The Logic of Partitions

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Abstract

This book is an introduction to the logic of partitions on a set as well as the (quantum) logic of partitions (direct-sum decompositions or DSDs) on a vector space. Partitions of a set are categorically dual to subsets of a set. Thus the logic of partitions is, in that sense, the dual to the Boolean logic of subsets (usually presented as the special case of propositional logic). Since partitions can be seen as the inverse image partitions of random variables or numerical attributes (without the actual values but retaining the information as to when the values are the same or different), partition logic is the logic of random variables or numerical attributes (abstracted from the actual values). On the lattice of partitions of an arbitrary unstructured set, there is a rich algebraic structure of dual operations of implication and co-implication–resembling a non-distributive version of Heyting and co-Heyting algebras.

Subsets linearize to subspaces of a vector space and the usual quantum logic is the logic of the (closed) subspaces of the Hilbert spaces used in quantum mechanics (QM). Set partitions linearize to DSDs of a vector space so the logic of partitions linearizes to the logic of DSDs that can then be specialized to the Hilbert spaces of QM. Since each diagonalizable linear operator, e.g., the observables of QM, on a vector operator determines a DSD of eigenspaces so the quantum logic of DSDs is the logic of observables (abstracted from the actual eigenvalues).

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1.1 Introduction to partitions

1.1.1 The two mathematical logics of subsets and partitions

There are fundamentally two mathematical logics. One the Boolean logic of subsets [7], usually presented today in the special case of propositional logic, which has many sublogics and extensions, the most important being the intuitionistic logic usually modeled by the open subsets of a topological space. The other co-fundamental mathematical logic is the topic of this book, the logic of partitions. We are using "logic" in a mathematical sense as being *about* basic mathematical objects, subsets of a universe set or partitions on a universe set.[1] Logic, in this mathematical sense, is not *about* propositions, although, as with any mathematical theory, it involves propositions about the basic

[1] We are not using the word "logic" for any syntactic axiom system using logical connectives, but as a theory about certain fundamental mathematical notions. Indeed, today a Hilbert-style axiom system for partition logic has yet to be developed.

objects, e.g., that an element is in a subset or that a distinction is made by a partition. Moreover, by taking the universe set to be the one element set 1, with two subsets $\emptyset$ and 1, there is a special case of subset logic, namely propositional logic, that can be interpreted as being about propositions with $\emptyset$ representing falsehood and 1 representing truth.

In the nineteenth century, what is now called "propositional logic," was developed as the logic of subsets.

> The algebra of logic has its beginning in 1847, in the publications of Boole and De Morgan. This concerned itself at first with an algebra or calculus of classes, to which a similar algebra of relations was later added. Though it was foreshadowed in Boole's treatment of "Secondary Propositions," a true propositional calculus perhaps first appeared from this point of view in the work of Hugh MacColl, beginning in 1877. [10, pp. 155-56]

Today the original subset version of propositional logic seems to be most often noted in the context of the category-theoretic treatment.

> The propositional calculus considers "Propositions" p, q, r,... combined under the operations "and","or", "implies", and "not", often written as $p \wedge q$, $p \vee q$, $p \Rightarrow q$, and $\neg p$. Alternatively, if P, Q, R,... are subsets of some fixed set U with elements u, each proposition p may be replaced by the proposition $u \in P$ for some subset $P \subset U$; the propositional connectives above then become operations on subsets; intersection $\wedge$, union $\vee$, implication ($P \Rightarrow Q$ is $\neg P \vee Q$), and complement of subsets.[40, p. 48]

Why should partition logic be paired with subset logic as the two fundamental mathematical logics? The answer to that question awaited the development of category theory and the associated notion of duality developed in the 1940's ([14]; [38])–although it was foreshadowed in the parallelism or 'duality' between the subobjects (subsets, subgroups,...) and quotient objects (quotient sets, quotient groups,...) in pre-category-theoretic abstract algebra. The category-theoretic dual to a subset or more generally a subobject or 'part' is a quotient set (or equivalently, an equivalence relation or partition) or more generally a quotient object. "The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition." [36, p. 85] F. William Lawvere and Robert Rosebrugh go on to treat "Logic as the Algebra of Parts" [36, p. 193] but do not suggest a dual logic as the algebra of partitions.

Partition logic is at the same mathematical level as subset logic since the semantic models for both are constructed from (partitions on or subsets of) arbitrary unstructured universe sets with no topologies, no ordering relations, and no compatibility or accessibility relations on the sets.

Since the Boolean logic of subsets was known in the nineteenth century and category-theoretic duality was known since the middle of the twentieth century, why did it take until the early twenty-first century for the logic of partitions to be developed? ([15]; [16]) One partial answer is simply the almost exclusive presentation of Boolean logic as propositional logic, and "propositions" do not have a mathematical dual (only a complement). The basic object of Boolean logic, i.e., subsets, needed to presented at the right level of generality in order for the dual notion of partitions to be clear.

Another partial answer was the lack to the logical operations on partitions beyond the join and meet. Those lattice-theoretic operations on partitions were known in the nineteenth century (e.g., Richard Dedekind and Ernest Schröder), but a logic to compare to subset logic awaited at least the operation of implication on partitions, if not the whole set of 16 binary logical operations. Yet, throughout the twentieth century, the only operations on partitions that were defined and studied were the lattice operations of join and meet. In a 2001 paper commemorating Gian-Carlo Rota, the three authors first note the fundamentality of partitions and then acknowledge the sole operations of join and meet.

Equivalence relations are so ubiquitous in everyday life that we often forget about their proactive existence. Much is still unknown about equivalence relations. Were this situation remedied, the theory of equivalence relations could initiate a chain reaction generating new insights and discoveries in many fields dependent upon it.

This paper springs from a simple acknowledgement: the only operations on the family of equivalence relations fully studied, understood and deployed are the binary join $\vee$ and meet $\wedge$ operations.[9, p. 445]

Gian-Carlo Rota indeed had the idea of developing a logic of equivalence relations or partitions, but without the implication operation, the only identities would be those formulas that hold on all lattices of partitions. Moreover, it was known that partitions are so versatile that the only identities that hold on all lattices of partitions or equivalence relations are the general lattice-theoretic identities [50]. Hence without the implication or other logical operations on partitions, the only way to develop a specific logic of equivalence relations was to focus on a specific type such as commuting equivalence relations [13], and that is what Rota and colleagues did [25].

1.1.2 The duality of elements and distinctions

The duality between subsets and partitions can be traced back to a more basic duality between the elements of a subset and the distinctions of a partition ('its' and 'dits'). In the category *Set* of sets and functions, the objects are sets and the morphisms are functions between sets. The notion of a function is naturally defined using the dual notions of elements and distinctions.

Given two sets X and Y, consider a binary relation $R \subseteq X \times Y$.

The relation R is said to *transmit elements* if for all $x \in X$, there is an ordered pair $(x, y) \in R$ for some $y \in Y$.

The relation R is said to *reflect elements* if for all $y \in Y$, there is an ordered pair $(x, y) \in R$ for some $x \in X$.

The relation R is said to *transmit distinctions* if for any $(x, y) \in R$ and $(x', y') \in R$, if $x \neq x'$, then $y \neq y'$.

The relation R is said to *reflect distinctions* if for any $(x, y) \in R$ and $(x', y') \in R$, if $y \neq y'$, then $x \neq x'$.

It might be noted that the definitions of "reflect" and "transmit" just interchange the roles of X and Y. Then a binary relation R is said to be *functional* or to define a *function* if it is defined everywhere on X, i.e., transmits elements, and if it is single-valued, i.e., reflects distinctions. The dual "turn-arround-the-arrows" notion of a morphism in the opposite category Set^{op} is obtained by interchanging elements and distinctions, i.e., a *cofunction* is a binary relation R that transmits distinctions and reflects elements. In this manner, the notion of duality in *Set*, that provides the duality between subsets and partitions, can be traced back to elements and distinctions [22]. It might also be noted that when R transmits elements and reflects distinctions so that it is a function $f : X \to Y$, then the two special types of functions, injective (one-to-one) and surjective (onto) are defined respectively as transmitting distinctions and reflecting elements. Each function $f : X \to Y$ has an associated subset, namely the image $f(X) \subseteq Y$, and an associated partition, namely the inverse-image or coimage (or kernel) $\left\{f^{-1}(y)\right\}_{y \in f(X)}$.

1.1.3 Partitions and equivalence relations

A *partition* $\pi = \{..., B, ..., B', ...\}$ on a universe set U (arbitrary cardinality unless otherwise specified) is a set of subsets, called *blocks* (or sometimes *cells*), $B, B', ...$ of U that are pairwise disjoint and whose union is U. An alternative definition is that a partition $\pi = \{..., B, ..., B', ...\}$ is a set of subsets of U such that every subset $S \subseteq U$ can be uniquely represented as a union of subsets of the blocks. If the union of the blocks did not exhaust U, then the difference $U - \cup_{B \in \pi} B \neq \emptyset$ would not be represented by a union of subsets of the blocks. And if any $B \cap B' \neq \emptyset$, then that non-empty subset could be represented in two ways by subsets of the blocks.

A *distinction* or *dit* of a partition π is an *ordered* pair of elements (u, u') in different blocks of the partition, while an *indistinction* or *indit* of π is an ordered pair of elements (u, u') in the same block. The *ditset* $\operatorname{dit}(\pi) \subseteq U \times U$ of π is the sets of distinctions, and the inditset $\operatorname{indit}(\pi) = U \times U - \operatorname{dit}(\pi)$ is the set of indistinctions of π:

$$\operatorname{indit}(\pi) = \cup_{B \in \pi} B \times B \text{ and } \operatorname{dit}(\pi) = \cup_{B, B' \in \pi, B \neq B'} B \times B'.$$

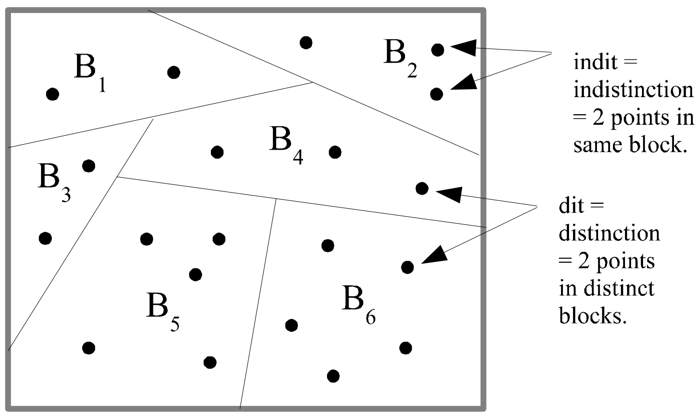

Figure 1.1: Distinctions and indistinctions of a partition

An *equivalence relation* $E \subseteq U \times U$ on U is a binary relation that is reflexive (for all $u \in U$, $(u, u) \in E$), symmetric (for any $u, u' \in U$, if $(u, u') \in E$, then $(u', u) \in E$), and transitive (for any $u, u', u'' \in U$, if $(u, u') \in E$ and $(u', u'') \in E$, then $(u, u'') \in E$). As a binary relation $\operatorname{indit}(\pi) \subseteq U \times U$ on U, the inditset is the equivalence relation associated with π, and the ditset $\operatorname{dit}(\pi) = U \times U - \operatorname{indit}(\pi)$ is its complement. A *partition relation* R on U (also called an "apartness relation") is the complement of an equivalence relation in $U \times U$ so it is irreflexive (i.e., $(u, u) \notin R$ for all $u \in U$), symmetric, and intransitive (i.e., for any $(u, u'') \in R$ and any $u' \in U$ either $(u, u') \in R$ or $(u', u'') \in R$, or, in other words, $U \times U - R$ is transitive). Every equivalence relation on U is the inditset of a partition on U (take the blocks as the equivalence classes) and vice-versa. Every partition relation on U is the ditset of a partition on U and vice-versa.

There is a suggestive analogy between equivalence and partition relations and the closed and open subsets of a topological space T. The complement of a closed set of T is an open set, and the intersection of an arbitrary number of closed subsets is closed. Given an arbitrary subset $S \subseteq T$, the topology on T defines the closure $\overline{S}$ (i.e., the intersection of all closed subsets containing S) which is the smallest closed set containing S. And given S, there is the largest open subset $\operatorname{int}(S) \subseteq S$ contained in S which is obtained by the complement of the closure of the complement, i.e., $\operatorname{int}(S) = \left(\overline{S^c}\right)^c$.

Partition logic does not assume any topology or other structure on the universe U. Nevertheless, an arbitrary subset $S \subseteq U \times U = U^2$ has a naturally defined reflexive, symmetric, and transitive (RST) closure $\overline{S}$ which could be obtained as the intersection of all equivalence relations ($\approx$ "closed subsets") containing S. That is the smallest equivalence relation or inditset containing S. Similarly, given any subset $S \subseteq U \times U$, the largest partition relation or ditset contained in S is obtained as $\operatorname{int}(S) = \left(\overline{S^c}\right)^c$. Since the intersection of an equivalence relation is an equivalence relation, we know that the union of partition relations or ditsets is always a partition relation or ditset. Thus $\operatorname{int}(S)$ can also be obtained as the union of all the partition relations contained in S. Since $\operatorname{int}(S) = \left(\overline{S^c}\right)^c = \operatorname{dit}(\pi)$ for some partition π on U, we also have: $\operatorname{indit}(\pi) = \overline{S^c}$.

But the analogy is not complete; otherwise partition logic would just be a special case of the intuitionistic logic of open subsets. In particular, the RST closure operation is not topological in the sense that the union of RST-closed subsets of $U \times U$, i.e., the union of equivalence relations, is not necessarily a RST-closed subset (an equivalence relation). As will be later seen, there are valid formulas of partition logic that are not valid in intuitionist logic and there are intuitionistic validities that are not valid in partition logic–although the valid formulas of both these logics are properly contained in the classical tautologies of subset logic.

1.2 The join and meet operations on partitions

1.2.1 The set-of-blocks definitions of join and meet

The lattice operations of join and meet of partitions will first be defined in the traditional (i.e., nineteenth century) way in terms of their blocks, and then in some equivalent new ways that generalize easily to the other logical operations. Let $\Pi(U)$ be the set of all partitions on U (where to avoid trivialities, we assume $|U| \geq 2$).

Given two partitions $\pi = \{,,, B, ..., B', ...\}$ and $\sigma = \{..., C, ..., C', ...\}$ in $\Pi(U)$, the *join* $\pi \vee \sigma$ is the partition whose blocks are the non-empty intersections $B \cap C$ for $B \in \pi$ and $C \in \sigma$. The union of two ditsets is a ditset, and indeed:

$$\mathrm{dit}(\pi) \cup \mathrm{dit}(\sigma) = \mathrm{dit}(\pi \vee \sigma).$$

The *meet* of π and σ, written $\pi \wedge \sigma$, may be defined using the notion of two blocks $B \in \pi$ and $C \in \sigma$ having a non-empty intersection, written $B \between C$ (following Ore [41]). If two blocks intersect, think of them as blobbing together like two touching drops of a liquid. Then we could have a sequence $B \between C \between B' \between C' \between ...$ and the blocks of the meet are the minimal unions of such sets of overlapping blocks that have no overlaps with blocks outside the union. Hence the blocks of the meet are the smallest subsets of U that are a union of some blocks of π and also a union of some blocks of σ. It was previously noted that the RST-closure operation was not topological in the sense that the union of two inditsets is not necessarily an inditset. Hence the intersection of two ditsets is not necessarily a ditset, but we can take the interior $\mathrm{int}[\mathrm{dit}(\pi) \cap \mathrm{dit}(\sigma)]$. To prove that is the ditset of the meet, consider two elements u and u' as being directly equated, $u \sim u'$ if u and u' are in the same block of π or σ so the set of directly equated pairs is: $\mathrm{indit}(\sigma) \cup \mathrm{indit}(\pi)$. Then u and u^* are in the same block of the meet in $\Pi(U)$ if there is a finite sequence $u = u_1 \sim u_2 \sim ... \sim u_n = u^*$ that indirectly equates u and u^*. The operation of indirectly equating two elements is just the closure operation in the closure space so the set of pairs indirectly equated, i.e., equated in the meet $\sigma \wedge \pi$ in $\Pi(U)$, is:

$$\mathrm{indit}(\sigma \wedge \pi) = \overline{(\mathrm{indit}(\sigma) \cup \mathrm{indit}(\pi))}.$$

The complementary subset of $U \times U$ is the dit set of the meet of the partitions:

$$\mathrm{dit}(\sigma \wedge \pi) = \mathrm{indit}(\sigma \wedge \pi)^c = \overline{(\mathrm{indit}(\sigma) \cup \mathrm{indit}(\pi))}^c = \mathrm{int}(\mathrm{dit}(\sigma) \cap \mathrm{dit}(\pi)).$$

The partial order on the set of partitions $\Pi(U)$ on U that makes these operations in the least upper bound and greatest lower bound respectively is the refinement partial ordering: σ is *refined by* π or π *refines* σ, written $\sigma \precsim \pi$, if for every $B \in \pi$, there is a block $C \in \sigma$ such that $B \subseteq C$. In terms of functions, π and σ define the canonical surjections $U \to \pi$ and $U \to \sigma$ (take each element of U to its block in π or σ), $\sigma \precsim \pi$ if and only if (iff) there exists a function $\pi \to \sigma$ to make the following triangle commute, i.e., $U \to \pi \to \sigma = U \to \sigma$:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \pi \arrow[dl, "\exists"'] \\ \sigma & \end{tikzcd}$$

$$\sigma \precsim \pi.$$

Refinement is a partial ordering in the sense that it is reflexive, transitive, and anti-symmetric (i.e., if $\sigma \precsim \pi$ and $\pi \precsim \sigma$, then $\sigma = \pi$). Moreover, the refinement relation on partitions is just the inclusion partial order on their ditsets (or partition relations):

$$\sigma \precsim \pi \text{ iff } \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi).$$

Then it follows from $\operatorname{dit}(\sigma) \cup \operatorname{dit}(\pi) = \operatorname{dit}(\pi \vee \sigma)$ that the join $\pi \vee \sigma$ is the least upper bound on π and σ. Similarly, it follows from $\operatorname{int}[\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)] = \operatorname{dit}(\pi \wedge \sigma)$ that the meet is the greatest lower bound on π and σ. It should be noted that many of older texts ([4]; [28]) actually deal with the lattice of equivalence relations with the partial order of inclusion even thought it may be called the "lattice of partitions." Since that is the opposite partial order, the join and meet operations are interchanged.

The join and meet operations on partitions make $\Pi(U)$ into a lattice. Moreover, the lattice $\Pi(U)$ always has a maximum element or top $\mathbf{1}_U = \{\{u\}\}_{u \in U}$, called the *discrete partition*, whose blocks are the singletons of the elements of U so that $\operatorname{dit}(\mathbf{1}_U) = U \times U - \Delta$ where $\Delta = \operatorname{indit}(\mathbf{1}_U)$ is the diagonal set of self-pairs (u, u) for $u \in U$. And the lattice also has a minimum partition or bottom $\mathbf{0}_U = \{U\}$, called the *indiscrete partition*,[2] with only one block U so that $\operatorname{dit}(\mathbf{0}_U) = \emptyset$ and $\operatorname{indit}(\mathbf{0}_U) = U \times U$.

