac{m}{1+i}\right)^n\right]}{1+i-m} - \frac{b}{m-1} a_n$$

Case 2 for m = 1+i ≠1: $$V_n = nc + \frac{b\left[n - a_n\right]}{i}$$

Case 3 for m = 1, i ≠ 0: $$V_n = \left[c + (n+1)b\right] a_n - \frac{b\left[n - a_n\right]}{i}$$

Case 4 for m = 1, i = 0: $$V_n = nc + \frac{bn(n+1)}{2} \; .$$

General Linear Recurrence Valuation Formula

Real estate appraisal often considers an income stream of the special form

$$d,\ d - y_1 h,\ d - y_2 h, \ldots,\ d - y_{n-1} h$$

for constants d and h. The stream has the present value

$$V^* = \frac{d}{(1+i)^1} + \frac{d - y_1 h}{(1+i)^2} + \cdots + \frac{d - y_{n-1} h}{(1+i)^n} = d a_n - \frac{h}{1+i} V_{n-1}.$$

Using the recurrence relation for the V_k's, we have:

$$V^* = da_n - \frac{h}{1+i}\left[V_n - ba_n - \frac{mc}{1+i}\right]\frac{1+i}{m}$$

which simplifies to the formula for V* in terms of V_n which, in turn, can be evaluated in the previous four cases:

$$V^* = \left[d + \frac{bh}{m}\right]a_n - \frac{hV_n}{m} + \frac{hc}{1+i} \ .$$

Application 1: The Straight Line Changing Annuity Formula

The formula for valuing the linear changing annuity stream d, d-h, d-2h, ..., d-(n-1)h can be obtained by taking m = b = 1 and c = 0 so that $y_k = k$. Using the previous formula of V* and V_n in case 3 when $m = 1 \neq 1+i$, we have:

$$\begin{aligned} V^* &= \sum_{k=1}^{n} \frac{d-(k-1)h}{(1+i)^k} = [d+h]a_n - hV_n \\ &= [d+h]a_n - h(n+1)a_n + \frac{h[n-a_n]}{i} \\ &= [d-nh]a_n + \frac{h[n-a_n]}{i} \end{aligned} \ .$$

which was the previously known formula for valuing the straight line (constant amount) changing income stream.

Application 2: The Constant Ratio Changing Annuity Formula

Suppose an income stream starts with 1 at the end of year one and then grows at a rate of g for n years. To apply the general formula, take b = 0 and m = 1+g. In order to start with $y_1 = 1$, we must take $y_0 = c = 1/(1+g)$ so that $y_k = (1+g)^{k-1}$. Using the general formula in case 1, we have

$$\begin{aligned} V_n &= \left[\frac{1}{1+g}\right]\frac{(1+g)\left[1-\left(\frac{1+g}{1+i}\right)^n\right]}{i-g} \\ &= \frac{\left[1-\left(\frac{1+g}{1+i}\right)^n\right]}{i-g} \end{aligned}$$

which is the usual formula for evaluating the constant ratio changing annuity.

Application 3: The Ellwood J Factor and Ellwood R Formulas

Recall that

$$s_n = s(n,i) = (1+i)^{n-1} + (1+i)^{n-2} + ... + (1+i)^1 + 1$$
$$= \frac{(1+i)^n - 1}{i} = (1+i)^n \, a_n = \frac{1}{SFF(n,i)}$$

is the accumulation of one per period. It is useful to first use the general formula to derive the value of the stream of incomes s_1, s_2, ..., s_n at the end of years 1, 2, ..., n. In this case, m = 1+i, b = 1, and c = 0. Then the formula yields in case 2:

$$\sum_{k=1}^{n} \frac{s_k}{(1+i)^k} = \sum_{k=1}^{n} a_k = \frac{[n - a_n]}{i}.$$

The Ellwood J premise is that the income stream will change by an amount ΔI over n years after starting with a (hypothetical) value at time 0 of I (where Δ is the relative change in I). The change, however, occurs in a particular way. At the end of the k^{th} year the income is $I+s_k h$ for some fixed h. Since we must have the income at the end of the n^{th} year as $I+s_n h = I+\Delta I$ we can quickly solve for h as $h = \Delta I/s_n$. The actual income stream starts at the end of year 1 so its value is:

$$V^* = \sum_{k=1}^{n} \frac{I + s_k h}{(1+i)^k} = I a_n + h \sum_{k=1}^{n} \frac{s_k}{(1+i)^k}$$

Using the previous formula for the present value of s_k's income steam and the definition of h, we have

$$V^* = I a_n + \frac{\Delta I}{s_n} \frac{[n - a_n]}{i}$$
$$= I\left[a_n + \frac{\Delta}{s_n} \frac{[n - a_n]}{i} \right]$$

so the reciprocal of the term in the square brackets is the capitalization rate R that would yield the value as $V^* = I/R$. The cap rate R can then be simplified as follows.

$$R = \frac{1}{\left[a_n + \frac{\Delta}{s_n}\frac{[n - a_n]}{i}\right]}$$

$$= \frac{1 - (1+i)^{-n} + (1+i)^{-n}}{\left[a_n + \frac{\Delta n a_n}{s_n\left[1-(1+i)^{-n}\right]} - \frac{\Delta a_n}{s_n i}\right]}$$

$$= \frac{i a_n + a_n / s_n}{a_n\left\{1 + \Delta\left[\frac{1}{s_n}\left(\frac{n}{1-(1+i)^{-n}} - \frac{1}{i}\right)\right]\right\}}$$

Thus the capitalization rate R can be simplified to:

$$R = \frac{i + 1/s_n}{1 + \Delta J} \quad \text{where} \quad J = \frac{1}{s_n}\left(\frac{n}{1-(1+i)^{-n}} - \frac{1}{i}\right)$$

is the Ellwood J factor. We have only been considering income streams defined by certain formulas. Thus we have not considered any extra term at the end of year n for the terminal value of some underlying asset. In other words we are assuming that any underlying asset wastes away to value zero at the end of year n. Otherwise, the "1" in the numerator of the expression for R would be replaced by the relative drop Δ_o in the overall value of the asset ($\Delta_o = 1$ in our case).