Let $\mathcal{O}(U \times U)$ be the set of ditsets (partition relations) as the 'open' subsets of $U \times U$ ordered by inclusion and with the join being union and the meet being the interior of the intersection. The result is a lattice isomorphic to the lattice of partitions $\Pi(U)$ under the mapping $\pi \longmapsto \operatorname{dit}(\pi)$ so the lattice of partitions is represented by the 'open' subsets of $U \times U$:

$$\Pi(U) \cong \mathcal{O}(U \times U),$$

The dual counterpart to the lattice of partitions $\Pi(U)$ on U is the lattice of subsets $\wp(U)$ of U. The duality carries through to the elements and distinctions on each side as indicated in Table 1.1.

Its & Dits	Lattice $\wp(U)$ of subsets of U	Lattice of partitions $\Pi(U)$ on U
Its or Dits	Elements of subsets	Distinctions of partitions
Partial order	Inclusion of subsets	Inclusion of ditsets
Join	Union of subsets	Union of ditsets
Meet	Subset of common elements	Ditset of common dits
Top	Subset U with all elements	Partition $\mathbf{1}_U$ with all distinctions
Bottom	Subset $\emptyset$ with no elements	Partition $\mathbf{0}_U$ with no distinctions

Table 1.1: Elements and Distinctions (Its & Dits) duality between two lattices

The lattice of subsets is degenerate if the top equals the bottom as in the case of $\wp(\emptyset)$ so it is commonly assumed that $|U| \geq 1$ when working with the lattice or Boolean algebra of subsets of U. In the partition case, there two degenerate cases where the top equals the bottom, namely for $U = \emptyset$ or $U = 1$ (the one element set). The partition $\{1\}$ is the inverse-image of any function $1 \to Y$ and the empty partition $\emptyset$ is the inverse-image of the unique empty function $\emptyset \to Y$. Hence, unless otherwise stated, we will assume $|U| \geq 2$.

1.2.2 The ditset definitions of join and meet

The previous definitions of the join and meet were given in terms of the blocks of the partitions so they might be called the "set-of-blocks" definitions. But then it was noted that $\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$ and $\operatorname{dit}(\pi \wedge \sigma) = \operatorname{int}[\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)]$. That means that we can define the join and meet in terms of the ditsets and the interior operation. That gives us the ditset definitions of the lattice operations.

[2] It is nicknamed the 'blob' since it absorbs everything it meets: $\mathbf{0}_U \wedge \pi = \mathbf{0}_U$.

To see the equivalence in the join definitions, if (u, u') is in one of the ditsets, say $\operatorname{dit}(\pi)$, then $u \in B$ and $u' \in B'$ for some $B \neq B'$, so $u \in B \cap C$ for some $C \in \sigma$ and $u' \in B' \cap C'$ for some $C' \in \sigma$ (even when $C = C'$) so (u, u') is a dit of the set-of-blocks defined $\pi \vee \sigma$. Conversely if (u, u') is a dit of the set-of-blocks definition, i.e., $u \in B \cap C$ and $u' \in B' \cap C'$, and even if $B = B'$ or $C = C'$ but not both, then (u, u') is in one of the ditsets.

The ditset definition of the meet $\pi \wedge \sigma$ immediately establishes it as the greatest lower bound on π and σ so to prove the equivalence with the set-of-blocks definition, it suffices to show that the set-of-blocks approach also gives the greatest lower bound. Consider any lower bound $\tau = \{..., D, ..., D', ...\}$ such that $\tau \precsim \pi, \sigma$. Then for any $B \in \pi$, there is a $D \in \tau$ with $B \subseteq D$ and for any $C \in \sigma$, there is a $D' \in \tau$ with $C \subseteq D'$. But consider any block in $\pi \wedge \sigma$ which is an exact union of blocks of π and at the same time an exact union of blocks of σ. Consider any B contained in that meet-block. It is contained in a block $D \in \tau$, but B intersects some $C \in \sigma$, i.e., $B \between C$, and thus $B \cap C \subseteq D$. But that means that the block of τ that C is contained must be the same D. And the reasoning is similar down the chain of intersections $B \between C \between B' \between ...$ so all the π-blocks and the σ-blocks in that chain of intersections must be contained in the same $D \in \tau$, thus that block of the meet $\pi \wedge \sigma$ (the union of those intersecting blocks) must be contained in that $D \in \tau$ so $\tau \precsim \pi \wedge \sigma$, i.e., the set-of-blocks defined meet is the greatest lower bound in the refinement partial ordering.

1.2.3 The graph-theoretic definitions of join and meet

Every partition π on U defines a graph $Gph(\pi)$ on the elements of U as the vertices (or nodes) and which is simple (at most one link or arc between any two vertices), undirected, and whose set of links or arcs is simply $\operatorname{indit}(\pi)$. Thus the graph $Gph(\mathbf{0}_U)$ associated with the indiscrete partition $\mathbf{0}_U$ is the complete graph $K(U)$ on U (a link between any two vertices plus a loop at each vertex) and $Gph(\mathbf{1}_U)$ is the graph with only the loops at the vertices.

The graph-theoretic approach to defining logical operations on partitions [21] starts with the complete graph $K(U)$. Then for any partition π on U and any link $u - u'$, mark the link with $T\pi$ if $(u, u') \in \operatorname{dit}(\pi)$ and with $F\pi$ if $(u, u') \in \operatorname{indit}(\pi)$. Doing the same with another partition σ on U will result in each link being labeled with 'truth-values' such as $u \overset{T\sigma, F\pi}{\text{——}} u'$. Then for the join, meet, or any binary logical (truth-functional) operation, label the link with the appropriate truth value for that operation. For instance, for the join or disjunction, we have in Table 1.2:

π	σ	$\pi \vee \sigma$
$T\pi$	$T\sigma$	$T(\pi \vee \sigma)$
$T\pi$	$F\sigma$	$T(\pi \vee \sigma)$
$F\pi$	$T\sigma$	$T(\pi \vee \sigma)$
$F\pi$	$F\sigma$	$F(\pi \vee \sigma)$

Table 1.2: Truth table for the join.

Hence to define the partition join, we would label the link $u \overset{T\sigma, F\pi}{\text{——}} u'$ with $T(\pi \vee \sigma)$ and so forth for all the links. Then we delete all the 'truth' links labeled with $T(\pi \vee \sigma)$ leaving only the 'false' links labeled with $F(\pi \vee \sigma)$. The connected components of that 'false-graph' $Gph(\pi \vee \sigma)$ is the partition on U defined by the truth-table for the join. For the meet, we use the truth-table for the meet or conjunction, and for any other binary or n-ary truth-functional operation, we use the truth-table for that operation to define the corresponding operation on partitions.

To see that this graph-theoretic approach gives the same operation as the set-of-blocks or ditset definitions, we note that a link $u - u'$ is labeled $F(\pi \vee \sigma)$ if and only if it was labeled $F\pi$ and $F\sigma$ which means that $(u, u') \in \operatorname{indit}(\pi) \cap \operatorname{indit}(\sigma) = \operatorname{indit}(\pi \vee \sigma)$ so the definitions are equivalent.

For the meet or conjunction operation, the truth table is Table 1.3:

π	σ	$\pi \wedge \sigma$
$T\pi$	$T\sigma$	$T(\pi \wedge \sigma)$
$T\pi$	$F\sigma$	$F(\pi \wedge \sigma)$
$F\pi$	$T\sigma$	$F(\pi \wedge \sigma)$
$F\pi$	$F\sigma$	$F(\pi \wedge \sigma)$

Table 1.3: Truth table for the meet.

so the links with an $F(\pi \wedge \sigma)$ assigned to them are the ones where (u, u') is an indit of π or σ or both, i.e., $(u, u') \in \mathrm{indit}(\pi) \cup \mathrm{indit}(\sigma)$. Now $\mathrm{dit}(\pi \wedge \sigma) = \mathrm{int}[\mathrm{dit}(\pi) \cap \mathrm{dit}(\sigma)]$ so $\mathrm{indit}(\pi \wedge \sigma) = \overline{[\mathrm{indit}(\pi) \cup \mathrm{indit}(\sigma)]}$, i.e., $\mathrm{indit}(\pi \wedge \sigma)$ is the smallest equivalence relation containing $\mathrm{indit}(\pi) \cup \mathrm{indit}(\sigma)$. But we have just shown that the $F_{\pi \wedge \sigma}$-links are the indits of $\mathrm{indit}(\pi) \cup \mathrm{indit}(\sigma)$. The partition determined by that graph is its connected components and those components are the same if we take the transitive closure of the graph (i.e., put in a link $u - u'$ if there is any path of links connecting u to u'), and that transitive closure just corresponds to the closure $\overline{[\mathrm{indit}(\pi) \cup \mathrm{indit}(\sigma)]} = \mathrm{indit}(\pi \wedge \sigma)$. Hence the graph-theoretic definition of the partition meet agrees with the ditset definition.

For examples consider the two partitions $\pi = \{\{a, b\}, \{c, d\}\}$ and $\sigma = \{\{a\}, \{b, c, d\}\}$ on $U = \{a, b, c, d\}$. The complete graph with the labeled links prior to considering any particular logical operation is given in Figure 1.2.

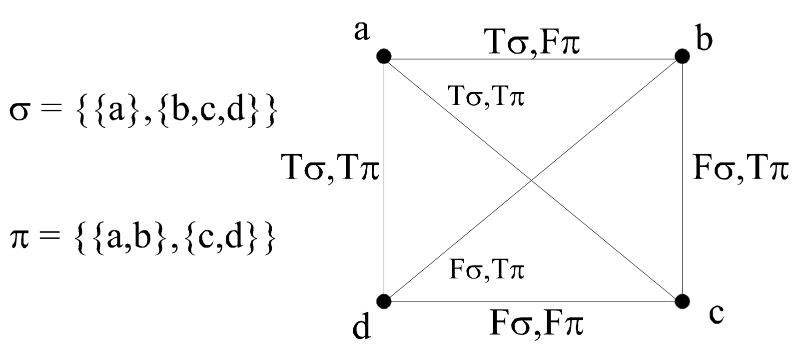

Figure 1.2: Graph with links labeled by truth-values from π and σ.

If we use the truth table for the join, than only the $d - c$ link has the $F\sigma, F\pi$ values for $F(\pi \vee \sigma)$ so it is the only surviving link when all the true-links are deleted and indicated by the thickened link in Figure 1.3. Then the connected components for $Gph(\pi \vee \sigma)$ are $\{a\}$, $\{b\}$, and $\{c, d\}$ which are the blocks of $\pi \vee \sigma$.

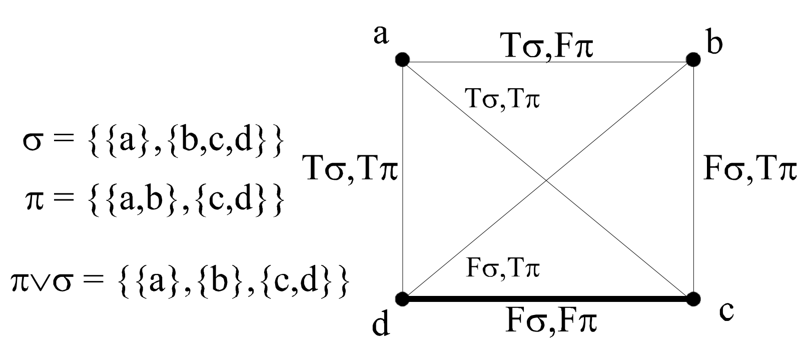

Figure 1.3: Connected components of graph with only $d - c$ link are blocks of $\pi \vee \sigma$.

If we use the truth table for the conjunction then the thickened links in Figure 1.4 are the false-links. The two links $a-d$ and $a-c$ are missing in the false-graph $Gph(\pi \wedge \sigma)$ but all the vertices are connected by false-links so there is only one connected component and thus the meet $\pi \wedge \sigma$ is the indiscrete partition $\mathbf{0}_U$.

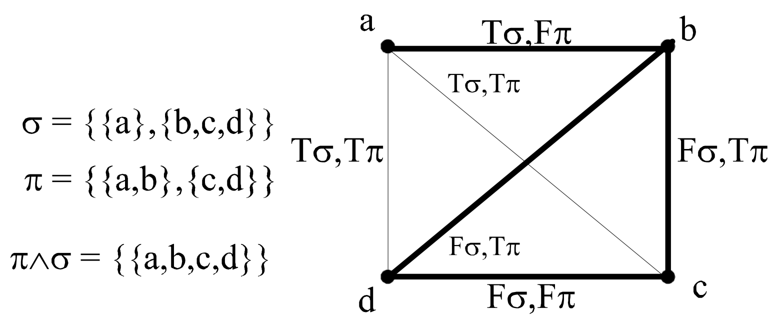

Figure 1.4: All vertices connected in graph $Gph(\pi \wedge \sigma)$ so $\pi \wedge \sigma$ is the indiscrete partition.

1.2.4 The complete Boolean subalgebra definitions of join and meet

Oystein Ore [41] noted a one-to-one correspondence between complete Boolean subalgebras of the powerset Boolean algebra $\wp(U)$ and partitions on U. Given a partition π on U, the corresponding complete Boolean subalgebra $\mathcal{B}(\pi)$ is that generated by arbitrary unions of the blocks of π, and given a complete Boolean subalgebra $\mathcal{B}$ of $\wp(U)$, the atoms or minimal subsets form a partition $\pi_{\mathcal{B}}$ on U. By this approach, it is the meet that has the easiest definition:

$$\mathcal{B}(\pi \wedge \sigma) = \mathcal{B}(\pi) \cap \mathcal{B}(\sigma)$$

since the atoms or minimal subsets of that intersection Boolean algebra would be the smallest subsets that are a union of blocks of π and at the same time a union of blocks of σ. The union of two subalgebras is not necessarily a subalgebra but it generates one by taking arbitrary unions and intersections of subsets in the subalgebras. The atoms in that generated complete Boolean subalgebra are the non-empty intersections $B \cap C$ for $B \in \pi$ and $C \in \sigma$ so we have:

$$\mathcal{B}(\pi \vee \sigma) = \overline{(\mathcal{B}(\pi) \cup \mathcal{B}(\sigma))}.$$

1.2.5 The adjunctive characterizations of join and meet

The definitions can also be approached using the concepts of category-theoretic logic. Given two partial orders P and Q, two order-preserving functions $F: P \to Q$ and $G: Q \to P$ form an *adjunction* [2, p. 191] (or *Galois connection* [4, p. 124]) if for any $p \in P$ and $q \in Q$, the following equivalence holds:

$$F(p) \leq_Q q \text{ iff } p \leq_P G(q).$$

Then F is said to be the *left adjoint* and G the *right adjoint* in the adjunction. Taking $q = F(p)$,the element $GF(p) \in P$, called the *unit*, is the least element $G(q)$ such that $p \leq_P G(q)$. And taking $p = G(q)$, the element $FG(q) \in Q$, called the *counit*, is the greatest element $F(p)$ such that $F(p) \leq_Q q$.

For instance, take $P = \Pi(U)$ ordered by refinement and $Q = \wp(U \times U)$ ordered by inclusion with $F: P \to Q$ as $\pi \longmapsto \operatorname{dit}(\pi)$ and $G: Q \to P$ as $V \longmapsto \lambda(\operatorname{int}(V))$ where $\lambda(\operatorname{int}(V))$ reads "the partition whose ditset is $\operatorname{int}(V)$." Then we have the adjunction:

$$\operatorname{dit}(\pi) \subseteq V \text{ iff } \pi \precsim \lambda(\operatorname{int}(V)).$$

For $\pi \in \Pi(U)$, the unit is $\lambda(\operatorname{int}(\operatorname{dit}(\pi))) = \pi$ which is trivially the least $\lambda(\operatorname{int}(V))$ such that $\pi \precsim \lambda(\operatorname{int}(V))$. For $V \in \wp(U \times U)$, the counit is $\operatorname{dit}(\lambda(\operatorname{int}(V))) = \operatorname{int}(V)$ which is the great ditset such that $\operatorname{dit}(\pi) \subseteq V$. A subset $V \in \wp(U \times U)$ is said to be *open* if the over-and-back operation of taking V to the counit $\operatorname{dit}(\lambda(\operatorname{int}(V))) = \operatorname{int}(V)$ is the identity, i.e., $V = \operatorname{int}(V)$. The over-and-back operation of taking each π to its unit $\lambda(\operatorname{int}(\operatorname{dit}(\pi)))$ is already the identity so the maps F and G establish an isomorphism [4, p. 124, Theorem 20] between the partitions in $\Pi(U)$ and the open subsets in $\wp(U \times U)$ which we previously established:

$$\Pi(U) \cong \mathcal{O}(U \times U).$$

It might also be noted that the adjunction works just as well if we cut down the partial orders to the corresponding upper segments, $[\pi, \mathbf{1}_U]$ and $[\operatorname{dit}(\pi), U \times U]$.

The powerset $\wp(U)$ is a category with the maps being the inclusion maps. There is then a diagonal functor $\Delta : \wp(U) \to \wp(U) \times \wp(U)$ from the powerset into the product category of the powerset times itself where $S \longmapsto (S, S)$. The diagonal functor has a left adjoint $\cup : \wp(U) \times \wp(U) \to \wp(U)$ that takes $(S, T) \longmapsto S \cup T$. That adjunction is equivalent to the statement:

$$S \cup T \subseteq W \text{ iff } (S, T) \subseteq (W, W).$$

Dually, the diagonal functor has a right adjoint that defines the intersection or meet: $\cap : \wp(U) \times \wp(U) \to \wp(U)$ where $(S, T) \longmapsto S \cap T$. That adjunction is equivalent to the statement:

$$(W, W) \subseteq (S, T) \text{ iff } W \subseteq S \cap T.$$

That is how the join and meet of sets can be defined using adjunctions [39, p. 96].

To apply this approach to partitions, we mimic it with the set of partitions $\Pi(U)$ on U partially ordered by refinement (or, equivalently, the set of ditsets $\mathcal{O}(U \times U)$ partially ordered by inclusion) replacing $\wp(U)$. Then the left adjoint to the diagonal functor $\tau \longmapsto (\tau, \tau)$ would satisfy:

$$\pi \vee \sigma \precsim \tau \text{ iff } (\pi, \sigma) \precsim (\tau, \tau)$$

which simply states that the join $\pi \vee \sigma$ is the least upper bound of π and σ in $\Pi(U)$ and that characterizes the join. Similarly, the right adjoint to the diagonal functor would satisfy:

$$(\tau, \tau) \precsim (\pi, \sigma) \text{ iff } \tau \precsim \pi \wedge \sigma$$

which simply states that the meet $\pi \wedge \sigma$ is the greatest lower bound of π and σ in $\Pi(U)$ and that characterizes the meet. The adjunctive approach thus characterizes the join and meet of partitions respectively as the least upper bound and greatest lower bound–which we have shown to exist by the previous methods.