Our previous presentation of the Ellwood mortgage analysis with a constant income stream can now be easily modified to accommodate an income stream changing according to the Ellwood J premise used above. Carrying over the relevant notation from our previous mortgage analysis, the value equation is:

$$V = \sum_{k=1}^{H} \frac{I + s_k h - R_m MV}{(1+Y)^k} + (1+Y)^{-H}\left[(1+\Delta_o)V - MVbal(12H)\right] + MV$$

where $s_k = s(k,Y)$ and $h = I\Delta/s_H$. Using the previous result

$$\sum_{k=1}^{H} \frac{s_k}{(1+Y)^k} = \frac{H - a_H}{Y}$$

where $a_H = a(H,Y)$, the value equation can be simplified to:

$$V = I a_H - R_m MV a_H + \frac{I\Delta\left[H - a_H\right]}{s_H Y} + (1+Y)^{-H}\left[(1+\Delta_o)V - MVbal(12H)\right] + MV.$$

Collecting the V terms on the left-hand side yields

$$V\left[1+R_m MVa_H - \frac{1+\Delta_o}{(1+Y)^{-H}} - \frac{Mbal(12H)}{(1+Y)^{-H}} - M\right] = I\left[a_H + \frac{\Delta\left[H-a_H\right]}{s_H Y}\right].$$

Then we can skip some algebra since the square brackets on the left-hand side are developed exactly as in the previous treatment of the Ellwood mortgage analysis and the square brackets on the right-hand side are developed like the treatment of Ellwood J factor above. Thus we can quickly arrive at the V = I/R formula with

$$R = \frac{Y - MC - \Delta_o\, SFF}{1+\Delta J}$$

Ellwood's R with Changes in Income and Asset Value

where Ellwood's C = Y + P SFF - R_m as before and SFF = SFF(H,Y) = $1/s_H$.

The Straight Line and Hoskold Capitalization Rates

There is some controversy in the field of real estate appraisal over the status of the so-called "straight line" method (also called "Ring" method) and the Hoskold method of determining direct capitalization rates.

Method to Determine Capitalization Rate	Return of Investment	+ Return on Investment	= Capitalization Rate R
Straight Line Method	SFF(n,0)	i	i + 1/n
Hoskold Method @ i_s	SFF(n,i_s)	i	i + SFF(n,i_s)
Annuity Method @ i	SFF(n,i)	i	1/a(n,i)

We will show that the straight line and Hoskold capitalization rates will, when divided into the first year's income, give the correct present value only for certain ***declining*** income streams.

The Straight Line Capitalization Formula

We will show that the straight line formula (as well as the Hoskold formula) applies to certain declining income streams from an income property (without any reference to a sinking fund). Sinking funds are relevant as a heuristic device because one can "motivate" the declining income stream as the combined income stream yielded by the composite investment of an income property giving a level income stream plus a sinking fund with a sub-standard interest rate. The decline in the total or composite income stream is precisely equal to the interest rate losses due to the reinvestment at a substandard interest rate. This sinking fund would usually be a hypothetical or "as if" device. The decline in the income stream is "as if" part of the proceeds of a level stream were reinvested at a "safe" rate below the prevailing interest rate.

Consider a declining income stream with d as the first year's income which then declines by the amount h each year for n years. The present value of the income stream at the discount rate i is:

$$V = \frac{d}{(1+i)^1} + \frac{d-h}{(1+i)^2} + \frac{d-2h}{(1+i)^3} + \cdots + \frac{d-(n-1)h}{(1+i)^n}.$$

The straight line changing annuity formula for this sum was previously derived.

$$V = [d-nh]a(n,i) + \frac{h[n-a(n,i)]}{i}.$$

The formula can, of course, be applied as well to straight line rising income streams by considering h as being negative.

The straight line capitalization formula can be obtained as a special case. We consider the hypothetical composite investment consisting of an income property with level income I and reinvest of the capital recovery portion of income in a mattress sinking fund. Suppose that the income only from a property is constant amount I for n years. At the end of each year part of the proceeds are reinvested in a sinking fund at the ultra-safe or "mattress" interest rate of zero. The value of the composite investment, property plus sinking fund, is V. At the end of each year, SFF(n,0)V = V/n is invested in the zero-interest sinking fund. Thus at the end of second year, there is an interest loss of h = iV/n. At the end of each subsequent year, there is an additional loss of h = iV/n. Thus the combined income stream is precisely of the straight line changing annuity kind with d = I and h = iV/n. Applying the valuation formula, we have:

$$\begin{aligned} V &= \left(I - n\frac{iV}{n}\right)a(n,i) + \frac{\frac{iV}{n}\left(n - a(n,i)\right)}{i} \\ &= Ia(n,i) - iVa(n,i) + V - \frac{Va(n,i)}{n}. \end{aligned}$$

Solving for V yields the straight line formula:

$$V = \frac{Ia(n,i)}{\left(i + \frac{1}{n}\right)a(n,i)} = \frac{I}{i + \frac{1}{n}} = \frac{I}{i + SFF(n,0)}.$$

Straight Line Capitalization Formula

Thus the specific declining income stream appropriate for the straight line formula can be motivated as the composite result of a constant income stream plus reinvestment of part of the proceeds each year in a mattress sinking fund. It is unlikely that an appraiser will be asked to appraise the composite investment of a level income property plus a mattress sinking fund. Thus it is easy to see that the sinking fund in this case is only a heuristic or hypothetical device to motivate the decline in the income stream "as if" they were the interest losses from a mattress sinking fund. The sinking fund is just as hypothetical in the Hoskold case which follows.