1.3 The implication operation on partitions

1.3.1 Analogies with Heyting and bi-Heyting algebras

In the development of the logic of partitions, it is important to see analogies with a Heyting algebra (or intuitionistic propositional calculus) where the principal model is the algebra of open subsets of a topological space. We are treating the logic of partitions entirely from the semantic point of view, i.e., reasoning about partitions, not using a set of axioms. In that sense, the analogy is between the logic of open subsets of a topological space (the standard semantic model of a Heyting algebra) and the logic of partitions on a set. There are at best "analogies" since Heyting algebras are distributive lattices while the lattice of partitions is not distributive. Heyting algebras have a negation (i.e., the largest open subset disjoint from a given subset) and an implication (where the negation $\neg\sigma$ is the "implication to zero $\sigma \Rightarrow 0$). The set of negated elements in a Heyting algebra (i.e., the regular open subsets in the standard topological semantic interpretation) form a Boolean algebra.

In the logic of partitions, there is also a negation and an implication where the negation is the implication to zero $\sigma \Rightarrow \mathbf{0}_U$. In both cases, the negation can be relativized to a fixed element π (a given open subset or partition) where the π-negation is just the implication with a fixed consequent $\overset{\pi}{\neg}\sigma = \sigma \Rightarrow \pi$. In the logic of partitions as in the logic of open subsets, the negated and, in general, π-negated elements form a Boolean algebra.

There is a formal axiomatic dual to the Heyting algebra axioms and it is the set of axioms for a co-Heyting algebra where the standard semantic model is the logic of closed subsets (complements of open subsets) of a topological space. The partitions on a set $\Pi(U)$ can be represented as the open subsets or ditsets of $U \times U$ when taken as a closure space with the RST-closure operation. The complements of the open subsets are the closed subsets or inditsets of $U \times U$ which are the equivalence relations on U. Hence there is a dual logic of equivalence relations analogous to the logic of closed subsets of a topological space. The dual to the implication operation on partitions or partition relations (ditsets) is the difference operation on equivalence relations. The negation is the "difference from 1" or $1 - \sigma$ and the relativized version is the "difference from π" or $\pi - \sigma$. That dual logic of equivalence relations will be outlined here but it contains nothing really new since it is just the logic of partitions looked at from a complementary point of view. Moreover, to compare formulas between the logic of subsets (all subsets in the Boolean case and open subsets in the intuitionistic case) and the logic of partitions, we need to stick to the approach initiated above (e.g., the logic of partition relations or ditsets), not the logic of equivalence relations. For instance, the standard formula for modus ponens, $(\sigma \wedge (\sigma \Rightarrow \pi)) \Rightarrow \pi$ is $\pi - (\sigma \vee (\pi - \sigma))$ in the dual logic of equivalence relations. Hence our main development is along the lines of partitions represented by partition relations or ditsets–which is analogous to Heyting algebras or the logic of open subsets of a topological space.

Since co-Heyting algebras are axiomatized, one can consider other co-Heyting algebras that just the algebra of closed subsets ([34], [35]). Moreover, one can consider a bi-Heyting algebra or Heyting-Brouwer (HB) logic which has both implication and difference (or co-implication) operations ([43]; [52]). We will show that there is also a difference or co-implication operation on partitions–a dual structure on $\Pi(U)$–that is *not* just the complementary view in terms of equivalence relations. For instance, the operations on equivalence relations do not define any new operations on partitions since they are the same operations looked at in complementary terms. But the dual operations of co-negation and co-implication do define new operations on partitions.The logic of partitions has much of the rich dual structure of a Heyting-Brouwer logic or bi-Heyting algebra except that it is more intricate and complex since partition logic is not distributive, and it is based on an arbitrary unstructured universe set U (with no topologies or orderings).

1.3.2 The set-of-blocks definition of implication on partitions

In the Boolean algebra $\wp(U)$, for subsets $S, T \subseteq U$, the implication or conditional operation on subsets is defined as:

$$S \supset T = S^c \cup T.$$

One key property is that when the implication equals the top U of the Boolean algebra, then the partial order holds between the subsets:

$$S \supset T = U \text{ iff } S \subseteq T.$$

Another way to approach the set implication $S \supset T$ is to first consider indicator or characteristic function that indicates the degree to which S is *not* contained in T. That indicator function indicates

1 on $u \in U$ if u is an element of S that is not contained in T and indicates 0 otherwise. That function is just χ_{S-T}, the indicator function for $S - T = S \cap T^c$. Then the indicator function for the extent to which S *is* contained in T would be its negation:

$$1 - \chi_{S-T} = \chi_{(S-T)^c} = \chi_{S^c \cup T} = \chi_{S \supset T}.$$

Thus the set implication $S \supset T$ is a subset indicating the extent to which S is contained in T so when it is all true, i.e., $\chi_{S \supset T} = \chi_U$, then and only then $S \subseteq T$.

That means, in the partition case, that if the implication partition $\sigma \Rightarrow \pi$ was equal to the top, the discrete partition $\mathbf{1}_U$, then and only then would $\sigma \precsim \pi$ hold. That refinement relation holds if and only if for every $B \in \pi$, there is a $C \in \sigma$ such that $B \subseteq C$. One candidate definition that precisely satisfies that criterion is to take $\sigma \Rightarrow \pi$ to be the same as π except that when there is a $C \in \sigma$ such that $B \subseteq C$, then B is replaced by its singletons $\{u\}$ for $u \in B$. Thus if B is 'discretized' by being replaced by its singletons then it becomes the local version of the discrete partition $\mathbf{1}_B$. If B is not contained in any block of σ, then it remains as B which is the local version of the indiscrete partition $\mathbf{0}_B$. Hence in this definition of $\sigma \Rightarrow \pi$, the implication serves as a characteristic or indicator function with values or blocks $\mathbf{1}_B$ or $\mathbf{0}_B$ according to whether or not each block $B \in \pi$ is or is not contained in a block of σ. Hence this implication $\sigma \Rightarrow \pi$ indicates the degree to which π refines σ so when it is all true, then refinement holds:

$$\sigma \Rightarrow \pi = \mathbf{1}_U \text{ iff } \sigma \precsim \pi.$$

If $f, g : U \to \mathbb{R}$ are random variables on U, and $\pi = \left\{f^{-1}(r)\right\}_{r \in f(U)}$ and $\sigma = \left\{g^{-1}(r)\right\}_{r \in g(U)}$ are the inverse-image or coimage (or kernel) partitions determined by f and g, then when $\sigma \precsim \pi$, the random variable f is said to be a *sufficient statistic* for the random variable g, i.e., in an experiment, if the value of f is known, then that is sufficient to know the value of g [33, p, 31 where the opposite partial order is used.]. The implication $\sigma \Rightarrow \pi$ is thus a partition that indicates *the extent to which f is sufficient for g*, so when $\sigma \Rightarrow \pi = \mathbf{1}_U$, then f is sufficient for g.

No special attention need be paid to the complete Boolean subalgebra treatment of the partition implication since it is a trivial variation of the set-of-blocks definition. The complete Boolean subalgebra $\mathcal{B}(\sigma \Rightarrow \pi)$ is generated from $\mathcal{B}(\pi)$ when each atom $B \in \pi$ is replaced by its discretization whenever there is a $C \in \sigma$ such that $B \subseteq C$.

1.3.3 The ditset definition of the partition implication

The ditset definition of the partition meet suggests a general way to define other logical operations on partitions: apply the set definition from subset logic to the ditsets (e.g., for the meet, take the intersection $\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)$) and if the result is not a ditset, then take its interior as the ditset of the partition operation (e.g., $\operatorname{dit}(\pi \wedge \sigma) = \operatorname{int}[\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)]$). Applying that method to the problem of defining the partition implication yields the definition:

$$\operatorname{dit}(\sigma \Rightarrow \pi) = \operatorname{int}[\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi)].$$

At first that ditset definition looks totally different from the set-of-blocks definition; it would threaten the naturalness of partition logic if there were several plausible definitions of the key operation of implication. But the definitions are the same. Let $\sigma \overset{*}{\Rightarrow} \pi$ temporarily stand for the set-of-blocks definition.

Proposition 1 *$\sigma \Rightarrow \pi = \sigma \overset{*}{\Rightarrow} \pi$.*

Proof: By the two definitions, $\operatorname{dit}(\pi) \subseteq \operatorname{dit}(\sigma \Rightarrow \pi) = \operatorname{int}[\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi)]$ and $\operatorname{dit}(\pi) \subseteq \operatorname{dit}\left(\sigma \overset{*}{\Rightarrow} \pi\right)$ with the reverse inclusions holding between the indit sets. We prove the proposition by showing that $\operatorname{dit}\left(\sigma \overset{*}{\Rightarrow} \pi\right) \subseteq \operatorname{dit}(\sigma \Rightarrow \pi)$ and that $\operatorname{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right) \subseteq \operatorname{indit}(\sigma \Rightarrow \pi)$ where:

$$\text{indit}(\sigma \Rightarrow \pi) = \overline{\text{dit}(\sigma) \cap \text{dit}(\pi)^c} = \overline{(\text{indit}(\pi) \cap \text{indit}(\sigma)^c)} = \overline{(\text{indit}(\pi) - \text{indit}(\sigma))}.$$

Now suppose that $(u,u') \in \text{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right)$ (where $u \neq u'$) where $\text{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right) \subseteq \text{indit}(\pi)$ so that $u, u' \in B$ for some block $B \in \pi$. Moreover if B were contained in any block $C \in \sigma$, then $(u,u') \in \text{dit}\left(\sigma \overset{*}{\Rightarrow} \pi\right) = \text{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right)^c$ contrary to assumption so B is not contained in any $C \in \sigma$. If u and u' were in different blocks of σ then $(u,u') \notin \text{indit}(\sigma)$ so that (u,u') would not be subtracted off in the formation of $\text{indit}(\sigma \Rightarrow \pi) = \overline{(\text{indit}(\pi) - \text{indit}(\sigma))}$ and thus would be in $\text{indit}(\sigma \Rightarrow \pi)$ which was to be shown. Now suppose that u and u' are in the same block $C \in \sigma$. Thus (u,u') was subtracted off in $\text{indit}(\pi) - \text{indit}(\sigma)$ and we need to show that it is restored in the closure $\overline{(\text{indit}(\pi) - \text{indit}(\sigma))}$. Since $u, u' \in B \cap C$ but B is not contained in any one block of σ, there is another σ-block C' such that $B \cap C' \neq \emptyset$. Let $u'' \in B \cap C'$. Then (u,u'') and (u',u'') are not in $\text{indit}(\sigma)$ since $u, u' \in C$ and $u'' \in C'$ but those two pairs are in $\text{indit}(\pi)$ since $u, u', u'' \in B$. Hence the pairs $(u,u''), (u',u'') \in \text{indit}(\pi) - \text{indit}(\sigma) = \text{indit}(\pi) \cap \text{dit}(\sigma)$ which implies that (u,u') must be in the closure $\text{indit}(\sigma \Rightarrow \pi) = \overline{(\text{indit}(\pi) - \text{indit}(\sigma))}$. That establishes $\text{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right) \subseteq \text{indit}(\sigma \Rightarrow \pi)$.

To prove the converse in the form $\text{dit}\left(\sigma \overset{*}{\Rightarrow} \pi\right) \subseteq \text{dit}(\sigma \Rightarrow \pi)$, assume $(u,u') \in \text{dit}\left(\sigma \overset{*}{\Rightarrow} \pi\right)$. Since $\text{dit}(\pi) \subseteq \text{dit}(\sigma \Rightarrow \pi)$, we would be finished if $(u,u') \in \text{dit}(\pi)$. Hence assume $(u,u') \notin \text{dit}(\pi)$ so that $u, u' \in B$ for some π-block B and (u,u') is one of the new dits added when $\sigma \overset{*}{\Rightarrow} \pi$ is formed from π in the set-of-blocks definition. Thus $B \subseteq C$ for some σ-block C so that $(u,u') \in \text{indit}(\sigma)$ and (u,u') is not in the difference $\text{indit}(\pi) - \text{indit}(\sigma) = \text{indit}(\pi) \cap \text{dit}(\sigma)$. It remains to show that it is not in the closure $\text{indit}(\sigma \Rightarrow \pi) = \overline{(\text{indit}(\pi) - \text{indit}(\sigma))}$. To be in the closure, there would have to be some sequence $u = u_1, u_2, ..., u_n = u'$ such that $(u_i, u_{i+1}) \in \text{indit}(\pi) - \text{indit}(\sigma) = \text{indit}(\pi) \cap \text{dit}(\sigma)$ for $i = 1, ..., n-1$. But since all the $(u_i, u_{i+1}) \in \text{indit}(\pi)$ and $u = u_1 \in B$, all the $u = u_1, u_2, ..., u_n = u' \in B$ and $B \subseteq C$ so all the pairs $(u_i, u_{i+1}) \in \text{indit}(\sigma)$ which contradicts those pairs being in the difference $\text{indit}(\pi) - \text{indit}(\sigma) = \text{indit}(\pi) \cap \text{dit}(\sigma)$. Hence (u,u') is not in the closure $\text{indit}(\sigma \Rightarrow \pi) = \overline{(\text{indit}(\pi) - \text{indit}(\sigma))}$ so (u,u') is in the complement $\text{dit}(\sigma \Rightarrow \pi) = \text{indit}(\sigma \Rightarrow \pi)^c$ which completes the proof of the proposition. □

Henceforth, $\sigma \Rightarrow \pi$ will refer to the partition implication defined either by the ditset definition or the ser-of-blocks definition.

1.3.4 The graph-theoretic definition of the partition implication

To define the partition implication using the graph-theoretic method, we simply use the truth table for the implication (or conditional), Table 1.4, to label the links in $K(U)$ so that we may then eliminate the true-links to obtain $Gph(\sigma \Rightarrow \pi)$.

σ	π	$\sigma \Rightarrow \pi$
$T\sigma$	$T\pi$	$T(\sigma \Rightarrow \pi)$
$T\sigma$	$F\pi$	$F(\sigma \Rightarrow \pi)$
$F\sigma$	$T\pi$	$T(\sigma \Rightarrow \pi)$
$F\sigma$	$F\pi$	$T(\sigma \Rightarrow \pi)$

Table 1.4: Implication truth table for the implication.

Then the connected components of $Gph(\sigma \Rightarrow \pi)$ give the partition implication $\sigma \Rightarrow \pi$.

In the previous example of $\sigma = \{\{a\}, \{b,c,d\}\}$ and $\pi = \{\{a,b\}, \{c,d\}\}$, the only link of $K(U)$ labeled with $T\sigma$ and $F\pi$ was $a - b$ so it is the only $F(\sigma \Rightarrow \pi)$ to provide a link in $Gph(\sigma \Rightarrow \pi)$ and thus the connected components of $Gph(\sigma \Rightarrow \pi)$ are $\{a,b\}$, $\{c\}$, and $\{d\}$ so those are the blocks off $\sigma \Rightarrow \pi$ as in Figure 1.5.

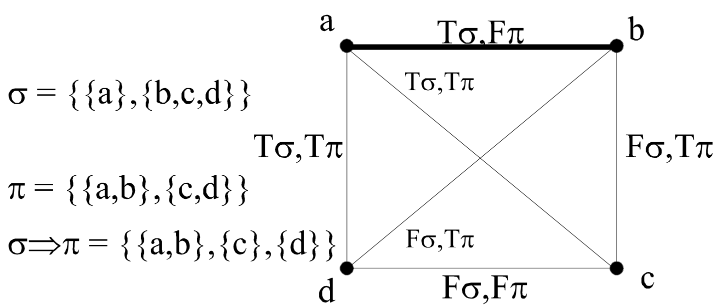

Figure 1.5: Graph-theoretic definition of $\sigma \Rightarrow \pi$.

Proposition 2 *Graph-theoretic definition = Set-of-blocks definition of* $\sigma \Rightarrow \pi$.

Proof: If $(u,u') \in \operatorname{dit}(\pi)$, then T_π is assigned to that link in $K(U)$ so u and u' are not connected in $Gph(\sigma \Rightarrow \pi)$. And if $(u,u') \in \operatorname{indit}(\pi)$ but also $(u,u') \in \operatorname{indit}(\sigma)$, then $T(\sigma \Rightarrow \pi)$ is assigned to the link in $K(U)$ so again there is no connection between u and u' in $Gph(\sigma \Rightarrow \pi)$. There is a link $u - u'$ in $Gph(\sigma \Rightarrow \pi)$ in and only in the following situation where $(u,u') \in \operatorname{indit}(\pi)$ and $(u,u') \in \operatorname{dit}(\sigma)$–which is exactly the situation when B is not contained in any block C of σ. Then for any other element $u'' \in B$ so that (u,u'') and $(u',u'') \in \operatorname{indit}(\pi)$, u''has three possible locations relative to σ, in C, in C', or outside both as pictured in Figure 1.6.

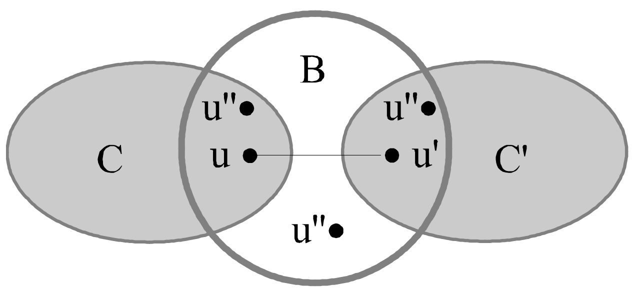

Figure 1.6: Possible locations for u''.

We must have either $(u,u'') \in \operatorname{dit}(\sigma)$ or $(u',u'') \in \operatorname{dit}(\sigma)$ so u'' is linked in $Gph(\sigma \Rightarrow \pi)$ to either u or to u', i.e., we have either $u \overline{T\sigma,F\pi} u''$or $u' \overline{T\sigma,F\pi} u''$. Thus all the elements of B are in the same connected component of the graph $Gph(\sigma \Rightarrow \pi)$ whenever B is not contained in any block of σ.

If, on the other hand, B is contained in some block C of σ, then any $u \in B$ cannot be linked to any other u' since that requires $F\pi$ assigned to the link $u - u'$ which requires $(u,u') \in \operatorname{indit}(\pi)$, i.e., u,u' both in B. And since $B \subseteq C$, they both belong to C so $F\sigma$ and thus $T(\sigma \Rightarrow \pi)$ is also assigned to that link and hence that link is eliminated in $Gph(\sigma \Rightarrow \pi)$. Thus when B is contained in a block $C \in \sigma$, then any point $u \in B$ is a disconnected component to itself in $Gph(\sigma \Rightarrow \pi)$ so B is discretized in the graph-theoretic construction of $\sigma \Rightarrow \pi$. Thus the graph-theoretic and set-of-blocks definitions of the partition implication are equivalent. □

1.3.5 The adjunctive approach to the partition implication

In the Boolean algebra of subsets, there is the "meet-with-S" functor $\wp(U) \to \wp(U)$ where $W \longmapsto W \cap S$ and the "implication-from-S" functor $\wp(U) \to \wp(U)$ where $T \longmapsto S \supset T$ which form an adjunction:

$$W \cap S \subseteq T \text{ iff } W \subseteq S \supset T$$

which characterizes $S \supset T$ as the largest subset such that modus ponens $((S \supset T) \cap S) \subseteq T$.