The Hoskold Formula

We must use case 1 in our more general valuation formula to evaluate the declining income stream that underlies the Hoskold formula. We will show that the Hoskold formula works for a certain declining income stream

$$I, I-y_1h, I-y_2h,..., I-y_{n-1}h$$

where $m = 1+i_s$, $b = 1$ and $c = 0$, and where i_s is a "safe" interest rate intermediate between i and 0. Then using the previous formula for V* with d = I, we have

$$V^* = \left[I + \frac{h}{1+i_s} \right] a_n - \frac{hV_n}{1+i_s}$$

so substituting in the formula for V_n (case 1 of $m \neq 1, 1+i$) yields after some algebra:

$$V^* = Ia_n - \frac{h}{i_s}\left[\frac{1-\left(\frac{1+i_s}{1+i}\right)^n}{i-i_s} - a_n \right].$$

V* Formula in Hoskold Case

To arrive at the specific declining income stream for the Hoskold case, we must fix h as the interest loss resulting from investing in the sub-standard sinking fund at the safe rate i_s. The declining stream is then motivated as the composite result of a constant income stream at the level d minus the interest losses in the safe sinking fund. The term subtracted from d in year k+1 for k = 1,...,k-1 is $y_k h$. Remembering that $m = 1+i_s$, $b = 1$, and $c = 0$ in this Hoskold case, the y_k term is:

$$y_k = (1+i_s)^{k-1} + \cdots + (1+i_s)^1 + 1 = s(k, i_s) = \frac{(1+i_s)^k - 1}{i_s} = \frac{1}{SFF(k, i_s)}$$

where the sinking fund factor $SFF(k,i_s)$ is the amount invested at the end of each year for k years to accumulate to 1 at the end of year k at the interest rate i_s. In our safe sinking fund, we must invest at the end of each year for n years the amount that will accumulate to V*, and that amount is $V^*SFF(n,i_s)$. After that amount is invested at the end of year 1, the interest rate loss at the end of year 2 from investing in the substandard sinking fund is $(i-i_s)V^*SFF(n,i_s)$ which should equal y_1h. At the end of year 3, there is the same loss on the amount invested at the end of year 2 but there is also the loss of what would have been the sinking fund accumulation on the previous loss. Thus the loss at the end of year 3 is

$$[(1+i_s)+1](i-i_s)V^*SFF(n,i_s) = (i-i_s)V^*SFF(n,i_s)s(2,i_s) = y_2 h\,.$$

By similar reasoning we see that the loss at the end of year k+1 is

$$(i-i_s)V^*SFF(n,i_s)s(k,i_s) = y_k h\,.$$

Since we know that $y_k = s(k,i_s)$, we see that

$$h = (i-i_s)V^*SFF(n,i_s) = \frac{(i-i_s)V^* i_s}{(1+i_s)^n - 1}$$

in the formula for V* in the Hoskold case.

Substituting h into the V* formula for the Hoskold case yields

$$V^* = Ia_n - \frac{h}{i_s}\left[\frac{1-\left(\frac{1+i_s}{1+i}\right)^n}{i-i_s} - a_n\right]$$

$$= Ia_n - \frac{(i-i_s)V^*SFF(n,i_s)}{i_s}\left[\frac{1-\left(\frac{1+i_s}{1+i}\right)^n}{i-i_s} - a_n\right]$$

which simplifies to

$$V^* = Ia_n + \frac{\left[\left(\frac{1+i_s}{1+i}\right)^n - \left(\frac{1}{1+i}\right)^n\right]V^*}{(1+i_s)^n - 1} - a_n V^*SFF(n,i_s)\ .$$

Collecting all the V* terms on the left side yields

$$V^*\left[\frac{(1+i_s)^n - 1 - \left(\frac{1+i_s}{1+i}\right)^n + \left(\frac{1}{1+i}\right)^n}{(1+i_s)^n - 1}\right] + a_n V^*SFF(n,i_s) = Ia_n$$

where the term in the square brackets simplifies to:

$$\frac{\left((1+i_s)^n-1\right)\left(1-\left(\frac{1}{1+i}\right)^n\right)}{(1+i_s)^n-1}=1-(1+i)^{-n}=ia_n\ .$$

Therefore we have $V^*[i + SFF(n,i_s)]a_n = Ia_n$ so we can cancel a_n and solve for the value V^* of the declining income stream $I, I-y_1h,...,I-y_{n-1}h$ (with $m = 1+i_s$, $b = 1$, and $c = 0$ in the definition of y_k) as:

$$V^*=\frac{I}{i+SFF(n,i_s)}\ .$$

The Hoskold Formula

Generalized Amortization Tables: The Main Theorem

We have relied mostly on the language of algebra. Since not all appraisers are fluent in that language, it might be useful to restate some of the results using amortization tables. We begin with a general result about amortization tables where the principal reductions P_1, P_2, ..., P_n are arbitrarily given along with the interest or discount rate i. The value V is the sum of the principal reductions. The incomes (or payments) per period are determined from this data. The Main Theorem is that the discounted present value of the incomes determined in this manner from the given P_k's is the value V which is the sum of the P_k's. For the results about the Ring and Hoskold methods, we consider amortization tables where the principal reductions or capital recovery entries are generated by a sinking fund at a rate r not necessarily the same as the discount rate i. When r = 0, we will have an amortization table for the straight line or Ring method which shows the declining income for that case. When $r = i_s$ between 0 and i, we have a Hoskold amortization table that shows the declining income for that case. When r = i, we have usual amortization table with level income or amortization payments. If r > i, we have an amortization table with involves capital recovery at a supra-standard rate r and which thus generates a rising income stream.

The principal or capital to be recovered is defined as the sum of those given principal reductions. Certain relationships hold between the columns in an amortization table. The interest in each year is the rate i times the balance or unrecovered capital from the previous year. The entry in the payment or income column is the sum of the interest and principal reduction (or capital recovery) columns. The entry in the balance (or unrecovered capital) column is the previous entry in the column minus the principal reduction (or capital recovery). The last entry in the balance or unrecovered capital column is zero.

Let P_1, P_2, ..., P_n be the given principal reductions, let $V = P_1+P_2+...+P_n$ be the sum, and let i be the discount rate. That is the only data given for the following general theorem about amortization tables.

General Amortization Table

Year	Income	= Interest +	Principal Reduction	Balance
1	$I_1 = P_1+i(P_1+...+P_n)$	iV	P_1	$V-P_1$
2	$I_2 = P_2+i(P_2+...+P_n)$	$i(V-P_1)$	P_2	$V-P_1-P_2$
...	...	...	...	...
k	$I_k = P_k+i(P_k+...+P_n)$	$i(V-P_1-...-P_{k-1})$	P_k	$V-P_1-...-P_k$
...	...	...	...	...
n	$I_n = P_n + iP_n$	$i(V-P_1-...-P_{n-1})$	P_n	$V-\Sigma P_k = 0$
			$\Sigma P_k = V$	

The other columns are all defined in terms of the given P_i's in the manner indicated. The incomes I_k's are determined as the sum of the Interest and Principal Reduction columns, and the general formula is

$$I_k = P_k + i(P_k +...+ P_n).$$

The Main Theorem is that the discounted present value of these incomes is the value V, the sum of the arbitrarily given P_k's.

$$\sum_{k=1}^{n} \frac{I_k}{(1+i)^k} = \sum_{k=1}^{n} P_k$$

Main Theorem on Amortization Tables

The proof in given in Appendix 2.

Amortization Tables with Sinking Fund Capital Recovery

Let V be the value of the investment (or loan) and n the number of years to recover the capital (or pay off the loan). Let i be the interest rate and r be the rate for the capital recovery sinking fund. The value of the first year's income (or payment) is I. The value V is related to the first year's income by the direct capitalization formula:

$$V = \frac{I}{i + SFF(n,r)}.$$

The new deposit in the sinking fund each year to recover the capital is SFF(n,r)V which is abbreviated SFFV. After the deposit at the end of the k^{th} year, the amount in the sinking fund is SFFVs(k,r) which abbreviated $SFFVs_k$ Therefore the capital recovery during the k^{th} year due to both the new deposit and the new interest is $SFFVs_k - SFFVs_{k-1} = SFFV(1+r)^{k-1}$ and that is the entry in the k^{th} row of the capital recovery (or principal reduction) column. Each year's income I_k beginning with $I_1 = I$ is the sum of the interest (or return on unrecovered capital) and the capital recovered (return of capital) for that year.

Amortization Table with Sinking Fund Capital Recovery

Year	Income	= Interest +	Capital Recovered	Balance
1	I	iV	SFFV	V(1-SFF)
2	I_2	iV(1-SFF)	SFFV(1+r)	$V(1-SFFs_2)$
3	I_3	$iV(1-SFFs_2)$	$SFFV(1+r)^2$	$V(1-SFFs_3)$
...	...	...	...	...
n	I_n	$iV(1-SFFs_{n-1})$	$SFFV(1+r)^{n-1}$	$V(1-SFFs_n)$

Since SFF = $1/s_n$ the last entry in the Balance or Unrecovered Capital column is 0. The sum of the Capital Recovered column is

$$\begin{aligned}&SFFV+SFFV(1+r)+SFFV(1+r)^2+\ldots+SFFV(1+r)^{n-1}\\&=SFFVs_n=V\end{aligned}$$

as desired. The incomes I_k are obtained as the sum of the Interest and Capital Recovered columns. It is useful to compute the first few incomes.

$$\begin{aligned}&I_2=iV(1-SFF)+SFFV(1+r)\\&=iV+SFFV-iSFFV+rSFFV\\&=I-(i-r)SFFV\end{aligned}$$

The income for the 2nd year is I minus (i–r)SFFV which is the interest loss on the sinking fund deposit of SFFV.

The third year's income is calculated as follows.

$$\begin{aligned}&I_3=iV(1-SFFs_2)+SFFV(1+r)^2\\&=iV-iVSFF+SFFV(1+r)-(i-r)SFFV(1+r)\\&=I_2-(i-r)SFFV(1+r)\\&=I-(i-r)SFFVs_2.\end{aligned}$$

Thus we see that each year's income I_k is I minus the interest losses on the sinking fund (assuming $r < i$) where the latter can be calculated as $(i–r)SFFVs_k$, the accumulation s_k on the interest losses $(i – r)$ on the sinking fund deposits SFFV:

$$I_k = I – (i – r)SFFVs_k.$$

Since these incomes I_k are the same as those obtained in our previous analysis of the Hoskold case, the Main Theorem on Amortization Tables now gives us another proof that the present value of these incomes is the value $V = I/[i+SFF(n,r)]$ when $r = i_s$.

In the straight line or Ring case of $r = 0$, SFF = 1/n and $s_k = k$ so the declining income is given by $I_k = I - i(V/n)k$. The income stream declines by a constant amount iV/n each year independent of k. In the Hoskold case, the drop in the income stream from I_k to I_{k+1} is $(i-r)SFFV(s_{k+1} - s_k) = (i-r)SFFV(1+r)^k$ which depends on k. Thus the Hoskold requires the formula more general than the constant amount changing annuity formula. The drop in the income stream in each period is (1+r) times the previous drop. This is illustrated in the following table based on the Hoskold situation where $0 < r < i$. The change in income accelerates at the sinking fund rate of r (as we see in the right-most column of the spreadsheet).

Amortization Table with Sinking Fund Capital Recovery: Hoskold Case

1st Income = 100.00 n = 5

i = 10% = Discount Rate V = 355.90

r = 5% = Sinking Fund Rate **% Change in**

Year	Income	Interest	Capital Recovery	Balance	ΔI	ΔI
1	100.00	35.59	64.41	291.49		
2	96.78	29.15	67.63	223.86	3.2205	
3	93.40	22.39	71.01	152.85	3.3815	5.00%
4	89.85	15.29	74.56	78.29	3.5506	5.00%
5	86.12	7.83	78.29	0.00	3.7281	5.00%
		Sum =	355.90			

In the straight line or Ring case, we set the sinking fund rate to 0.

Amortization Table with Sinking Fund Capital Recovery: Straight Line Case

1st Income = 100.00 n = 5

i = 10% = Discount Rate V = 333.33

r = 0% = Sinking Fund Rate **% Change in**

Year	Income	Interest	Capital Recovery	Balance	ΔI	ΔI
1	100.00	33.33	66.67	266.67		
2	93.33	26.67	66.67	200.00	6.6667	
3	86.67	20.00	66.67	133.33	6.6667	0.00%
4	80.00	13.33	66.67	66.67	6.6667	0.00%
5	73.33	6.67	66.67	0.00	6.6667	0.00%
		Sum =	333.33			

When r = i, we have an ordinary amortization table where i – r = 0 so the interest loss is 0 and the income is constant.