The analogous approach to the partition implication would be to find similar adjoints so that:

$$\tau \wedge \sigma \precsim \pi \text{ iff } \tau \precsim \sigma \Rightarrow \pi.$$

But this statement is false for the partition implication as defined above. The simplest non-trivial set of partitions is that on the three element set $U = \{a, b, c\}$ where we may take $\tau = \{\{a, b\}, \{c\}\}$, $\sigma = \{\{a, c\}, \{b\}\}$, and $\pi = \{\{a\}, \{b, c\}\}$. Then $\tau \wedge \sigma = \mathbf{0}_U$ so the left-hand side $\mathbf{0}_U \preceq \pi$ is true. But $\sigma \Rightarrow \pi = \pi$ (since no non-singleton block of π is contained in a block of σ), so the right-hand side is $\tau \precsim \pi$ which is false.

There is another way to see that the adjunction does not exist for partitions. In the Boolean algebra of subsets (or the Heyting algebra of open subsets of a topological space), that adjunction implies the distributivity of the algebra [8, p. 6]. But partition lattices $\Pi(U)$ are the standard examples of non-distributive lattices. In fact, it was an embarrassing moment for American mathematics in the nineteenth century when the mathematican-philosopher Charles Saunders Peirce claimed [42] to have proved the distributivity of all lattices but omitted the 'proof' as being too tedious. Europeans (e.g., Richard Dedekind and Ernest Schröder) soon besieged him with the example of the simplest non-trivial lattice on a three-element set shown in Figure 1.7.

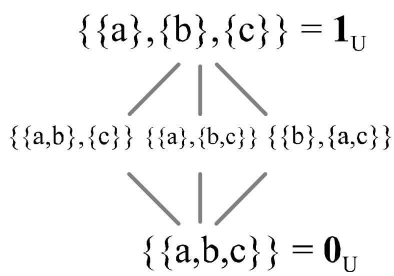

Figure 1.7: Simplest non-distributive partition lattice on $U = \{a, b, c\}$.

Using the same τ, σ, and π, $\pi \vee \sigma = \mathbf{1}_U$ so $\tau \wedge (\pi \vee \sigma) = \tau$. But $\tau \wedge \pi = \tau \wedge \sigma = \mathbf{0}_U$ so $(\tau \wedge \pi) \vee (\tau \wedge \sigma) = \mathbf{0}_U$.

But there is another approach via adjunctions. Starting with the set $U \times U$, for any $S \subseteq U \times U$, there is the usual adjunction for the Boolean conditional $\wp(U \times U) \rightleftarrows \wp(U \times U)$:

$$T \cap S \subseteq P \text{ iff } T \subseteq S \supset P$$

for any subsets $T, P \in \wp(U \times U)$. Moreover, the dit-set representation $\Pi(U) \to \wp(U \times U)$ where $\tau \longmapsto \operatorname{dit}(\tau)$ has a right adjoint where $P \in \wp(U \times U)$ is taken to the partition $G(P)$ whose dit set is $\operatorname{int}(P)$:

$$\operatorname{dit}(\tau) \subseteq P \text{ iff } \tau \precsim G(P).$$

Composing the two right adjoints $\wp(U \times U) \to \wp(U \times U) \to \Pi(U)$ gives a functor taking $P \in \wp(U \times U)$ to $G_S(P)$ which is the partition whose dit set is $\operatorname{int}(S \supset P) = \operatorname{int}(S^c \cup P)$. Its left adjoint is obtained by composing the two left adjoints $\Pi(U) \to \wp(U \times U) \to \wp(U \times U)$ to obtain a functor taking a partition τ to $F_S(\tau) = \operatorname{dit}(\tau) \cap S$:

$$F_S(\tau) = \operatorname{dit}(\tau) \cap S \subseteq P \text{ iff } \operatorname{dit}(\tau) \subseteq S \supset P \text{ iff } \tau \precsim G_S(P).^3$$

[3]Thanks to Toby Kenney for suggesting this simplified presentation of this adjunction.

Specializing $S = \operatorname{dit}(\sigma)$ and $P = \operatorname{dit}(\pi)$ gives $G_{\operatorname{dit}(\sigma)}(\operatorname{dit}(\pi))$ as the partition whose ditset is $\operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi))$ which we know from above is the partition implication $\sigma \Rightarrow \pi$, i.e., $G_{\operatorname{dit}(\sigma)}(\operatorname{dit}(\pi)) = \sigma \Rightarrow \pi$. Using these restrictions, the adjunction gives the iff statement characterizing the partition implication:

$$\operatorname{dit}(\tau) \cap \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi) \text{ iff } \tau \preceq \sigma \Rightarrow \pi.$$

Characterization of $\sigma \Rightarrow \pi$.

Taking $\tau = \sigma \Rightarrow \pi$, we see that $\sigma \Rightarrow \pi$ is the most refined partition such that modus ponens holds. That is, the partition implication $\sigma \Rightarrow \pi$ is the most refined partition τ such that $\operatorname{dit}(\tau) \cap \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$ and thus that $(\sigma \Rightarrow \pi) \wedge \sigma \precsim \pi$ since $\operatorname{int}[\operatorname{dit}(\sigma \Rightarrow \pi) \cap \operatorname{dit}(\sigma)] = \operatorname{dit}((\sigma \Rightarrow \pi) \wedge \sigma)$. And $(\sigma \Rightarrow \pi) \wedge \sigma \precsim \pi$ is equivalent to $[(\sigma \Rightarrow \pi) \wedge \sigma] \Rightarrow \pi = \mathbf{1}_U$.

There are some other ways to state this characterization of the partition implication. Since arbitrary unions of ditsets are ditsets, we have:

$$\operatorname{dit}(\sigma \Rightarrow \pi) = \bigcup \{\operatorname{dit}(\tau) : \operatorname{dit}(\tau) \cap \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)\}.$$

Or taking the join in the complete lattice $\Pi(U)$,

$$\sigma \Rightarrow \pi = \bigvee \{\tau : \operatorname{dit}(\tau) \cap \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)\}.$$

Henceforth, the partition lattice $\Pi(U)$ equipped with the implication operation will be referred to as the *partition algebra* $\Pi(U)$.

1.4 Negation and other operations on partitions

1.4.1 Negation in partition logic

From the Boolean logic of subsets and the intuitionistic logic of open subsets, we have the suggestion to define the negation of a partition $\neg\sigma$ as the implication to zero, i.e., $\sigma \Rightarrow \mathbf{0}_U$. But in partition logic, this is immediately seen (using the set-of-blocks definition) to be rather trivial since for any $\sigma \neq \mathbf{0}_U$, $\sigma \Rightarrow \mathbf{0}_U = \mathbf{0}_U$ and for $\sigma = \mathbf{0}_U$, $\mathbf{0}_U \Rightarrow \mathbf{0}_U = \mathbf{1}_U$. In intuitionistic logic, the negation of an open subset is the largest open subset disjoint from the given open set. In the ditset representation, that would mean the ditset of the negation of a partition is the largest ditset disjoint from a given ditset. But in the partition case, for any $\sigma \neq \mathbf{0}_U$, i.e., for any non-empty ditset $\operatorname{dit}(\sigma)$, there is *no* non-empty ditset disjoint from it. That means that any two non-empty ditsets always have a non-empty intersection, Thus for $\pi, \sigma \neq \mathbf{0}_U$, there is always a pair $\{u, u'\}$ that are in different blocks of both π and σ. This may also be stated as: if $\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma) = \emptyset$, then $\operatorname{dit}(\pi) = \emptyset$ or $\operatorname{dit}(\sigma) = \emptyset$. Or in terms of inditsets or equivalence relations: if $\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma) = U \times U$, then $\operatorname{indit}(\pi) = U \times U$ or $\operatorname{indit}(\sigma) = U \times U$.

That is an interesting result in its own right, and in view of the connections between partitions and graphs, it is a known result in graph theory. If $\sigma \neq \mathbf{0}_U$, then the graph on U with the links $\operatorname{indit}(\sigma)$ is disconnected. The largest graph on U with links disjoint from that $\operatorname{indit}(\sigma)$-graph is the complementary graph whose links are given by $\operatorname{dit}(\sigma)$. The graph theorem is that the complementary graph of a disconnected graph is connected [51, p. 30], so with each vertex connected by links to every other vertex, the partition corresponding to a connected graph is $\mathbf{0}_U$, i.e., $\neg\sigma = \sigma \Rightarrow \mathbf{0}_U = \mathbf{0}_U$.

Theorem 1 (Common dits) *For any partitions $\pi \neq \mathbf{0}_U \neq \sigma$, $\operatorname{dit}(\pi) \cap \operatorname{dit}(\pi) \neq \emptyset$.*

Proof: Since π is not the blob $\mathbf{0}_U$, consider two elements u and u' distinguished by π but identified by σ; otherwise $(u, u') \in \operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)$ and we are finished. Since σ is also not the blob, there must be a third element u'' not in the same block of σ as u and u'.

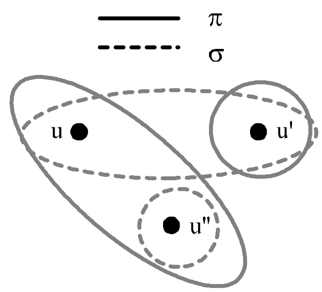

Figure 1.8: (u', u'') as a common dit

But since u and u' are in different blocks of π, the third element u'' must be distinguished from one or the other or both in π, e.g., distinguished from u' in both partitions as in Figure 1.8. Hence (u, u'') or (u', u'') must be distinguished by both partitions and thus must be in $\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)$. $\square$

One implication is that if $\sigma \neq \mathbf{0}_U$, then $\overline{\operatorname{dit}(\sigma)} = U \times U$, i.e., the RST-closure of any non-empty ditset is $U \times U$, since $\left(\overline{\operatorname{dit}(\sigma)}\right)^c$ is the ditset of the largest ditset disjoint from $\operatorname{dit}(\sigma)$ but that is $\operatorname{dit}(\mathbf{0}_U) = \emptyset$.

In spite of some similarities between partition logic and intuitionistic propositional logic, e.g., as modeled by open subsets of a topological space, there are some subtle differences. Both are lattices. In a lattice, a *pseudo-complement* [6, p.103] of an element σ (if it exists) is the maximal element σ^* such that $\sigma \wedge \sigma^* = 0$. If every element has a pseudo-complement, then the lattice is *pseudo-complemented*. In the lattice of open subsets of a topological space, the pseudo-complement of an open subset σ is the largest open subset disjoint from it, i.e., the interior of the complement, and that pseudo-complement might be denoted $\sigma \Rightarrow 0$. In the lattice of partitions $\Pi(U)$, for any partition σ, there is a largest ditset disjoint from $\operatorname{dit}(\sigma)$ and it is the ditset for the partition $\sigma \Rightarrow \mathbf{0}_U$. But this partition is not a pseudo-complement and $\Pi(U)$ is not pseudo-complemented. This is particularly striking since $\sigma \Rightarrow \mathbf{0}_U$ can be defined as having the largest ditset $\operatorname{dit}(\tau)$ such that $\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau) = \emptyset$. But that is *not* the same as the maximal partition τ such that $\sigma \wedge \tau = \mathbf{0}_U$ (recall that $\operatorname{dit}(\sigma \wedge \tau) = \operatorname{int}[\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau)]$).

Example: To show that $\sigma \Rightarrow \mathbf{0}_U$ is not the pseudo-complement of σ, let $\tau = \{\{a, b\}, \{c\}\}$ and $\sigma = \{\{a, c\}, \{b\}\}$ as in the previous example on $U = \{a, b, c\}$. Then $\tau \wedge \sigma = \mathbf{0}_U$ and $(\tau \wedge \sigma) \wedge ((\tau \wedge \sigma) \Rightarrow \mathbf{0}_U) = \mathbf{0}_U$. If $\sigma \Rightarrow \mathbf{0}_U$ was the pseudo-complement of σ, then it is the largest (most refined) partition π to satisfy $\pi \wedge \sigma = \mathbf{0}_U$ so

$$\tau \wedge ((\tau \wedge \sigma) \Rightarrow \mathbf{0}_U) \overset{?}{\precsim} \sigma \Rightarrow \mathbf{0}_U.$$

But that refinement is false since $(\tau \wedge \sigma) \Rightarrow \mathbf{0}_U = \mathbf{0}_U \Rightarrow \mathbf{0}_U = \mathbf{1}_U$ and $\tau \wedge \mathbf{1}_U = \tau$ so the LHS is $\tau \neq \mathbf{0}_U$ while the RHS is $\mathbf{0}_U$. The negation $\sigma \Rightarrow \mathbf{0}_U$ has the largest ditset disjoint from $\operatorname{dit}(\sigma)$ and $\sigma \wedge (\sigma \Rightarrow \mathbf{0}_U) = \mathbf{0}_U$. We also have $\tau \wedge ((\tau \wedge \sigma) \Rightarrow \mathbf{0}_U) \wedge \sigma = \mathbf{0}_U$, but the ditsets are not even disjoint since $\operatorname{dit}(\tau \wedge ((\tau \wedge \sigma) \Rightarrow \mathbf{0}_U)) = \operatorname{dit}(\tau) = \{(a, c), (b, c), ...\}$ and $\operatorname{dit}(\sigma) = \{(a, b), (c, b), ...\}$ so $\operatorname{dit}(\tau \wedge ((\tau \wedge \sigma) \Rightarrow \mathbf{0}_U)) \cap \operatorname{dit}(\sigma) = \{(b, c), (c, b)\}$.

1.4.2 Relative negation in partition logic

In intuitionistic propositional logic (or in a Heyting algebra), the negated elements, called the *regular* elements (since they correspond to regular open sets, i.e., the interiors of closed sets, in the topological interpretation), form a Boolean algebra. In a partition algebra, the negated elements $\sigma \Rightarrow \mathbf{0}_U$ also form a Boolean algebra but is the trivial two-element one consisting of $\mathbf{0}_U$ and $\mathbf{1}_U$. But the general implication $\sigma \Rightarrow \pi$ could be seen as an indicator function with values $\mathbf{1}_B$ and $\mathbf{0}_B$ indicating whether each block $B \in \pi$ is a subset of a block of σ or not. This suggests a non-trivial notion of negation where the indicator function's values are reversed, and that is given by the partition $(\sigma \Rightarrow \pi) \Rightarrow \pi$.

Hence we consider the implication $\sigma \Rightarrow \pi$ as the *relative negation* of σ by π and denote this π-*negation* by $\overset{\pi}{\neg}\sigma = \sigma \Rightarrow \pi$. Then the formula $(\sigma \Rightarrow \pi) \Rightarrow \pi$ is just the double π-negation $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$.The triple π-negation reverses the indicator values again to return to the original single π-negation so that:

$$\overset{\pi}{\neg}\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma = \overset{\pi}{\neg}\sigma.$$

A π-negated partition is said to be π-*regular.* All the π-regular partitions are in the upper segment $[\pi, \mathbf{1}_U] = \{\tau : \pi \precsim \tau\}$, which is the lattice of partitions between π and $\mathbf{1}_U$ (including the endpoints). Let the set of π-regular partitions be denoted $\mathcal{B}[\pi, \mathbf{1}_U] \subseteq \Pi(U)$. We have just seen that the π-negation of a π-regular partition as like the Boolean negation in $\mathcal{B}[\pi, \mathbf{1}_U]$ by considering each π-regular partition as an indicator function. Do the join and meet operations in $\Pi(U)$ act like the Boolean join and meet in $\mathcal{B}[\pi, \mathbf{1}_U]$? For any other partitions $\sigma, \tau \in \Pi(U)$, the partition join $\sigma \Rightarrow \pi \vee \tau \Rightarrow \pi$ of two π-regular partition can be analyzed using a truth table of its indicator values as in Table 1.51.

$\sigma \Rightarrow \pi$	$\tau \Rightarrow \pi$	$(\sigma \Rightarrow \pi) \vee (\tau \Rightarrow \pi)$
$\mathbf{1}_B$	$\mathbf{1}_B$	$\mathbf{1}_B$
$\mathbf{1}_B$	$\mathbf{0}_B$	$\mathbf{1}_B$
$\mathbf{0}_B$	$\mathbf{1}_B$	$\mathbf{1}_B$
$\mathbf{0}_B$	$\mathbf{0}_B$	$\mathbf{0}_B$

Table 1.5: Join operation on π-regular partitions

The blocks of the join are the non-empty intersections of the blocks of the two partitions. In any of the cases where a block B was discretized to $\mathbf{1}_B$, the singleton blocks $\{u\}$ for $u \in B$ would always yield the same singleton blocks when intersected with any other blocks so all the cases where a block B was atomized to $\mathbf{1}_B$, the join has that same value as indicated in Table 1.5 (the first three cases). If B was not atomized in either π-regular partition, then $B = \mathbf{0}_B$ would be the intersection in the join. But is $(\sigma \Rightarrow \pi) \vee (\tau \Rightarrow \pi)$ a π-regular partition in $\mathcal{B}[\pi, \mathbf{1}_U]$? Since $(\sigma \Rightarrow \pi) \vee (\tau \Rightarrow \pi)$ has values like a π-regular partition, it would equal its double π-negation which is a π-regular partition.

The same exercise can be carried out for the meet of two π-regular partitions.

$\sigma \Rightarrow \pi$	$\tau \Rightarrow \pi$	$(\sigma \Rightarrow \pi) \wedge (\tau \Rightarrow \pi)$
$\mathbf{1}_B$	$\mathbf{1}_B$	$\mathbf{1}_B$
$\mathbf{1}_B$	$\mathbf{0}_B$	$\mathbf{0}_B$
$\mathbf{0}_B$	$\mathbf{1}_B$	$\mathbf{0}_B$
$\mathbf{0}_B$	$\mathbf{0}_B$	$\mathbf{0}_B$

Table 1.6: Meet operation on π-regular partitions.

Whenever one of the values for a π-regular partition $\sigma \Rightarrow \pi$ or $\tau \Rightarrow \pi$ is $\mathbf{0}_B$ in the first two columns of Table 1.6, then it absorbs the other B-value $\mathbf{1}_B$ or $\mathbf{0}_B$, so the result for that row is also $\mathbf{0}_B$ (the last three rows). In the remaining case of two $\mathbf{1}_B$ values, the singletons only intersect with themselves so that value is $\mathbf{1}_B$ in the meet. Since the meet has only the values of $\mathbf{0}_B$ and $\mathbf{1}_B$, its double π-negation has the same values and is a π-regular partition.

The exercise might also be carried out for the implication of two π-regular partitions as indicated in Table 1.7.

$\sigma \Rightarrow \pi$	$\tau \Rightarrow \pi$	$(\sigma \Rightarrow \pi) \Rightarrow (\tau \Rightarrow \pi)$
$\mathbf{1}_B$	$\mathbf{1}_B$	$\mathbf{1}_B$
$\mathbf{1}_B$	$\mathbf{0}_B$	$\mathbf{0}_B$
$\mathbf{0}_B$	$\mathbf{1}_B$	$\mathbf{1}_B$
$\mathbf{0}_B$	$\mathbf{0}_B$	$\mathbf{1}_B$

Table 1.7: Implication operation on π-regular partitions.