Amortization Table with Sinking Fund Capital Recovery: Ordinary Case r = i

1st Income = 100.00 n = 5

i = 10% = Discount Rate V = 379.08

r = 10% = Sinking Fund Rate

Year	Income	Interest	Capital Recovery	Balance	ΔI
1	100.00	37.91	62.09	316.99	
2	100.00	31.70	68.30	248.69	0.0000
3	100.00	24.87	75.13	173.55	0.0000
4	100.00	17.36	82.64	90.91	0.0000
5	100.00	9.09	90.91	0.00	0.0000
		Sum =	379.08		

The Internal Rate of Return

The Many Flaws and Few Benefits of IRR's

What is the criteria to use to measure the benefits of an investment project? It is the net present value or NPV of the project computed using a discount rate appropriate for the riskiness of the project. There is an old real estate saying that there are three things which determine the value of real estate for retail purposes: location, location, and location. In a similar manner, we can say there are three investment measuring devices: NPV, NPV, and NPV. The internal rate of return or IRR is not one of them.

Why analyze IRR at all? The IRR is important because it is widely used by practitioners and textbook writers. However, many of those who recommend the IRR concept seem to be unaware or only vaguely aware of the many problems with IRR's. Hence it is necessary to reiterate the many fallacies in the use of IRR's and to show the limited domain where IRR's can be correctly applied.

Definition of IRR

An investment project is defined by a series of cash flows C_0, C_1, C_2, ..., C_n, ... where C_t is the cashflow at the end of time t (time periods are taken as years). A negative cashflow C_t is an investment into the project and a positive cashflow C_t is a payout from the project. Given the discount rate i (the opportunity cost of capital to be invested in projects of similar riskiness), the ***net present value NPV of a project*** C_0, C_1, C_2, ..., C_n is:

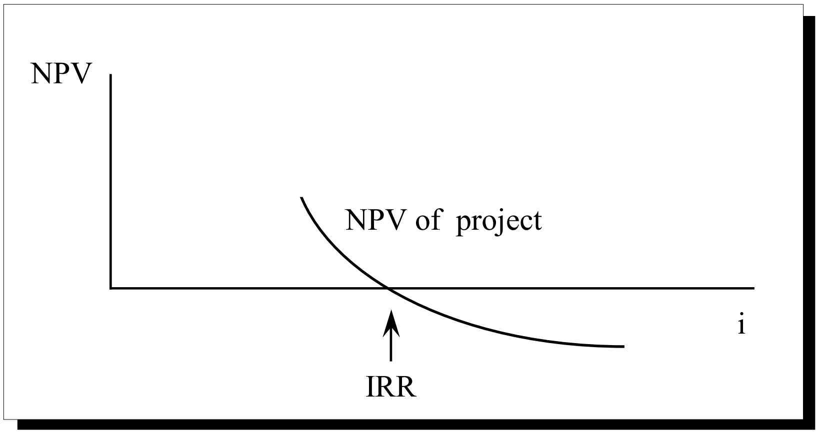
$$NPV = C_0 + \sum_{k=1}^{n} \frac{C_k}{(1+i)^k}$$

where we might write NPV(i) to make explicit the use of i as the discount rate in the definition of NPV. An ***internal rate of return IRR of the project*** can be defined as a rate which sets the net present value to zero:

$$NPV(IRR) = C_0 + \sum_{k=1}^{n} \frac{C_k}{(1+IRR)^k} = 0 .$$

While we may speak of "the" IRR of a project, there are some projects which have multiple IRR's.

If we graph NPV on the vertical axis and the discount rate i on the horizontal axis, then the IRR is the discount rate at which the NPV curve cuts the horizontal axis.



Examples of IRR's

There is no simple formula for finding an IRR. Except in a few simple cases, IRR's (as the roots of a polynomial) are best computed through an iterative procedure of ever closer approximation. Fortunately, such numerical computational procedures are now built into most hand-held financial calculators so finding IRR's is no longer a practical problem.

To construct an example with an IRR = .20 or 20%, choose any initial investment of say $1000 (so that $C_0 = -1000$), and then take the cashflows as the interest $200 until the final time period when the principal is return as well.

Project	C_0	C_1	C_2	C_3	IRR	NPV @ 10%	NPV @ 12%
A	–1000	200	200	1200	20%	$248.69	$192.15
B	–1000	500	500	500	23.38%	$243.43	$200.92
C	–1000	120	120	1120	12%	$49.74	$0

Pitfall 1 in Using IRR's: The Negative of a Project has the same IRR

One of the simplest "rules" you will find in the literature is that an investment project is profitable (that is, has positive NPV) if its IRR is greater than the interest rate i. But this cannot be true without additional assumptions since the negative of a project has the same IRR. Reversing all the cashflows reverses the role of the lender and borrower. For instance consider the negative of project A.

Project	C_0	C_1	C_2	C_3	IRR	NPV @ 10%	NPV @ 12%
–A	1000	–200	–200	–1200	20%	$–248.69	$–192.15

If the discount rate were, say, 10% or 15% then the project –A has a greater IRR of 20% but a negative NPV at those discount rates. In order for $i < IRR$ to imply $0 < NPV$, it is sufficient to assume that NPV declines as the discount rate increases, i.e., that the NPV curve slopes downward from left to right. Thus we have the rule:

If the NPV of a project declines as the discount rate i increases then $i < IRR$ implies $0 < NPV$.

Pitfall 2 in Using IRR's: "Choose the Project with the Highest IRR"

When considering the choice of projects one must be explicit about the interrelationships between the projects. Is it a situation where one can choose several projects out of a set of projects (i.e., choose all projects with positive NPV) or is one restricted to choosing only one project out of the set (i.e., choose the project with highest NPV). The alleged rule "Choose the project with the highest IRR" is usually applied in the situation where one can only choose one project out of the set of alternatives (e.g., build only one building on a site).

It is easy to see the fallacy if the projects are of quite different scale. Suppose one project turns \$100 into \$200 in one year for an IRR of 100% while another project turns \$1000 into \$1500 in a year for an IRR of only 50%. If one must choose one project or the other (and cannot repeat the first project ten times), then clearly the second project is more profitable (assuming a discount rate less than 50%) even though it has the lower IRR.

To be taken seriously, the "Highest IRR" rule should be amended to read: "Among projects with the same required investment capital, choose the project with the highest IRR." This amended rule is also wrong as can be seen by comparing projects A and B in the previous table. Both have the same invested capital of \$1000 and project B has the higher IRR (23.38% versus 20%). But at the discount rate of 10% (or lower rates), project A has the higher NPV so it is the best project at certain discount rates.

Perhaps the "Highest IRR" rule seems attractive because many practitioners incorrectly extrapolate the rule from the case of one-year projects (only one cash payout) to multi-year projects. The Highest IRR rule works for projects with the same initial capital investment and only one cash payout at the end of the period. Then it is, of course, true that the project with the highest cash payout is the best project (although both projects might have negative NPV at high discount rates).

If there is a multi-year payout, then projects begin to differ in more subtle ways. Some projects pay out early while others pay out later but in greater amounts. To know which is best, one must know how heavily to discount the future payouts--which means one must use the discount rate to compute the NPV. Thus it is easy to see that the multi-year highest IRR rule could not possibly be valid since it makes no mention of the discount rate.

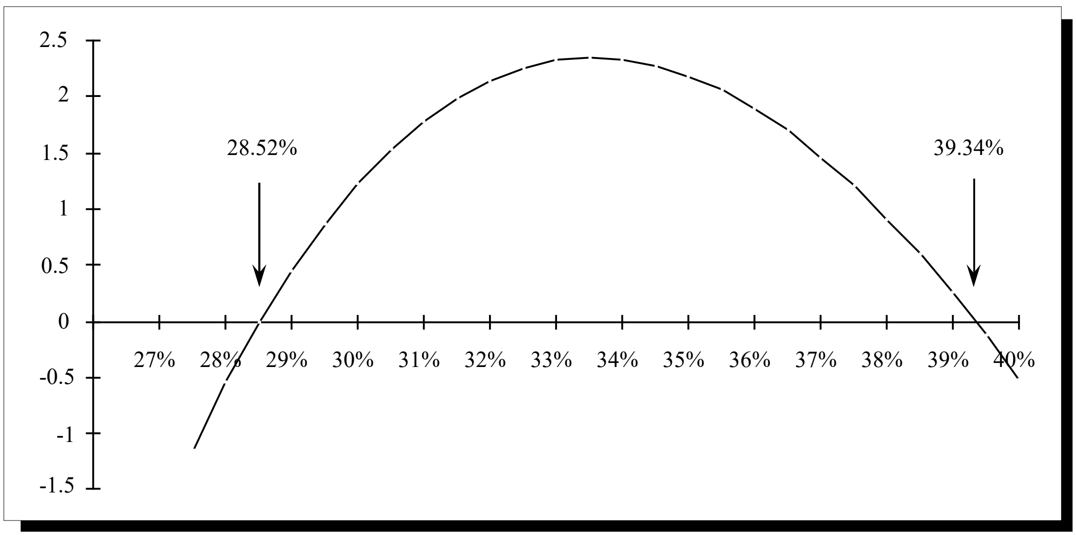
One can only ignore the discount rate when all the cash payouts from one project exceed the payouts at the same times from the other equal-investment project (which is why one could use the highest IRR rule for one-period projects).

Pitfall 3 in Using IRR's: Multiple IRR's

It is unfortunately possible for a project to have two or more IRR's. However, this can only happen if the cashflows changes signs more than once (e.g., go from negative to positive and then back to negative). Then the NPV curve could cross the horizontal axis twice giving two IRR's.

Project	C_0	C_1	C_2	C_3	IRR_1	IRR_2	NPV @ 30%
D	–1000	1450	1500	–2200	28.52%	39.34%	$1.59

Project D starts out with an investment of $1000, has two positive cash payouts, and then has a large negative closing cost of $2200 (e.g., cleaning up the environment after a project is finished).



The project has two IRR's at about 28.52% and 39.34%. In between, the project has a small positive NPV.

It might be noted that a project might have no IRR instead of multiple IRR's. For instance, if we lower the payout C_2 in project D from 1500 to 1450, then the NPV curve shifts down enough that it does not cross the horizontal axis at all so it has no IRR.

Criterion for Pair-wise Choice Between Projects

We have placed most of the emphasis on the fallacies and pitfalls in using IRRs. When can IRR's be used to make choices between investment projects? Under certain assumptions, the IRR concept can be used to make a choice between two mutually exclusive projects. We will assume that for both projects, the NPV's decline as the discount rate increases.

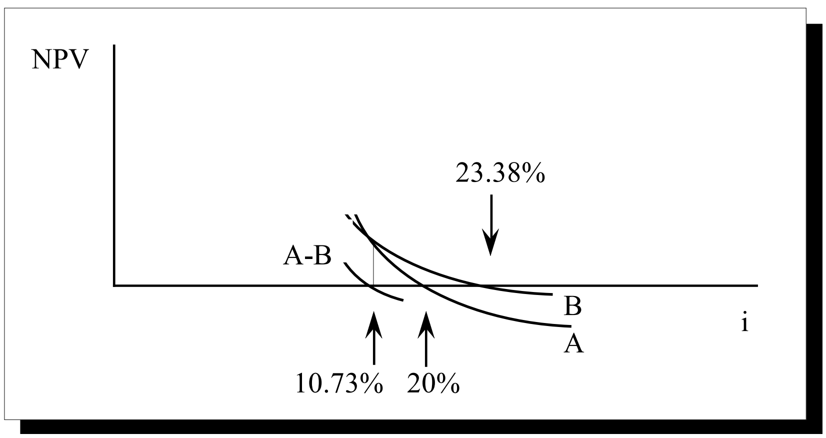
Suppose we are given a choice between two projects such as projects A and B previously considered.