For any row where the consequent $\tau \Rightarrow \pi$ has the value $\mathbf{1}_B$, it is already atomized so that value for the overall implication is also $\mathbf{1}_B$. When both the values are $\mathbf{0}_B$ (i.e., the last row), then since $B \subseteq B$, the overall implication has the value $\mathbf{1}_B$ in that case. In the remaining case with antecedent $\mathbf{1}_B$ and consequent $\mathbf{0}_B$, B is not contained in its atomized version, so it remains B and thus that row has the value $\mathbf{0}_B$. And it is equivalent to its double π-negation so that implication is also π-regular.

In this manner, we see that all the partition operations on the π-regular elements are the same as the Boolean operations on the indicator values $\mathbf{1}_B$ and $\mathbf{0}_B$ for $B \in \pi$, so $\mathcal{B}[\pi, \mathbf{1}_U]$ is a Boolean algebra under the partition operations and will be called the *Boolean core* for π. The Boolean core $\mathcal{B}[\pi, \mathbf{1}_U]$ is not technically a subalgebra since the Boolean core $\mathcal{B}[\pi, \mathbf{1}_U] \subseteq [\pi, \mathbf{1}_U] = \{\tau : \pi \precsim \tau\}$ is contained in the upper segment $[\pi, \mathbf{1}_U]$ of $\Pi(U)$ determined by π and the bottom of the Boolean core $\mathcal{B}[\pi, \mathbf{1}_U]$ is π. It is customary to rule out the degenerate Boolean algebra 1 where the top equals the bottom so that means $\pi \neq \mathbf{1}_U$ for the Boolean algebras $\mathcal{B}[\pi, \mathbf{1}_U]$.

If the partition π has any singleton blocks $B = \{u\}$ for $u \in U$, then they are always included in a block of any other partition. Moreover they are the same as their atomized version so $\mathbf{1}_B = \mathbf{0}_B$ when B is a singleton. Thus the construction of the Boolean core $\mathcal{B}[\pi, \mathbf{1}_U]$ ignores singleton blocks; they are like a useless appendage onto any π-regular partition and do not change under negation. Let π_{ns} stand for π with the singleton blocks removed. Then the π-regular partitions $\varphi \in \mathcal{B}[\pi_{ns}, \mathbf{1}_U]$ are characterized by the characteristic or indicator function of their non-singleton blocks with values $\mathbf{1}_B$ or $\mathbf{0}_B$, which could be viewed as an normal characteristic function $\chi(\varphi) : \pi_{ns} \to 2$. The Boolean core $\mathcal{B}[\pi_{ns}, \mathbf{1}_U]$ is isomorphic to the powerset Boolean algebra $\wp(\pi_{ns})$ under the correspondence: $\varphi \longleftrightarrow \chi(\varphi) : \pi_{ns} \to 2$.

We now have two Boolean algebras associated with each partition $\pi \in \Pi(U)$ so we should expect a close relationship. In the complete Boolean subalgebra approach to defining the lattice operations, $\mathcal{B}(\pi)$ was the Boolean subalgebra of $\wp(U)$ generated by the unions of the blocks of π, singleton or not, so it is isomorphic to the powerset Boolean algebra $\wp(\pi)$. The difference between $\mathcal{B}(\pi)$ and $\mathcal{B}[\pi_{ns}, \mathbf{1}_U]$ is solely due to the different treatment of the singletons since $\mathbf{0}_B = \mathbf{1}_B$ for singleton blocks B. There are two versions of an arbitrary union of π-blocks in $\mathcal{B}(\pi)$, namely with or without a singleton block. Hence if we multiply $\mathcal{B}[\pi_{ns}, \mathbf{1}_U]$ by the two-element Boolean algebra 2 for every singleton block in π, then we obtain $\mathcal{B}(\pi)$:

$$\mathcal{B}(\pi) \cong \mathcal{B}[\pi_{ns}, \mathbf{1}_U] \times \prod_{\{u\} \in \pi} 2.$$

1.4.3 The Sheffer stroke, not-and, or nand operation on partitions

In addition to the lattice operations and implication, we will analyze the Sheffer stroke, not-and, or nand operation denoted $\pi|\sigma$. The ditset definition would be:

$$\operatorname{dit}(\pi|\sigma) = \operatorname{int}[(\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma))^c] = \operatorname{int}[\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma)]$$

so that:

$$\operatorname{indit}(\pi|\sigma) = \overline{[\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)]}.$$

In the graph-theoretic definition, the links in $K(U)$ marked with $F(\pi|\sigma)$ would be those with $T\pi$ and $T\sigma$ as indicated in the truth-table Table 1.8 for the nand. After deleting all the links marked with $F\pi$ or $F\sigma$ to obtain $Gph(\pi|\sigma)$, the connected components would be the partition $\pi|\sigma$.

π	σ	$\pi\|\sigma$
$T\pi$	$T\sigma$	$F(\pi\|\sigma)$
$T\pi$	$F\sigma$	$T(\pi\|\sigma)$
$F\pi$	$T\sigma$	$T(\pi\|\sigma)$
$F\pi$	$F\sigma$	$T(\pi\|\sigma)$

Table 1.8: Truth-table for the partition nand $\pi|\sigma$.

For the set-of-blocks definition, each $u \in U$ is contained in some block $B \cap C$ of $\pi \vee \sigma$ and each different u' is contained in a block $B' \cap C'$. If $B \neq B'$ and $C \neq C'$, then (u, u') is a dit of both π and σ so:

$$(u, u') \in (B \cap C) \times (B' \cap C') \subseteq \operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma) = (\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma))^c.$$

Two vertices are connected in $Gph(\pi|\sigma)$ if and only if they are in the closure $\overline{\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)}$ and thus they are a distinction if and only if they are in the complement of the closure which is the interior: $\operatorname{int}[\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau)]$. Thus the graph-theoretic and ditset definitions of $\pi|\sigma$ agree. To obtain the set-of-blocks definition, note that when u and u' are linked in $Gph(\pi|\sigma)$ because $B \neq B'$ and $C \neq C'$, then all the elements of $B \cap C$ and $B' \cap C'$ are in the same block of the nand $\pi|\sigma$. But for any non-empty $B \cap C$, if there is *no* other block $B' \cap C'$ of the join with $B \neq B'$ and $C \neq C'$, then the elements of $B \cap C$ would not even be connected with each other so they would be singletons in the nand. Hence for the set-of-blocks definition of the nand $\pi|\sigma$, the blocks of the nand partition are formed by taking the unions of any join blocks $B \cap C$ and $B' \cap C'$ which differ in both "components" but by taking as singletons the elements of any $B \cap C$ which does not differ from any other join block in both components.

Example: Let $\pi = \{\{a, b\}, \{c, d, e\}\}$ and $\sigma = \{\{a, b, c\}, \{d, e\}\}$. In Figure 1.9, all the arcs in the complete graph $K(U)$ on five vertices are labeled according to the status of the two endpoints in the thickened lines. In the graph $Gph(\pi|\sigma)$ with only the thickened links, there are two connected components giving the blocks of the nand: $\pi|\sigma = \{\{a, b, d, e\}, \{c\}\}$.

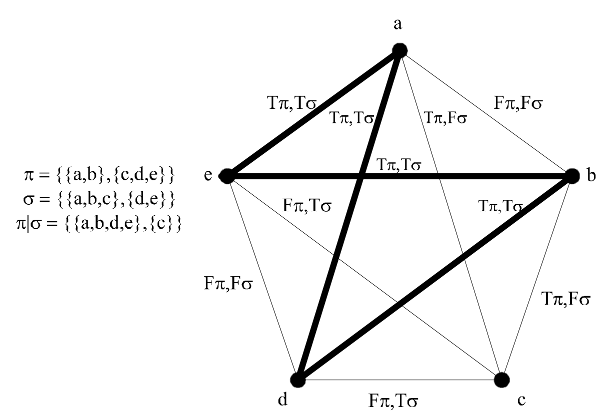

Figure 1.9: Graph-theoretic definition of $\pi|\sigma$.

By the set-of-blocks definition, the two blocks of the join $\{a, b\} = \{a, b\} \cap \{a, b, c\} = B \cap C$ and $\{d, e\} = \{c, d, e\} \cap \{d, e\} = B' \cap C'$ have both $B \neq B'$ and $C \neq C'$, so all those elements are in one block $\{a, b, d, e\}$ of the nand. And $\{c\} = \{c, d, e\} \cap \{a, b, c\}$ but there are no other blocks in the join $\pi \vee \sigma = \{\{a, b\}, \{c\}, \{d, e\}\}$ that differ in both blocks intersected, so $\{c\}$ is a singleton in the nand.

An *atom* in the lattice of partitions $\Pi(U)$ is a partition π so that there is no partition between it and the blob $\mathbf{0}_U$, i.e., if $\mathbf{0}_U \precsim \varphi \precsim \pi$, then $\varphi = \mathbf{0}_U$ or $\varphi = \pi$. The atoms are the partitions with exactly two blocks.

Example: An interesting special case are atoms where one block is a singleton such as $\pi = \{\{u\}, U - \{u\}\}$ and $\sigma = \{\{u'\}, U - \{u'\}\}$ for $u \neq u'$. Then the join has only three blocks $\pi \vee \sigma = \{\{u\}, \{u'\}, U - \{u, u'\}\}$ where $\{u\} = \{u\} \cap (U - \{u'\})$ and $\{u'\} = (U - \{u\}) \cap \{u'\}$. Since those intersections differ in both components, $\{u, u'\}$ is a block in the nand. But $U - \{u, u'\} = U - \{u\} \cap U - \{u'\}$ and there are no other non-empty intersections that differ in both blocks intersected so that block $U - \{u, u'\}$ is atomized or discretized in the nand, i.e., $\pi|\sigma = \{\{u, u'\}, \{u''\}, ...\}$. On the graph-theoretic approach, the only link $F_{\pi|\sigma}$ in $Gph(\pi|\sigma)$ is $u \overset{T_\pi, T_\sigma}{\text{——}} u'$ so u and u' are in the same block and all other $u'' \in U$ are singletons. Thus $\pi|\sigma$ is a coatom of $\Pi(U)$, i.e., a partition so there is no partition strictly between it and the top $\mathbf{1}_U$.

Example: The universe set $U = \{Tom, John, Jim\}$ consists of three people and there are two partitions: α which distinguishes people according to the first letter of their name so that $\alpha = \{\{Tom\}, \{John, Jim\}\}$, and ω which distinguishes people according to the last letter of their name so that $\omega = \{\{Tom, Jim\}, \{John\}\}$. Then the meet $\alpha \wedge \omega$ would identify people who are directly and indirectly identified by the two partitions. Tom and John are not directly identified but are indirectly identified: $Tom \overset{\omega}{\sim} Jim \overset{\alpha}{\sim} John$ so that $\sigma \wedge \omega = \mathbf{0}_U$. But since the meet is $\mathbf{0}_U$, the orthogonal nand of the two partitions could be non-zero, and in fact $\alpha|\omega = \{\{Tom, John\}, \{Jim\}\}$. Thus the fact that Tom and John are directly distinguished by both the first and last letters of their names, i.e., $T_\alpha T_\omega$ on the link $Tom - John$, results in them not being distinguished, i.e., $F_{\alpha|\omega}$. by the nand partition. In this case, $\neg\alpha \vee \neg\omega = \mathbf{0}_U$ and $\neg(\alpha \wedge \omega) = \mathbf{1}_U$.

A number of results about the nand can be obtained using the graph-theoretic characterization. Truth means distinctions. Hence for any ordered pair $(u, u') \in \operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)$, the $u - u'$ link in $K(U)$ would be marked $T\pi, T\sigma$ and thus $F_{\pi|\sigma}$ so u and u' will be in the same block of $\pi|\sigma$. By the Common-Dits result, any two non-blob partitions have dits in common, so for any $\pi, \sigma \neq \mathbf{0}_U$, $\pi|\sigma \neq \mathbf{1}_U$. Moreover, if $\pi|\sigma \neq \mathbf{1}_U$, then for some link in $Gph(\pi|\sigma)$ has $T\pi T\sigma$ assigned to it so that the Common-Dits Theorem implies $\pi, \sigma \neq \mathbf{0}_U$. And if π or σ is $\mathbf{0}_U$, then there can be no common dits (since the blob has no distinctions), so for any π:

$$\pi|\mathbf{0}_U = \mathbf{1}_U.$$

That is the partition version of the set relation $S|\emptyset = U$ since for $S, T \subseteq U$, $S|T = S^c \cup T^c$.

In subset logic, negation can be defined in terms of the nand: $S|S = S^c = S \supset \emptyset$. In partition logic, for $\sigma = \mathbf{0}_U$, we have already noted that $\mathbf{0}_U|\mathbf{0}_U = \mathbf{1}_U = \mathbf{0}_U \Rightarrow \mathbf{0}_U$. If $\sigma \neq \mathbf{0}_U$, then for any two blocks $C, C' \in \sigma$ with $u \in C$ and $u' \in C'$, then the link $u - u'$ has T_σ, T_σ assigned to it and thus $F_{\sigma|\sigma}$ so that u and u' are in the same block of $\sigma|\sigma$. Similarly any other $u'' \in C$ is linked to u' so u and u'' are in the same connected component of $Gph(\sigma|\sigma)$. By symmetry, any other $u''' \in C'$ is in the same connected component, and this is true for any $C, C' \in \sigma$, so we have:

$$\sigma|\sigma = \mathbf{0}_U = \sigma \Rightarrow \mathbf{0}_U = \neg\sigma.$$

In subset logic, two subsets are disjoint (or orthogonal) if their meet (intersection) is the bottom zero element $\emptyset$, or equivalently (by the DeMorgan laws) if the union of their complements was the top element U. But in partition logic, these relationship are more subtle. Two partitions φ and φ' on U are said to be π-*orthogonal* if $\overset{\pi}{\neg}\varphi \vee \overset{\pi}{\neg}\varphi' = \mathbf{1}_U$. They are *orthogonal* if $\neg\varphi \vee \neg\varphi' = \mathbf{1}_U$. Orthogonality and π-orthogonality give a partition version of "disjointness."

Lemma 2 *φ and φ' are orthogonal, i.e., $\neg\varphi \vee \neg\varphi' = \mathbf{1}_U$, iff $\varphi|\varphi' = \mathbf{1}_U$.*

Proof: If $\neg\varphi \vee \neg\varphi' = \mathbf{1}_U$, then $\operatorname{int}(\operatorname{indit}(\varphi)) \cup \operatorname{int}(\operatorname{indit}(\varphi')) = \operatorname{dit}(\mathbf{1}_U) = U^2 - \Delta$. By the monotonicity of the interior operator, $\operatorname{int}(\operatorname{indit}(\varphi)) \cup \operatorname{int}(\operatorname{indit}(\varphi')) \subseteq \operatorname{int}(\operatorname{indit}(\varphi) \cup \operatorname{indit}(\varphi')) = \operatorname{dit}(\varphi|\varphi')$ so $\varphi|\varphi' = \mathbf{1}_U$. Conversely, if $\varphi|\varphi' = \mathbf{1}_U$, then one of the partitions have to be the blob $\mathbf{0}_U$ so $\neg\varphi$ or $\neg\varphi'$ is $\mathbf{1}_U$ and so is the join. $\square$

The contrapositive (negation of each side) of the Lemma is just a restatement of the Common-dits Theorem since $\neg\varphi \vee \neg\varphi' \neq \mathbf{1}_U$ just says that neither φ nor φ' is the blob $\mathbf{0}_U$ and the other side of the equivalence $\varphi|\varphi' \neq \mathbf{1}_U$ just says that $\operatorname{dit}(\varphi) \cap \operatorname{dit}(\varphi') \neq \emptyset$.

Thus orthogonality is also characterized by: $\varphi|\varphi' = \mathbf{1}_U$. And thus orthogonality means that if one partition is not $\mathbf{0}_U$, then the other must be $\mathbf{0}_U$. Orthogonality thus immediately implies $\varphi \wedge \varphi' = \mathbf{0}_U$ but not the reverse. In the previous example, the meet of $\pi = \{\{u\}, U - \{u\}\}$ and $\sigma = \{\{u'\}, U - \{u'\}\}$ is $\pi \wedge \sigma = \mathbf{0}_U$ and $\neg\mathbf{0}_U = \mathbf{1}_U$ but $\pi|\sigma \neq \mathbf{1}_U$ ($\{u, u'\}$ is a block in the nand) since neither is the blob. Also this means that the negation $\neg(\pi \wedge \sigma)$ is not necessarily the same as the nand $\pi|\sigma$. Since $\operatorname{dit}(\pi \wedge \sigma) = \operatorname{int}[\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)]$ and $\operatorname{dit}(\pi|\sigma) = \operatorname{int}[(\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma))^c]$, they cannot have a dit in common, so their nand must be the discrete partition:

$$(\pi \wedge \sigma) \mid (\pi|\sigma) = \mathbf{1}_U$$
"and" orthogonal to "nand".

The same example shows that $\neg\pi \vee \neg\sigma$ (which is $\mathbf{0}_U$) is not necessarily the same as $\pi|\sigma$. The formulas are equivalent in subset logic, but in partition logic, the relationship is:

$$\neg\pi \vee \neg\sigma \precnsim \pi|\sigma \precnsim \neg(\pi \wedge \sigma)$$

since

$$\begin{gathered}\operatorname{int}[\operatorname{indit}(\pi)] \cup \operatorname{int}[\operatorname{indit}(\sigma)] \subseteq \operatorname{int}[\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma)] \\ \subseteq \operatorname{int}[\operatorname{indit}(\pi \wedge \sigma)] = \operatorname{int}\left[\overline{\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma)}\right].\end{gathered}$$

Hence the "strong" DeMorgan law $\neg\sigma \vee \neg\tau = \neg(\sigma \wedge \tau)$ does not hold in partition logic. But the weak DeMorgan law, $\neg(\pi \vee \sigma) = \neg\pi \wedge \neg\sigma$, does hold (since if $\pi = \mathbf{0}_U = \sigma$, then both sides are $\mathbf{1}_U$, and if one is not $\mathbf{0}_U$, then both sides are $\mathbf{0}_U$).

Since the three formulas $\neg\pi \vee \neg\sigma$, $\pi|\sigma$, and $\neg(\pi \wedge \sigma)$ are classically equivalent, i.e., in subset logic, they will have the same truth table, and the truth table is used to define the *atomic* partition operations in the graph-theoretic method. That does not mean that the three formulas are equivalent in partition logic since the two formulas,$\neg\pi \vee \neg\sigma$ and $\neg(\pi \wedge \sigma)$, are not atomic but are compound formulas using other atomic partition operations. In subset logic, there are 16 different truth tables for binary operations, and all compound formulas with only two variables will have a truth table equivalent to one of the 16 possible atomic operations. In fact, there are smaller subsets of atomic operations that can be used to define all the others in subset logic, the most famous being the nand operation (or its negation neither-nor) by itself. But we have just seen that the binary operations on partitions are not closed under the 16 atomic definitions definable by the graph-theoretic method from the 16 possible truth tables. For instance, $\neg\pi \vee \neg\sigma$ and $\neg(\pi \wedge \sigma)$ also define new binary operations on partitions–but not atomic ones.

Partitions are more complex objects than subsets, and, accordingly, partition logic is much more complex than subset logic. We will see below that four atomic partition operations suffice to define all 16 of the truth-table definable partition atomic operations, but all the compound formulas with only two variables define a much larger universe of binary partition operations. Since there is a countably infinite number of finite compound binary formulas, it is not even currently known if the number of binary partition operations defined by the 16 logical atomic operations is infinite or finite.

In contrast to binary partition logical operations, the unary (including the 0-ary constants) operations are quite manageable. There are four possible truth tables for unary operations, which give $\mathbf{0}_U, \sigma, \neg\sigma$, and $\mathbf{1}_U$. These are the four graph-theoretic definable atomic unary operations. What happens when we form compound formulas? The negation applied to the top or bottom just gives the other, and the negation of σ is $\neg\sigma$. But the double negation $\neg\neg\sigma$ is not necessarily the same as σ so it is a new unary operation. But the triple negation is the same as the single negation so the number of unary operations, atomic or compound are the five ones, the four atomic ones $\mathbf{0}_U, \sigma, \neg\sigma$, and $\mathbf{1}_U$ and the compound operation $\neg\neg\sigma$. If we allow other binary operations, then there are other unary operations such as: $\sigma \vee \neg\sigma$.

Just as the unary operation $\neg\sigma$ is usefully generalized by the binary operation $\overset{\pi}{\neg}\sigma = \sigma \Rightarrow \pi$, so the binary operation $\sigma|\tau$ might be usefully generalized by the ternary operation $\sigma|_\pi\tau$, which is the nand operation relative to π. The basic idea is that the nand relativized by π is like the usual nand but relativized to the upper segment $[\pi, \mathbf{1}_U]$ defined by π. Hence its ditset must include $\operatorname{dit}(\pi)$ so the natural change from $\operatorname{dit}(\sigma|\tau) = \operatorname{int}[\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau)]$ to the ditset definition of $\sigma|_\pi\tau$ is:

$$\operatorname{dit}(\sigma|_\pi\tau) = \operatorname{int}[\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau) \cup \operatorname{dit}(\pi)].$$

Since all the ditset definitions can be seen as being the interior of some Boolean subset combination of ditsets and inditsets (whether the interior operation is needed or not), the inditsets are always the RST-closures of negated Boolean combination of inditsets and ditsets, e.g.,

$$\operatorname{indit}(\sigma|_\pi\tau) = \overline{[\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau) \cap \operatorname{indit}(\pi)]}.$$

That closure operation corresponds to taking the transitive closure of the connected component in the graph $Gph(\sigma|_\pi\tau)$ for the equivalent graph-theoretic definition of the ternary operation. Hence we can read off from the ditset definition's inditset what the $F_{\sigma|_\pi\tau}$ conditions are, namely just $T\sigma$, $T\tau$, and $F\pi$. Hence we see that the truth-table for that ternary operation is given in Table 1.9 which is the truth table for the classical (not partition) formula $\sigma|\tau \vee \pi$.

σ	τ	π	$\sigma\|\tau \vee \pi$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

Table 1.9: Truth table for graph-theoretic definition of $\sigma|_\pi\tau$.

Starting with that truth table, the graph $Gph(\sigma|_\pi\tau)$ would only have $F_{\sigma|_\pi\tau}$ links with $T\sigma T\tau F\pi$ on them so the transitive closure would be $\operatorname{indit}(\sigma|_\pi\tau) = \overline{[\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau) \cap \operatorname{indit}(\pi)]}$ and thus the ditset and graph-theoretic definitions agree.

Since $\operatorname{dit}(\pi)$ was included in the ditset definition, $\operatorname{dit}(\sigma|_\pi\tau) = \operatorname{int}[\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau) \cup \operatorname{dit}(\pi)]$, we have $\pi \precsim \sigma|_\pi\tau$. As in the case of the unrelativized nand, we have three classically equivalent formulas, $(\sigma \Rightarrow \pi) \vee (\tau \Rightarrow \pi) = \overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau$, $\sigma|_\pi\tau$, and $(\sigma \wedge \tau) \Rightarrow \pi = \overset{\pi}{\neg}(\sigma \wedge \tau)$ that are different in partition logic where:

$$\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau \precsim \sigma|_\pi\tau \precsim \overset{\pi}{\neg}(\sigma \wedge \tau)$$

since:

$$\begin{gathered}\operatorname{int}(\operatorname{indit}(\sigma) \cup \operatorname{dit}(\pi)) \cup \operatorname{int}(\operatorname{indit}(\tau) \cup \operatorname{dit}(\pi)) \\ \subseteq \operatorname{int}[\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau) \cup \operatorname{dit}(\pi)] \\ \subseteq \operatorname{int}[\operatorname{int}(\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau))^c \cup \operatorname{dit}(\pi)] \\ = \operatorname{int}\left[\overline{\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau)} \cup \operatorname{dit}(\pi)\right].\end{gathered}$$

It is immediate that

$$\begin{gathered}\operatorname{dit}(\sigma|_\pi\sigma) = \operatorname{int}[\operatorname{int}(\operatorname{dit}(\sigma) \cap \operatorname{dit}(\sigma))^c \cup \operatorname{dit}(\pi)] \\ = \operatorname{int}[\operatorname{indit}(\sigma) \cup \operatorname{dit}(\pi)] = \operatorname{dit}(\sigma \Rightarrow \pi)\end{gathered}$$

so that:

$$\sigma|_\pi\sigma = \overset{\pi}{\neg}\sigma.$$

It is also clear that:

$$\sigma|_\pi\pi = \mathbf{1}_U$$

since $\operatorname{dit}(\sigma|_{\pi}\pi) = \operatorname{int}[\operatorname{indit}(\sigma) \cup \operatorname{indit}(\pi) \cup \operatorname{dit}(\pi)] = \operatorname{int}[U \times U] = \operatorname{dit}(\mathbf{1}_U)$.

As previously defined, two partitions σ and τ on U are said to be *π-orthogonal* if $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau = \mathbf{1}_U$. The double π-negation $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ of any partition σ is its Booleanization, i.e., the closest element of the Boolean core $\mathcal{B}[\pi, \mathbf{1}_U]$ to σ in the sense that $\sigma \precsim \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ and for any other $\overset{\pi}{\neg}\varphi \in \mathcal{B}[\pi, \mathbf{1}_U]$ with $\sigma \precsim \overset{\pi}{\neg}\varphi$, then $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \precsim \overset{\pi}{\neg}\varphi$. This Booleanization determines a characteristic function $\chi_\sigma : \pi \to 2$ where $\chi_\sigma(B) = 0$ if the B-component of $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ was $\mathbf{0}_B$ (i.e., B is contained in a block of σ) and $\chi_\sigma(B) = 1$ if the B-component of $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ was $\mathbf{1}_B$ (i.e., the discretized version of B).

Lemma 3 *σ and τ are π-orthogonal iff $\sum_{B\in\pi} \chi_\sigma(B)\,\chi_\tau(B) = 0$.*

Proof: σ and τ are π-orthogonal iff each $B \in \pi$ is contained in block of σ or a block of τ iff for each $B \in \pi$, $\chi_\sigma(B)\,\chi_\tau(B) = 0$. □

Every partition σ and its π-negation $\overset{\pi}{\neg}\sigma$ are π-orthogonal since $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma = \mathbf{1}_U$ which is just the law of excluded middle in the Boolean algebra $\mathcal{B}[\pi, \mathbf{1}_U]$.

Lemma 4 *σ and τ are π-orthogonal, i.e., $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau = \mathbf{1}_U$, iff $\sigma|_\pi\tau = \mathbf{1}_U$.*

Proof: Now $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau = \mathbf{1}_U$ iff for every $B \in \pi$, there is a $C \in \sigma$ or a $D \in \tau$, such that $B \subseteq C$ or $B \subseteq D$. Then for any link $u - u'$ with F_π assigned to it, i.e., $u, u' \in B$, $(u, u') \notin \operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau)$, so the link $u - u'$ could not have $F_{\sigma|_\pi\tau}$ assigned to it in $Gph(\sigma|_\pi\tau)$ so $\sigma|_\pi\tau = \mathbf{1}_U$. Conversely if $\sigma|_\pi\tau = \mathbf{1}_U$, then for any two $u, u' \in B \in \pi$, we much have F_σ or F_τ assigned to the link $u - u'$ which means that $u, u \in C$ for some $C \in \sigma$ or $u, u' \in D$ for some $D \in \tau$. But u, u' were arbitrary $u, u' \in B$ so all the elements of B must belong to the same C or the same D, and that holds for any $B \in \pi$, so $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau = \mathbf{1}_U$. □

The contrapositive (negation of both sides) of the Lemma is worth proving directly as a result of the Common Dits Theorem.

Corollary 1 *$\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau \neq \mathbf{1}_U$ iff $\sigma|_\pi\tau \neq \mathbf{1}_U$.*

Proof: If $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau \neq \mathbf{1}_U$, then there is a $B \in \pi$ that not contained in any $C \in \sigma$ or any $D \in \tau$ (which implies B is not a singleton) so the partition σ restricted to B, denoted $\sigma \upharpoonright B = \{B \cap C \neq \emptyset : C \in \sigma\}$ and similarly the partition τ restricted to B, $\tau \upharpoonright B$ are neither the blob when the universe is B. Hence the Common-dits Theorem applied to the universe B with the two non-blob partitions on it implies that $\operatorname{dit}(\sigma \upharpoonright B) \cap \operatorname{dit}(\tau \upharpoonright B) \neq \emptyset$ so such a distinction (u, u') is in $\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau) \cap \operatorname{indit}(\pi)$ so $\sigma|_\pi\tau \neq \mathbf{1}_U$. Conversely, $\sigma|_\pi\tau \neq \mathbf{1}_U$ implies such a pair $(u, u') \in \operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau) \cap \operatorname{indit}(\pi)$ and $(u, u') \in \operatorname{indit}(\pi)$ means there is a $B \in \pi$ such that $u, u' \in B$ while $(u, u') \in \operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau)$. Hence we could not have $B \subseteq C$ (since then $B \cap C = B$) for any $C \in \sigma$ nor have $B \subseteq D$ for any $D \in \tau$, and thus $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau \neq \mathbf{1}_U$. □

Note that while $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau \precsim \sigma|_\pi\tau \precsim \overset{\pi}{\neg}(\sigma \wedge \tau)$ and both $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau$ and $\overset{\pi}{\neg}(\sigma \wedge \tau)$ are in $\mathcal{B}[\pi, \mathbf{1}_U]$, there is no necessity for $\sigma|_\pi\tau$ to be in $\mathcal{B}[\pi, \mathbf{1}_U]$ as illustrated in Figure 1.10 even though it is always in the upper segment $[\pi, \mathbf{1}_U]$.

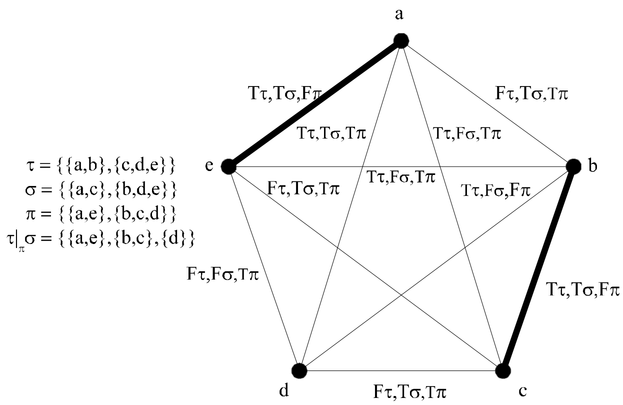

Figure 1.10: Example where $\tau|_\pi\sigma$ not in $\mathcal{B}[\pi, \mathbf{1}_U]$.

In the example of Figure 1.10, $\overset{\pi}{\neg}\tau \vee \overset{\pi}{\neg}\sigma = \pi$ and $\overset{\pi}{\neg}(\tau \wedge \sigma) = \mathbf{1}_U$, but $\pi \precsim \tau|_\pi\sigma = \{\{a,e\},\{b,c\},\{d\}\} \notin \mathcal{B}[\pi, \mathbf{1}_U]$.

1.4.4 The sixteen binary logical operations on partitions

There are $2^{(2^2)} = 16$ truth tables for binary operations. In terms of subsets $S, T \subseteq U$, the four rows in the truth table for a binary operation correspond to: $S \cap T$, $S \cap T^c$, $S^c \cap T$, and $S^c \cap T^c$ which give the four atomic areas in the Venn diagram for S and T. Then there are $2^4 = 16$ possible unions of some of the atomic areas to give the 16 logically definable subsets in terms of S and T.

Taking the universe to be $U \times U$ and S, T to be $\operatorname{dit}(\pi)$ and $\operatorname{dit}(\sigma)$, there are again 16 subsets definable in terms of these ditsets and their complementary inditsets. Some of those subsets will already be a ditset, e.g., $\operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma) = \operatorname{dit}(\pi \vee \sigma)$, but most will require the interior operation to obtain a ditset, e.g., $int[\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)] = \operatorname{dit}(\pi \wedge \sigma)$. In this manner, the sixteen atomic binary logical operations are defined on partitions using the ditset method. But, as already noted, there are many other binary operations constructed from compound formulas of the atomic operations, e.g., $\neg(\sigma \wedge \tau)$ and $\neg\sigma \vee \neg\tau$, that could just as well be called "logical." Since there are a countable number of finite compound formulas, it is not at present known if there are only a finite number of binary logical operations on partitions. However, there are some results if we focus on the sixteen atomic logical operations. Four of the operations, the join, meet, implication, and nand, suffice to define the other atomic operations.

Lemma 5 *For any subsets* $S, T \subseteq U \times U$, $\operatorname{int}[S \cap T] = \operatorname{int}[\operatorname{int}(S) \cap \operatorname{int}(T)]$.

Proof: Since $\operatorname{int}(S) \subseteq S$ and $\operatorname{int}(T) \subseteq T$, $\operatorname{int}[\operatorname{int}(S) \cap \operatorname{int}(T)] \subseteq \operatorname{int}[S \cap T]$. Conversely, $S \cap T \subseteq S, T$ so $\operatorname{int}(S \cap T) \subseteq \operatorname{int}(S) \cap \operatorname{int}(T)$ and since $\operatorname{int}(S \cap T)$ is open, $\operatorname{int}[S \cap T] \subseteq \operatorname{int}[\operatorname{int}(S) \cap \operatorname{int}(T)]$. $\square$

The four atomic subset areas, $S \cap T$, $S \cap T^c$, $S^c \cap T$, and $S^c \cap T^c$, correspond to the disjunctive normal form in subset logic. But the Lemma shows that it is conjunctive normal form (CNF) that might be more useful in partition logic–since the corresponding result for disjunctive normal form does not hold. Hence the strategy is to express each of the 15 (non-universal) regions definable in $U \times U$ from S and T as in Table 1.10 and then take $S = \operatorname{dit}(\sigma)$ and $T = \operatorname{dit}(\tau)$ and take the interior of the meet and use the Lemma.[4]

[4] The notation for some of the binary logical operations is taken from Church [10].

15 regions Conjunctive Normal Form	Binary operation on partitions
$(S \cup T) \cap (S^c \cup T) \cap (S \cup T^c) \cap (S^c \cup T^c)$	0
$(S^c \cup T) \cap (S \cup T^c) \cap (S^c \cup T^c)$	$\sigma \overline{\vee} \tau = \neg\sigma \wedge \neg\tau$
$(S \cup T) \cap (S \cup T^c) \cap (S^c \cup T^c)$	$\tau \not\Leftarrow \sigma = \sigma \wedge \neg\tau$
$(S \cup T^c) \cap (S^c \cup T^c)$	$\neg\tau = \tau \Rightarrow 0$
$(S \cup T) \cap (S^c \cup T) \cap (S^c \cup T^c)$	$\sigma \not\Leftarrow \tau = \neg\sigma \wedge \tau$
$(S^c \cup T) \cap (S^c \cup T^c)$	$\neg\sigma = \sigma \Rightarrow 0$
$(S \cup T) \cap (S^c \cup T^c)$	$\sigma \not\equiv \tau$
$S^c \cup T^c$	$\sigma \vert \tau$
$(S \cup T) \cap (S^c \cup T) \cap (S \cup T^c)$	$\sigma \wedge \tau$
$(S^c \cup T) \cap (T^c \cup S)$	$\sigma \equiv \tau$
$(S \cup T) \cap (S \cup T^c)$	σ
$S \cup T^c$	$\tau \Rightarrow \sigma$
$(S \cup T) \cap (S^c \cup T)$	τ
$S^c \cup T$	$\sigma \Rightarrow \tau$
$S \cup T$	$\sigma \vee \tau$

Table 1.10: Interior of Column 1 gives partition operation in Column 2

Using the Lemma, we may distribute the interior of the CNF subset across it with $S = \operatorname{dit}(\sigma)$ and $T = \operatorname{dit}(\tau)$, we get the expression for the 15 atomic operations in terms of the meet, join, implication, and nand.[5] The sixteenth operation, the constant **1**, can be obtained as $\sigma \Rightarrow \sigma$ or $\tau \Rightarrow \tau$.

Binary operation	Partition CNF for 15 binary operations
0	$= (\sigma \vee \tau) \wedge (\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma) \wedge (\sigma \vert \tau)$
$\sigma \overline{\vee} \tau = \neg\sigma \wedge \neg\tau$	$= (\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma) \wedge (\sigma \vert \tau)$
$\tau \not\Leftarrow \sigma = \sigma \wedge \neg\tau$	$= (\sigma \vee \tau) \wedge (\tau \Rightarrow \sigma) \wedge (\sigma \vert \tau)$
$\neg\tau = \tau \Rightarrow 0$	$= (\tau \Rightarrow \sigma) \wedge (\sigma \vert \tau)$
$\sigma \not\Leftarrow \tau = \neg\sigma \wedge \tau$	$= (\sigma \vee \tau) \wedge (\sigma \Rightarrow \tau) \wedge (\sigma \vert \tau)$
$\neg\sigma = \sigma \Rightarrow 0$	$= (\sigma \Rightarrow \tau) \wedge (\sigma \vert \tau)$
$\sigma \not\equiv \tau$	$= (\sigma \vee \tau) \wedge (\sigma \vert \tau)$
$\sigma \vert \tau$	$= \sigma \vert \tau$
$\sigma \wedge \tau$	$= (\sigma \vee \tau) \wedge (\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma)$
$\sigma \equiv \tau$	$= (\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma)$
σ	$= (\sigma \vee \tau) \wedge (\tau \Rightarrow \sigma)$
$\tau \Rightarrow \sigma$	$= \tau \Rightarrow \sigma$
τ	$= (\sigma \vee \tau) \wedge (\sigma \Rightarrow \tau)$
$\sigma \Rightarrow \tau$	$= \sigma \Rightarrow \tau$
$\sigma \vee \tau$	$= \sigma \vee \tau$

Table 1.10: Distributing interior across intersections gives partition CNF.