Project	C_0	C_1	C_2	C_3	IRR	NPV @ 10%	NPV @ 12%
A	–1000	200	200	1200	20%	$248.69	$192.15

B	–1000	500	500	500	23.38%	$243.43	$200.92
A–B	0	–300	–300	700	10.73%	$5.26	$–8.77

We have already noted that the decision will depend on the discount rates. At 10%, A is the best project--while at 12%, B is the best project. What is the cutoff interest rate at which one project is replaced by the other as the best project? The cutoff interest rate is found by considering the IRR of the "difference project" A–B. The IRR of A–B is about 10.73% which means that for interest rates below that (such as 10%), project A is best, while for interest above that rate (such as 12%), project B is best.

One might ask, why choose A–B as the difference project? Why not B–A? The answer is that the difference project should also satisfy our rule that the NPV declines as the discount rate increases. A–B satisfies the rule while B–A does not. This can be seen from the pattern of the signs in the cashflows. If the cashflows go from negative to positive as time increases, and do not reverse later on, then the NPV curve will slope downward. Since the difference project A–B has that property, we say the "A is later than B" in the sense that A's payouts are unambiguously later than the payouts from B.



Since A is later than B, it can be intuitively understood why A is better before--and B after, the cutoff point of 10.73%. As the discount rate increases above 10.73%, both projects lose NPV but A loses NPV faster since its payouts are later and will thus be hit harder by the higher discount rate. The reverse happens as the discount rate decreases below the cutoff point.

It is also possible to understand the pair-wise choice rule in terms of our previous result that a project (with downward sloping NPV) is profitable if its IRR exceeds the discount rate. The difference project A–B can be thought of as the project of converting from B to A. If the discount rate is below the cutoff point of 10.73%, which is the IRR of the difference project (with downward sloping NPV), then it is profitable to convert from B to A, i.e., A is better than B. If the discount rate exceeds the cutoff point, then it is unprofitable to convert from B to A, i.e., B is better than A.

Appendix 1: Proof of the General Linear Recurrence Formula

Consider the general linear recurrence relation defined by

$y_0 = c$ and $y_k = my_{k-1} + b$ for some constants m, b, and c.

The general solution has the form

$$y_n = m^n c + m^{n-1} b + \cdots + mb + b$$

which can be expressed by the formula

$$y_n = \begin{cases} m^n c + \dfrac{b\left[m^n - 1\right]}{m-1} \text{ for } m \neq 1 \\ c + nb \text{ for } m = 1. \end{cases}$$

Taking the k^{th} year's income as y_k for k = 1,...,n, the present value of the income stream is

$$V_n = \sum_{k=1}^{n} \frac{y_k}{(1+i)^k}.$$

A formula for this summation will be derived for each of the four cases where m equals or does not equal 1 and 1+i.

Case 1: m ≠ 1, 1+i

Expanding the summation yields:

$$V_n = \sum_{k=1}^{n} \frac{y_k}{(1+i)^k} = \sum_{k=1}^{n} \frac{m^k c + m^{k-1} b + \cdots + mb + b}{(1+i)^k}$$

$$= c\sum_{k=1}^{n}\left(\frac{m}{1+i}\right)^k + b\sum_{k=1}^{n}\frac{\left(m^k - 1\right)/(m-1)}{(1+i)^k}$$

$$= \left[\frac{b}{m-1} + c\right]\sum_{k=1}^{n}\left(\frac{m}{1+i}\right)^k - \frac{b}{m-1} a_n .$$

Since m ≠ 1+i, the summation in the last term can be simplified.

$$V_n = \left[\frac{b}{m-1} + c\right]\frac{m\left[1 - \left(\dfrac{m}{1+i}\right)^n\right]}{1+i-m} - \frac{b}{m-1} a_n$$

Case 1 Formula

Case 2: m = 1+i ≠ 1

In this case we can easily evaluate the summation

$$\sum_{k=1}^{n}\left(\frac{m}{1+i}\right)^{k} = n$$

and m-1 = i, so the last step of the Case 1 derivation can be easily modified to yield the desired formula.

$$V_n = nc + \frac{b\left[n - a_n\right]}{i}$$

Case 2 Formula

There is some other useful information that can be extracted in this case and that will be useful later. Since m = 1+i, we have that $y_k = (1+i)^k c + s(k,i)b = (1+i)^k c + s_k b$ so the value V_n can be expressed as follows:

$$V_n = \sum_{k=1}^{n}\frac{(1+i)^k c + s_k b}{(1+i)^k} = nc + b\sum_{k=1}^{n}\frac{s_k}{(1+i)^k} = nc + b\sum_{k=1}^{n} a_k \, .$$

From the case 2 formula we can thus derive the following:

$$\sum_{k=1}^{n}\frac{s_k}{(1+i)^k} = \sum_{k=1}^{n} a_k = \frac{n - a_n}{i}.$$

There is an interesting direct and intuitive proof of this formula using the perpetuity capitalization formula. If there is the constant amount n-a(n,i) at the end of each year in perpetuity, then the present value is the right-hand side term: [n-a(n,i)]/i.

The picture below illustrates this proof for the case of n = 4. There is an array of four 1's at t = 1, 2, ... in perpetuity. Consider the column of four 1's at t = 1 and the top box of four 1's that begins at t = 2. The value of that box of 1's at t = 1 is a_4 = a(4,i) and the value of the four 1's in the column at t = 1 is, of course, 4. Thus the value of those 1's minus the box is 4-a_4 at t = 1. Then consider the next column of four 1's at t =2 and the second box of 1's that begins in the second row at t = 3. The value of those 1's minus that box at t = 2 is again 4-a_4.

t =	1	2	3	4	5	6	7 ⋯
	1	1	1	1	1	1	1 ⋯
	1	1	1	1	1	1	1 ⋯
	1	1	1	1	1	1	1 ⋯
	1	1	1	1	1	1	1 ⋯

We continue in a similar way with the process cycling at t = 5. The four 1's in the column at t = 5 are coupled with the second box in the top row starting at t = 6. The value of those 1's minus that box is $4\text{-}a_4$ at t = 5. Since this pattern repeats itself forever, the present value is $[4 - a_4]/i$. But all the 1's in boxes occurred both positively (in their column) and negatively (in their box) so they cancel out. Thus only the 1's not in any box contribute to the total value, and their present value is clearly $a_1+a_2+a_3+a_4$. Thus we have shown that

$$\sum_{k=1}^{4} a(k,i) = \frac{[4 - a(4,i)]}{i}.$$

Although illustrated for the n = 4 case, the pattern of the proof clearly works for any n.