In addition to the two constants 0 and 1, and the four unary operations σ, τ, $\neg\sigma$, and $\neg\tau$, the ten binary operations of Table 1.10 occur in natural pairs: $\Rightarrow$ and $\not\Rightarrow$, $\Leftarrow$ and $\not\Leftarrow$, $\equiv$ and $\not\equiv$, $\vee$ and $\overline{\vee}$, and $\wedge$ and $\vert$. The relationship between the operations in the pairs is not negation (except for the join); it is orthogonality. If one of the operations applied to σ and τ is non-zero, the other must be zero (the indiscrete partition). There is another pairing of the operations by complemention-duality.

[5] There are other combinations which can be taken as primitive since the *inequivalence*, *symmetric difference*, *exclusive-or*, or *xor* $\sigma \not\equiv \tau$ can be used to define the nand operation: $((\sigma \vee \tau) \Rightarrow (\sigma \not\equiv \tau)) = \sigma \mid \tau$.

1.4.5 The sixteen binary logical operations on equivalence relations

For every operation on partitions or partition relations, there is a complementation-dual operation on equivalence relations. The lattice of partitions on U enriched by implication gives the partition algebra $\Pi(U)$. It is isomorphic to the algebra of open subsets (partition relations or ditsets) $\mathcal{O}(U \times U)$ of $U \times U$ under the mapping $\pi \leftrightarrow \operatorname{dit}(\pi)$. Thus $\mathcal{O}(U \times U)$ is anti-isomorphic (under order-reversing complementation) to the algebra of closed subsets or equivalence relations $\mathcal{C}(U \times U)$ on $U \times U$. The order-reversing isomorphism between $\Pi(U)$ and $\mathcal{C}(U \times U)$ is $\pi \leftrightarrow \operatorname{indit}(\pi)$. In order to keep notation analogous, we will refer to the equivalence relation $\operatorname{indit}(\pi)$ as π^d, the complementation-dual to π.

One could define the logical operations on equivalence relations directly but it is more convenient to define them by duality. The top of the dual algebra of equivalence relations (ordered by inclusion) is $\hat{\mathbf{1}}_U = \mathbf{0}_U^d = \operatorname{indit}(\mathbf{0}_U) = U \times U$, the universal equivalence relation and the bottom is $\hat{\mathbf{0}}_U = \mathbf{1}_U^d = \operatorname{indit}(\mathbf{1}_U) = \Delta$, the diagonal.

The operations on equivalence relations or inditsets is defined as the inditset of the dual (i.e., complementation-dual) operation on ditsets.

- Join: $\operatorname{indit}(\pi) \vee \operatorname{indit}(\sigma) = \operatorname{indit}(\pi \wedge \sigma) = \overline{(\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma))}$ which in the d-superscript notation is: $\pi^d \vee \sigma^d = (\pi \wedge \sigma)^d = \operatorname{indit}(\pi \wedge \sigma)$;
- Meet: $\operatorname{indit}(\pi) \wedge \operatorname{indit}(\sigma) = \operatorname{indit}(\pi \vee \sigma) = \operatorname{indit}(\pi) \cap \operatorname{indit}(\sigma)$ or $\pi^d \wedge \sigma^d = (\pi \vee \sigma)^d = \operatorname{indit}(\pi \vee \sigma)$; and
- Difference: $\operatorname{indit}(\pi) - \operatorname{indit}(\sigma) = \operatorname{indit}(\sigma \Rightarrow \pi) = \overline{(\operatorname{dit}(\sigma) \cap \operatorname{indit}(\pi))}$, or: $\pi^d - \sigma^d = (\sigma \Rightarrow \pi)^d = \operatorname{indit}(\sigma \Rightarrow \pi)$.

The dual of the implication for subsets $S \supset T = S^c \cup T$ by complementation is: $(S^c \cup T)^c = S \cap T^c = S - T$ hence the complementation-dual of implication is difference. That gives the definition of the dual algebra of equivalence relations $\mathcal{C}(U \times U)$ with the constants of $\hat{\mathbf{0}}_U$ and $\hat{\mathbf{1}}_U$ and the four primitive operations of meet, join, and difference.

For any compound formula φ using the three atomic partition operations, the dual equivalence relation φ^d is proven by standard methods to be $\operatorname{indit}(\varphi)$.

Proposition 3 $\varphi^d = \operatorname{indit}(\varphi)$.

Proof: The proof uses induction over the complexity of the formulas [where complexity is defined in the standard way in propositional logic]. If φ is one of the constants $\mathbf{0}_U$ or $\mathbf{1}_U$, then the proposition holds since: $\mathbf{0}_U^d = \widehat{\mathbf{1}}_U = \operatorname{indit}(\mathbf{0}_U)$ and $\mathbf{1}_U^d = \widehat{\mathbf{0}_U} = \operatorname{indit}(\mathbf{1}_U)$. If $\varphi = \alpha$ is atomic, then it is true by the definition: $\sigma^d = \operatorname{indit}(\sigma)$. If φ is a compound formula then the main connective in φ is one of the four primitive partition operations and the main connective in φ^d is one of the four primitive equivalence relation operations. Consider the case: $\varphi = \pi \wedge \sigma$ so that $\varphi^d = \pi^d \vee \sigma^d$. By the induction hypothesis, $\pi^d = \operatorname{indit}(\pi)$ and $\sigma^d = \operatorname{indit}(\sigma)$, and by the definition of the equivalence relation join: $\varphi^d = \pi^d \vee \sigma^d = \operatorname{indit}(\pi) \vee \operatorname{indit}(\sigma) = \overline{\{\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma)\}} = \operatorname{indit}(\varphi)$. The other three cases proceed in a similar manner. □

Substituting the dual of $\mathbf{0}_U$, namely $\hat{\mathbf{1}}_U$, for π^d gives the dual of negation, as: $\hat{\mathbf{1}}_U - \sigma^d$ which might be called "non-σ^d" and denoted as $-\sigma^d$.[6] The Common-Dits Theorem was equivalent to the result that if the union of two equivalence relations was the universal relation, then one of the equivalence relations was the universal relation $U \times U$. That is, $\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma) = (\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma))^c$ so if $\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma) = \emptyset$, then the Common-dits Theorem says that $\operatorname{dit}(\pi) = \emptyset$ or $\operatorname{dit}(\sigma) = \emptyset$, i.e., if $\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma) = U \times U$, then $\operatorname{indit}(\pi) = U \times U$ or $\operatorname{indit}(\sigma) = U \times U$. In the partition case, this implies that if $\sigma \neq \mathbf{0}_U$, then $\sigma \Rightarrow \mathbf{0}_U = \neg\sigma = \mathbf{0}_U$. In the equivalence relation case, if $\sigma^d \neq \hat{\mathbf{1}}_U$, then

[6] See Lawvere's discussion of "non-A" in co-Heyting algebras [35].

$\hat{\mathbf{1}}_U - \sigma^d = -\sigma^d = \hat{\mathbf{1}}_U$. Thus subtracting any equivalence relation less than $U \times U$ from $U \times U$ leaves $U \times U$. But subtracting $U \times U$ from itself leaves $\hat{\mathbf{0}}_U$.

The key result, derivable from the Common-dits Theorem is important enough to get its own proof.

Proposition 4 *The RST-closure* $\overline{\mathrm{dit}(\sigma)}$ *of non-empty ditset* $\mathrm{dit}(\sigma)$ *is the universal equivalence relation* $U \times U$*.*

Proof: Since $\mathrm{dit}(\sigma)$ is non-empty, there exists $C, C' \in \sigma$ with $C \neq C'$ and for any $u \in C$ and $u' \in C'$, $(u, u') \in \mathrm{dit}(\sigma)$. The RST-closure adds the diagonal Δ to the set so the only pairs that need to be added to make $U \times U$ are the pairs of $\mathrm{indit}(\sigma)$ such as $(u, u'') \in C$.

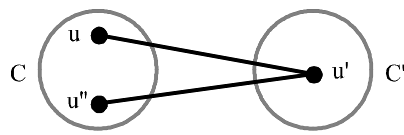

Figure 1.11: Point u connected to u'' by transitivity in RST-closure of $\mathrm{dit}(\sigma)$.

As in Figure 1.11, (u, u') and $(u', u'') \in \mathrm{dit}(\sigma)$ so by transitivity, any pair $(u, u'') \in \mathrm{indit}(\sigma)$ is also included in the RST-closure $\overline{\mathrm{dit}(\sigma)}$ so it is $U \times U$. □

In the case of $\sigma^d \neq \hat{\mathbf{1}}_U$ implying $-\sigma^d = \hat{\mathbf{1}}_U$, $\sigma^d = \mathrm{indit}(\sigma) \neq \mathrm{indit}(\mathbf{0}_U) = \hat{\mathbf{1}}_U$ means that $\mathrm{dit}(\sigma) \neq \emptyset$ so the closure $\overline{(\mathrm{dit}(\sigma) \cap \mathrm{indit}(\mathbf{0}_U))} = \overline{\mathrm{dit}(\sigma)}$ which by the Proposition is $U \times U = \mathrm{indit}(\mathbf{0}_U) = \hat{\mathbf{1}}_U$. When $\sigma = \mathbf{0}_U$, then $\mathrm{dit}(\sigma) = \emptyset$ whose RST-closure is the diagonal $\Delta = \mathrm{indit}(\mathbf{1}_U) = \mathbf{1}_U^d = \hat{\mathbf{0}}_U$, the bottom of the algebra $\mathcal{C}(U \times U)$ of equivalence relations.

The Lemma $\mathrm{int}[A \cap B] = \mathrm{int}[\mathrm{int}(A) \cap \mathrm{int}(B)]$ for $A, B \subseteq U \times U$ could also be expressed using the closure operation as $\overline{[A \cup B]} = \overline{[\overline{A} \cup \overline{B}]}$. Hence the conjunctive normal form treatment of the 15 binary operations on partitions in terms of the operations of $\vee$, $\wedge$, $\Rightarrow$, and $|$ dualizes to the disjunctive normal form (DNF) treatment of the 15 (dual) binary operations on equivalence relations in terms of the dual operations $\wedge$, $\vee$, $-$, and $\overline{\vee}$, which are the primitive operations in the algebra of equivalence relations $\mathcal{C}(U \times U)$.

The previous pair of tables giving the CNF treatment of the 15 operations on partitions complementation-dualize to give two similar tables for the DNF treatment of the 15 non-zero operations on equivalence relations. In the Table 1.11, let $S' = \mathrm{indit}(\sigma)$ and $T' = \mathrm{indit}(\tau)$ where $()^c$ is complementation in $U \times U$. We have also taken the liberty of writing the "converse non-implication" operation as the difference operation on both equivalence relations and partitions: $\tau^d - \sigma^d = \sigma^d \not\Leftarrow \tau^d$ and $\tau - \sigma = \sigma \not\Leftarrow \tau$.

15 regions Disjunctive Normal Form	Bin. op. on eq. rel.	Dual to
$S'^c \cap T'^c$	$\sigma^d \overline{\vee} \tau^d$	$\sigma \vert \tau$
$S' \cap T'^c$	$\sigma^d - \tau^d$	$\tau \Rightarrow \sigma$
$(S' \cap T'^c) \cup (S'^c \cap T'^c)$	$-\tau^d$	$\neg\tau$
$S'^c \cap T'$	$\tau^d - \sigma^d$	$\sigma \Rightarrow \tau$
$(S'^c \cap T') \cup (S'^c \cap T'^c)$	$-\sigma^d$	$\neg\sigma$
$(S'^c \cap T') \cup (S' \cap T'^c)$	$\sigma^d \not\equiv \tau^d$	$\sigma \equiv \tau$
$(S'^c \cap T') \cup (S'^c \cap T'^c) \cup (S' \cap T'^c)$	$\sigma^d \vert \tau^d$	$\sigma \overline{\vee} \tau$
$S' \cap T'$	$\sigma^d \wedge \tau^d$	$\sigma \vee \tau$
$(S' \cap T') \cup (S'^c \cap T'^c)$	$\sigma^d \equiv \tau^d$	$\sigma \not\equiv \tau$
$(S' \cap T') \cup (S' \cap T'^c)$	σ^d	σ
$(S' \cap T') \cup (S' \cap T'^c) \cup (S'^c \cap T'^c)$	$\tau^d \Rightarrow \sigma^d$	$\sigma - \tau$
$(S' \cap T') \cup (S'^c \cap T')$	τ^d	τ
$(S'^c \cap T') \cup (S'^c \cap T'^c) \cup (S' \cap T')$	$\sigma^d \Rightarrow \tau^d$	$\tau - \sigma$
$(S' \cap T') \cup (S' \cap T'^c) \cup (S'^c \cap T')$	$\sigma^d \vee \tau^d$	$\sigma \wedge \tau$
$(S' \cap T') \cup (S' \cap T'^c) \cup (S'^c \cap T') \cup (S'^c \cap T'^c)$	$\widehat{1}$	$\mathbf{0}$

Table 1.11: RST-closure of Column 1 gives equivalence relation binary operation in Column 2

The CNF expression for the partition symmetric difference or inequivalence is: $\sigma \not\equiv \tau = (\sigma \vee \tau) \wedge (\sigma|\tau)$ so that:

$$\begin{aligned}\operatorname{dit}(\sigma \not\equiv \tau) &= \operatorname{int}\left[\operatorname{int}(\operatorname{dit}(\sigma) \cup \operatorname{dit}(\tau)) \cap \operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\tau)^c)\right] \\ &= \operatorname{int}\left[(\operatorname{dit}(\sigma) \cup \operatorname{dit}(\tau)) \cap (\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\tau)^c)\right].\end{aligned}$$

Taking complements yields:

$$\begin{aligned}\operatorname{indit}(\sigma \not\equiv \tau) &= \overline{\left[(\operatorname{indit}(\sigma) \cap \operatorname{indit}(\tau)) \cup (\operatorname{indit}(\sigma)^c \cap \operatorname{indit}(\tau)^c)\right]} \\ &= \overline{\left[\overline{(\operatorname{indit}(\sigma) \cap \operatorname{indit}(\tau))} \cup \overline{(\operatorname{indit}(\sigma)^c \cap \operatorname{indit}(\tau)^c)}\right]} \\ &= \overline{\left[(\sigma^d \wedge \tau^d) \cup (\sigma^d \overline{\vee} \tau^d)\right]} \\ &= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right) \\ &= \sigma^d \equiv \tau^d.\end{aligned}$$

Thus the equivalence $\sigma^d \equiv \tau^d$ of equivalence relations has the disjunctive normal form: $\sigma^d \equiv \tau^d = \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right)$ in the "dual" logic of equivalence relations. The disjunctive normal forms for the 15 operations on equivalence relations is given in the following Table 1.12.

Binary operation	Equivalence relation DNF for 15 binary operations
$\sigma^d \bar{\vee} \tau^d$	$= \sigma^d \bar{\vee} \tau^d$
$\sigma^d - \tau^d$	$= \sigma^d - \tau^d$
$-\tau^d$	$= \left(\sigma^d - \tau^d\right) \vee \left(\sigma^d \bar{\vee} \tau^d\right)$
$\tau^d - \sigma^d$	$= \tau^d - \sigma^d$
$-\sigma^d$	$= \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d \bar{\vee} \tau^d\right)$
$\sigma^d \not\equiv \tau^d$	$= \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d - \tau^d\right)$
$\sigma^d \vert \tau^d$	$= \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d \bar{\vee} \tau^d\right) \vee \left(\sigma^d - \tau^d\right)$
$\sigma^d \wedge \tau^d$	$= \sigma^d \wedge \tau^d$
$\sigma^d \equiv \tau^d$	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d \bar{\vee} \tau^d\right)$
σ^d	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d - \tau^d\right)$
$\tau^d \Rightarrow \sigma^d$	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d - \tau^d\right) \vee \left(\sigma^d \bar{\vee} \tau^d\right)$
τ^d	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\tau^d - \sigma^d\right)$
$\sigma^d \Rightarrow \tau^d$	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d \bar{\vee} \tau^d\right)$
$\sigma^d \vee \tau^d$	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d - \tau^d\right) \vee \left(\tau^d - \sigma^d\right)$
$\widehat{1}$	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d - \tau^d\right) \vee \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d \bar{\vee} \tau^d\right)$

Table 1.12: Distributing closure across unions gives equivalence relation DNFs.

We should keep different notions of duality separate. The logic of partitions is dual to the logic of subsets in the sense that partitions are category-theoretically dual to subsets. But the logic of equivalence relations is dual to the logic of partitions in the sense of complementation duality–since equivalence relations are the complements to partition relations in $U \times U$. The duality between the Heyting algebra of open subsets is the complementation-dual to the co-Heyting algebra of closed subsets. That duality is analogous to the duality between the algebra of partitions $\Pi(U)$ (or partition relations $\mathcal{O}(U \times U)$) and the algebra of equivalence relations $\mathcal{C}(U \times U)$–except that the RST-closure operation is not topological (union of two RST-closed subsets is not necessarily RST-closed).

2 Partition tautologies

2.1 Subset, truth-table, and partition tautologies

We consider formulas composed using the operations of $\vee$, $\wedge$, $\Rightarrow$, and $\vert$ as well as the top and bottom constants so that all formulas can be interpreted as being in subset logic, propositional logic, or partition logic (as well as intuitionistic propositional logic or Heyting algebras). A *subset tautology* is a formula so that no matter what subsets of any universe U ($|U| \geq 1$) are substituted for the atomic variables, the formulas evaluates (using the subset interpretation of the operations) to the universe set U. A *truth-table tautology* is a formula so that no what truth values T or F are substituted for the atomic variables, the whole formula evaluates (using the truth-table definitions of the operations) to T.

All subset tautologies are truth-table tautologies since one can take $U = 1$, the one element set so that $\wp(1) = \{\emptyset, 1\}$ with $\emptyset$ taken as F and 1 as T. To see the converse, consider a subset formula with m different atomic variables for which can be substituted m subsets of U. Those m subsets define 2^m atomic areas in a Venn diagram which correspond to the 2^m rows in the truth table for the formula. Each element $u \in U$ must belong to one of the atomic areas so interpret $u \in S$ as T assigned to a subset S and $u \notin S$ as F assigned to a subset S. Then if the formula is a truth-table tautology, e.g. Table 2.1, all the T's in the final column will indicate that any $u \in U$ is included in the subset constructed from the formula so that subset must be U, i.e., a truth-table tautology is a subset tautology.

σ	ρ	$\sigma \Rightarrow \rho$	$\sigma \wedge (\sigma \Rightarrow \rho)$	$[\sigma \wedge (\sigma \Rightarrow \rho)] \Rightarrow \rho$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Table 2.1: Truth-table tautology of modus ponens

Then reinterpreting the T's and F's as indicating that a $u \in U$ belongs to an atomic area or not, gives Table 2.2 showing that a truth-table tautology is also a subset tautology, i.e., no matter what atomic area an element $u \in U$ belongs to, it belongs to the subset generated by the final formula for a truth-table tautology so that subset is U.