Case 3: m = 1, m ≠ 1+i

In this case, $y_k = c + nb$ so the summation for V_n yields:

$$V_n = ca_n + b\sum_{k=1}^{n} \frac{k}{(1+i)^k} \ .$$

In the summation of the terms $k/(1+i)^k$ each such term is the present value of a 1 in a column of the following triangular array (n rows and n columns).

t =	1	2	⋯	n
	1	1	⋯	1
		1	⋯	1
			⋱	⋮
				1

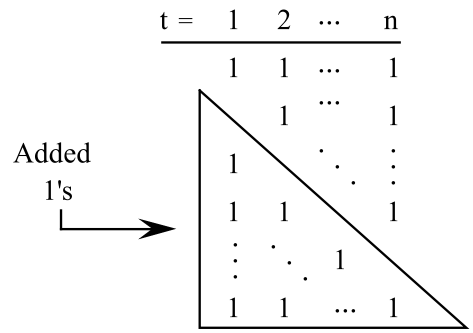
Summing the present values by rows, we have

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{k}{(1+i)^k} &= a_n + \frac{a_{n-1}}{(1+i)^1} + \cdots + \frac{a_1}{(1+i)^{n-1}} \\
&= \frac{1-1/(1+i)^n}{i} + \frac{1-1/(1+i)^{n-1}}{i(1+i)^1} + \cdots \frac{1-1/(1+i)^1}{i(1+i)^{n-1}} \\
&= \frac{(1+i)\,a_n}{i} - \frac{n/(1+i)^n}{i}.
\end{aligned}
$$

Adding and subtracting n/i allows us to simplify the sum to

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{k}{(1+i)^k} &= \frac{(1+i)\,a_n}{i} - \frac{n/(1+i)^n}{i} + \frac{n}{i} - \frac{n}{i} \\
&= (n+1)\,a_n - \frac{[n-a_n]}{i}.
\end{aligned}
$$

There is another way to arrive at this result. Suppose we complete the triangular array used above by continuing 1's down each column to form an $n \times n$ array and then add one more row of 1's at the bottom to form an $(n+1) \times n$ array.



There are then n+1 rows each with the present value a_n. But we must subtract the added 1's which have the present value $a_1+a_2+...+a_n = [n-a_n]/i$. Thus the original triangular array has the value of the difference:

$$
\sum_{k=1}^{n} \frac{k}{(1+i)^k} = (n+1)\,a_n - \frac{n-a_n}{i}.
$$

Substituting back into the formula for V_n and rearranging finishes case 3.

$$
V_n = [c+(n+1)\,b]\,a_n - \frac{b[n-a_n]}{i}
$$

Case 3 Formula

Case 4: m = 1 = 1+i

Since m = 1 and i = 0, the original summation can be quickly simplified.

$$V_n = \sum_{k=1}^{n}(c+kb) = nc + b\sum_{k=1}^{n}k$$

The summation 1+2+...+n is easily evaluated by adding it to itself written backwards:

$$\begin{array}{cccc} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \\ \hline n+1 & n+1 & \cdots & n+1 \end{array}$$

so the original sum is one-half that amount. Hence we arrive at the formula for the last case.

$$V_n = nc + \frac{bn(n+1)}{2}$$

Case 4 Formula

Appendix 2: Proof of the Main Theorem on Amortization Tables

To prove the result,

$$\sum_{k=1}^{n}\frac{I_k}{(1+i)^k}=\sum_{k=1}^{n}P_k$$

where $I_k = P_k + i(P_k+...+P_n) = (1+i)P_k + i(P_{k+1}+...+P_n)$ we need to evaluate the sum

$$\sum_{k=1}^{n}\frac{I_k}{(1+i)^k}=\sum_{k=1}^{n}\frac{(1+i)P_k+i(P_{k+1}+\cdots+P_n)}{(1+i)^k}.$$

To rearrange the sum, we consider the following table of the terms to be discounted at t=1,2,...,n. Each row gives the income for that time period, the sum of the table entries across the row times the P_j's at the head of the columns.

Time : Income	P_1	P_2	P_3	$\cdots$	P_k	$\cdots$	P_n
$t=1:I_1$	$1+i$	i	i	$\cdots$	i	$\cdots$	i
$t=2:I_2$		$1+i$	i	$\cdots$	i	$\cdots$	i
$\vdots$			$\ddots$	$\ddots$	$\vdots$	$\cdots$	$\vdots$
$t=k-1:I_{k-1}$				$1+i$	i	$\cdots$	i
$t=k:I_k$					$1+i$	$\cdots$	i
$\vdots$						$\ddots$	$\vdots$
$t=n:I_n$							$1+i$

We can now easily rewrite the sum as the discounted present value of the entries in the columns to obtain:

$$\sum_{k=1}^{n}\frac{I_k}{(1+i)^k}=\sum_{k=1}^{n}P_k\left(\frac{1}{(1+i)^k}+i\left[\frac{1}{(1+i)^k}+\cdots+\frac{1}{(1+i)^1}\right]\right)=\sum P_k\left(\frac{1}{(1+i)^k}+ia_k\right)$$

$$=\sum_{k=1}^{n}P_k\left(\frac{1}{(1+i)^k}+1-\frac{1}{(1+i)^k}\right)=\sum_{k=1}^{n}P_k$$

which completes the proof of the Main Theorem on Amortization Tables.