$u \in S$	$u \in R$	$u \in S \supset R$	$u \in S \cap (S \supset R)$	$u \in [S \cap (S \supset R)] \supset R$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Table 2.2: Truth-table tautology reinterpreted as showing the formula is a subset tautology.

As mentioned previously, one of the reasons for the retarded development of the logic of partitions, even though the duality between partitions and subsets has been well-known since the middle of the twentieth century, is that subset logic is almost always presented as just propositional logic where validities or tautologies are defined as truth-table tautologies and subset tautologies are ignored.

A 'partition tautology' is defined just like a subset tautology with partitions on U substituted for subsets of U. That is, a *partition tautology* is a formula such that no matter what partitions on a universe U ($|U| \geq 2$), the formula evaluates using the partition operations to the top, the discrete partition $\mathbf{1}_U$. It is also useful to define a *weak partition tautology* as a formula that under the same substitutions never evaluates to the bottom, the indiscrete partition $\mathbf{0}_U$. Then a partition tautology is obviously a weak one too. Moreover, it is easily seen that:

φ is a weak partition tautology iff $\neg\neg\varphi$ is a partition tautology.

The law of excluded middle formula $\sigma \vee \neg\sigma$ is the simplest example of a subset tautology that is also a weak partition tautology but not a ('strong') partition tautology.

The simplest non-degenerate partition algebra is $\Pi(2)$, where $2 = \{0, 1\}$ ("the" two-element set), the algebra with only two partitions, $\mathbf{0}_2 = \{\{0, 1\}\}$ and $\mathbf{1}_2 = \{\{0\}, \{1\}\}$. The simplest non-degenerate Boolean subset algebra is $\wp(1)$ where 1 is "the" one-element set which has only two subsets, $\emptyset$ and 1. The results of any set of partition operations on $\Pi(2)$ can be described by a 'truth table' since there are only two partitions. For instance, the partition operations on the modus ponens formula are given in Table 2.3.

σ	ρ	$\sigma \Rightarrow \rho$	$\sigma \wedge (\sigma \Rightarrow \rho)$	$[\sigma \wedge (\sigma \Rightarrow \rho)] \Rightarrow \rho$
$\mathbf{1}_2$	$\mathbf{1}_2$	$\mathbf{1}_2$	$\mathbf{1}_2$	$\mathbf{1}_2$
$\mathbf{1}_2$	$\mathbf{0}_2$	$\mathbf{0}_2$	$\mathbf{0}_2$	$\mathbf{1}_2$
$\mathbf{0}_2$	$\mathbf{1}_2$	$\mathbf{1}_2$	$\mathbf{0}_2$	$\mathbf{1}_2$
$\mathbf{0}_2$	$\mathbf{0}_2$	$\mathbf{1}_2$	$\mathbf{0}_2$	$\mathbf{1}_2$

Table 2.3: 'Truth table' for modus ponens formula in $\Pi(2)$.

The basic point is that the partition operations on $\Pi(2)$ have the same 'truth tables' as the propositional or subset operations in $\wp(1)$ so the two algebras are isomorphic in that sense:

$$\wp(1) \cong \Pi(2).$$

In particular, any formula that is a weak partition tautology would never evaluate to $\mathbf{0}_2$ so it must always evaluate to $\mathbf{1}_U$ in $\Pi(2)$ which is isomorphic, in turn, to $\wp(1)$ so it must always evaluate to 1 or T in the truth table for the formula. Hence we have proven:

Proposition 5 *Any weak partition tautology is a truth-table tautology.* $\square$

Corollary 2 *Any partition tautology is a truth-table tautology.* $\square$

For an example of truth-table tautology that is not a weak partition tautology, consider the simplest example of non-distributivity where $\tau = \{\{a,b\},\{c\}\}$, $\sigma = \{\{a,c\},\{b\}\}$, and $\pi = \{\{a\},\{b,c\}\}$ and the formula is:

$$[\tau \wedge (\pi \vee \sigma)] \Rightarrow [(\tau \wedge \pi) \vee (\tau \wedge \sigma)].$$

That formula is an ordinary tautology but for those partitions substituted for the variables, the result is $\mathbf{0}_U$ so it is not a weak partition tautology. Thus the weak partition tautologies (and thus the partition tautologies) are properly contained in the truth-table tautologies.

When comparing partition tautologies (weak or not), there is no inclusion either way with the valid formulas of intuitionistic propositional logic (or Heyting algebras). The accumulation formula, $\sigma \Rightarrow (\pi \Rightarrow (\pi \wedge \sigma))$, is valid in both classical and intuitionistic logic but not in partition logic. The law of excluded middle, $\sigma \vee \neg\sigma$, is a weak partition tautology, and the weak law of excluded middle, $\neg\sigma \vee \neg\neg\sigma$, is a partition tautology that is not intuitionistically valid.

2.2 The finite model property

If a formula φ is not valid according to subset or partition semantics, then a universe and subsets or partitions that do not evaluation to the top would be a *countermodel*. If a formula is not a subset (or truth-table) tautology, then it suffices to take $U = 1$ to give a subset countermodel, and in view of the isomorphism, $\wp(1) \cong \Pi(2)$, it suffices to take $U = 2$ to have a partition countermodel. But what about formulas that are subset tautologies but not partition tautologies? For subsets, it suffices to take $|U| = 1$ to find countermodels. Is there an n such that for all $|U| \leq n$, if φ evaluates to $\mathbf{1}_U$ for all partitions on U, then φ is a partition tautology for any U? The following result, adapted from lattice theory, shows that there is no such n.

Proposition 6 *There is no n such that if any φ has no partition countermodel on any universe U with $|U| \leq n$, then φ is a partition tautology.*

Proof: Consider any fixed $n \geq 2$. We use the standard device of a "universal disjunction of equations" [28, p. 316] to construct a formula ω_n that evaluates to $\mathbf{1}_U$ for any substitutions of partitions on U with $|U| \leq n$ and yet the formula is not a partition tautology. Let B_n be the Bell number, the number of partitions on a set U with $|U| = n$. Take the atomic variables to be π_i for $i = 0, 1, ..., B_n$ so that there are $B_n + 1$ atomic variables. Let ω_n be the join of all the equivalences between distinct atomic variables:

$$\omega_n = \bigvee \{\pi_i \equiv \pi_j : 0 \leq i < j \leq B_n\}.$$

Then for any substitution of partitions on U where $|U| \leq n$ for the atomic variables, there is, by the pigeonhole principle, some "disjunct" $\pi_i \equiv \pi_j = (\pi_i \Rightarrow \pi_j) \wedge (\pi_j \Rightarrow \pi_i)$ which has the same partition substituted for the two variables so the disjunct evaluates to $\mathbf{1}_U$ and thus the join ω_n evaluates to $\mathbf{1}_U$. Thus ω_n evaluates to $\mathbf{1}_U$ for any substitutions of partitions on any U where $|U| \leq n$. To see that ω_n is not a partition tautology, take $U = \{0, 1, ..., B_n\}$ and let π_i be the atomic partition which has i as a singleton and all the other elements of U as a block, i.e., $\pi_i = \{\{i\}, \{0, 1, ..., i-1, i+1, ..., B_n\}\}$.

Then $\pi_i \Rightarrow \pi_j = \pi_j$ and $\pi_j \wedge \pi_i = \mathbf{0}_U$ so that $\omega_n = \mathbf{0}_U$ for that substitution and thus ω_n is not even a weak partition tautology. $\square$

This result leaves open the question of whether or not partition logic has the finite model property. That is, if φ is not a partition tautology, then does it have a countermodel with a finite U?

If π is a partition on U and $S \subseteq U$, then $\pi \upharpoonright S = \{B \cap S \neq \emptyset : B \in \pi\}$ is a partition on S.

Lemma 6 *Let φ be a formula (always finite) involving the four basic connectives, join, meet, and implication, and the two constants $\mathbf{0}_U$ and $\mathbf{1}_U$. Then for any partitions $\pi_1, ..., \pi_n$ on a universe set U substituted for the atomic variables, if the partition, $\varphi(\pi_1, ..., \pi_n)$ has a proper indit (u, u') (i.e., $u \neq u'$), then the indit only involves a finite set of elements $U' \subseteq U$ such that (u, u') is also an indit of $\varphi(\pi_1 \upharpoonright U', ..., \pi_n \upharpoonright U')$.*

Proof: The proof uses induction over the complexity of the formula φ. At the atomic level, if (u, u') is an indit of a partition (including $\mathbf{0}_U$), then it is also an indit of the partition cut down to any finite U' containing u and u'. Assuming the hypothesis for formulas of lesser complexity, suppose that $\varphi = \sigma * \pi$ where $*$ is one of the three basic connectives.

- Join: $\operatorname{indit}(\sigma \vee \pi) = \operatorname{indit}(\sigma) \cap \operatorname{indit}(\pi)$ so an indit $(u, u') \in \operatorname{indit}(\sigma \vee \pi)$ only uses one indit in σ and π;
- Meet: $\operatorname{indit}(\sigma \wedge \pi) = \overline{(\operatorname{indit}(\sigma) \cup \operatorname{indit}(\pi))}$ so an indit in the closure requires a finite chain of links drawn from $\operatorname{indit}(\sigma)$ and $\operatorname{indit}(\pi)$ and each of those indits uses only a finite set of elements from U by the induction hypothesis; and
- Implication: $\operatorname{dit}(\sigma \Rightarrow \pi) = \operatorname{int}[\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi)]$ so $\operatorname{indit}(\sigma \Rightarrow \pi) = \overline{(\operatorname{dit}(\sigma) \cap \operatorname{indit}(\pi))}$ and thus it only involves a finite chain of indits for π which, in turn, uses only a finite set of elements from U.

Since a finite union of finite sets is finite, there is a finite $U' \subseteq U$ so that $(u, u') \in \operatorname{indit}(\varphi(\pi_1 \upharpoonright U', ..., \pi_n \upharpoonright U'))$.$\square$

Proposition 7 (Finite model property) *If a formula φ is not a partition tautology, then it always has a finite countermodel.*

Proof: If φ is not a partition tautology, then there is a universe U and partitions $\pi_1, ..., \pi_n$ on U such that when substituted for the atomic variables of φ, then the partition $\varphi(\pi_1, ..., \pi_n) \neq \mathbf{1}_U$. Hence that partition contains a non-singleton block $\{u, u', ...\}$. By the lemma, the indit (u, u') only involves a finite set $U' \subseteq U$ of elements so that (u, u') is also an indit of $\varphi(\pi_1 \upharpoonright U', ..., \pi_n \upharpoonright U')$ and thus the formula φ has a finite countermodel. $\square$

2.3 Generating partition tautologies using the Boolean core $\mathcal{B}[\pi, \mathbf{1}_U]$

For any fixed partition π on U, a π-regular partition in $\Pi(U)$ is a partition of the form $\sigma \Rightarrow \pi$ for any σ. We previously showed that the π-regular partitions form a Boolean algebra (BA) $\mathcal{B}[\pi, \mathbf{1}_U]$ where the partition operations of join, meet, and implication also act as the corresponding BA operations in $\mathcal{B}[\pi, \mathbf{1}_U]$ and where π is the bottom (or zero element) and $\mathbf{1}_U$ is the top of the BA. Hence any classical (= subset = truth-table) tautology, e.g., $\sigma \vee (\sigma \Rightarrow 0)$, expressed in the language of join, meet, implication, and the two constants, when formulated in $\mathcal{B}[\pi, \mathbf{1}_U]$ would always evaluate to $\mathbf{1}_U$ using the BA operations when any π-regular elements are substituted for the variables. But the BA operations are the same as the corresponding partition operations so classical tautologies in $\mathcal{B}[\pi, \mathbf{1}_U]$ yield partition tautologies.

This connection can be formalized by converting any classical tautology into the corresponding tautology in $\mathcal{B}[\pi, \mathbf{1}_U]$ and thus into a partition tautology. Given any classical formula using the connectives of $\vee$, $\wedge$, $\Rightarrow$ and the constants of 0 and 1, its *single π-negation transform* is obtained by replacing each atomic variable σ by its single π-negation $\overset{\pi}{\neg}\sigma = \sigma \Rightarrow \pi$ and by replacing the constant 0 by π and 1 by $\mathbf{1}_U$. The binary operations $\vee$, $\wedge$, and $\Rightarrow$ all remain the same. For instance, the single π-negation transform of the excluded middle formula $\sigma \vee \neg\sigma = \sigma \vee (\sigma \Rightarrow 0)$ is the weak excluded middle formula for π-negation:

$$(\sigma \Rightarrow \pi) \vee ((\sigma \Rightarrow \pi) \Rightarrow \pi) = \overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma.$$

Then the single π-negation transformed formula is a tautology in the BA $\mathcal{B}[\pi, \mathbf{1}_U]$, so it is also a partition tautology. This formula is also an example of a partition tautology that is not a valid formula of intuitionistic logic (either for $\pi = \mathbf{0}_U$ or in general).

Proposition 8 *The single π-negation transform of any classical tautology is a partition tautology.* $\square$

This process can be repeated using double negation. The *double π-negation transform* of a classical formula using the connectives of $\vee$, $\wedge$, $\Rightarrow$ and the constants of 0 and 1 is obtained by replacing any atomic variable σ by its double π-negation $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma = (\sigma \Rightarrow \pi) \Rightarrow \pi$ and by replacing the constant 0 by π and 1 by $\mathbf{1}_U$. The binary operations $\vee$, $\wedge$, and $\Rightarrow$ all remain the same.

Proposition 9 *The double π-negation transform of any classical tautology is a partition tautology.* $\square$

The double π-negation transform of excluded middle is the formula $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$. Since the π-negation has the effect of flipping the π-blocks B back and forth being locally equal to $\mathbf{0}_B$ or $\mathbf{1}_B$ (i.e., from being whole to being discretized), it is clear that $\overset{\pi}{\neg}\sigma = \overset{\pi}{\neg}\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ so the formula $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ is equivalent to $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\sigma$.

The BA of regular elements is not the only idea that can be usefully transplanted from intuitionistic logic to partition logic. The partition analogue of the Gödel transform [26] that produces an intuitionistic validity from each classical tautology can be constructed in the following manner. For any classical formula φ in the language of $\vee$, $\wedge$, and $\Rightarrow$ as well as 0 and 1, we define the *Gödel π-transform* φ^g_π of the formula as follows:

- If φ is atomic, then $\varphi^g_\pi = \varphi \vee \pi$;
- If $\varphi = 0$, then $\varphi^g_\pi = \pi$, and if $\varphi = 1$, then $\varphi^g_\pi = \mathbf{1}_U$;
- If $\varphi = \sigma \vee \tau$, then $\varphi^g_\pi = \sigma^g_\pi \vee \tau^g_\pi$;
- If $\varphi = \sigma \Rightarrow \tau$, then $\varphi^g_\pi = \sigma^g_\pi \Rightarrow \tau^g_\pi$; and
- if $\varphi = \sigma \wedge \tau$, then $\varphi^g_\pi = \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma^g_\pi \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\tau^g_\pi$.

When $\pi = 0$, then we write $\varphi^g_0 = \varphi^g$. We first consider the case for $\pi = 0$.

Lemma 7 *φ is a classical tautology iff φ^g is a weak partition tautology iff $\neg\neg\varphi^g$ is a partition tautology.*

Proof: The idea of the proof is that the partition operations on the Gödel 0-transform φ^g mimic the Boolean 0, 1-operations on φ if we associate the partition interpretation $\sigma^g = 0$ with the Boolean $\sigma = 0$ and $\sigma^g \neq 0$ with the Boolean $\sigma = 1$. We proceed by induction over the complexity of the formula φ where the induction hypothesis is that: $\varphi = 1$ in the Boolean case iff $\varphi^g \neq 0$ in the partition case, which could also be stated as: $\varphi = 0$ in the Boolean case iff $\varphi^g = 0$ in the partition case. If φ is atomic, then $\varphi^g = \varphi \vee 0 = \varphi$. The Boolean assignment $\varphi = 0$ (the Boolean truth value

0) is associated with the partition assignment of $\varphi = \mathbf{0}_U$ (the indiscrete partition) and for atomic φ, $\varphi = \varphi \vee \mathbf{0}_U = \varphi^g$ so the hypothesis holds in the base case.

For the join in the Boolean case, $\varphi = \sigma \vee \tau = 1$ iff $\sigma = 1$ or $\tau = 1$. In the partition case, $\varphi^g = \sigma^g \vee \tau^g \neq \mathbf{0}_U$ iff $\sigma^g \neq \mathbf{0}_U$ or $\tau^g \neq \mathbf{0}_U$, so by the induction hypothesis, $\varphi = \sigma \vee \tau = 1$ iff $\sigma = 1$ or $\tau = 1$ iff $\sigma^g \neq \mathbf{0}_U$ or $\tau^g \neq \mathbf{0}_U$ iff $\varphi^g = \sigma^g \vee \tau^g \neq \mathbf{0}_U$.

For the implication in the Boolean case, $\varphi = \sigma \Rightarrow \tau = 0$ iff $\sigma = 1$ and $\tau = 0$. In the partition case, $\varphi^g = \sigma^g \Rightarrow \tau^g = \mathbf{0}_U$ iff $\sigma^g \neq \mathbf{0}_U$ and $\tau^g = \mathbf{0}_U$. Hence using the induction hypothesis, $\varphi = \sigma \Rightarrow \tau = 1$ iff $\sigma = 0$ or $\tau = 1$ iff $\sigma^g = \mathbf{0}_U$ or $\tau^g \neq \mathbf{0}_U$ iff $\varphi^g = \sigma^g \Rightarrow \tau^g \neq \mathbf{0}_U$.

For the meet in the Boolean case, $\varphi = \sigma \wedge \tau = 1$ iff $\sigma = 1 = \tau$. In the partition case, $\varphi^g = \neg\neg\sigma^g \wedge \neg\neg\tau^g = \mathbf{1}_U$ iff $\neg\neg\sigma^g = \mathbf{1}_U = \neg\neg\tau^g$ iff $\sigma^g \neq \mathbf{0}_U \neq \tau^g$. By the induction hypothesis, $\varphi = \sigma \wedge \tau = 1$ iff $\sigma = 1 = \tau$ iff $\sigma^g \neq \mathbf{0}_U \neq \tau^g$ iff $\varphi^g = \neg\neg\sigma^g \wedge \neg\neg\tau^g = \mathbf{1}_U$ iff $\varphi^g = \neg\neg\sigma^g \wedge \neg\neg\tau^g \neq \mathbf{0}_U$.

Thus φ is a classical tautology iff under any Boolean interpretation, $\varphi = 1$ iff for any partition interpretation, $\varphi^g \neq \mathbf{0}_U$ iff φ^g is a weak partition tautology iff $\neg\neg\varphi^g$ is a partition tautology. □

In this case of $\pi = \mathbf{0}_U$, the negation $\neg\sigma = \sigma \Rightarrow \mathbf{0}_U$ is unchanged and, for atomic variables φ, $\varphi \vee \mathbf{0}_U = \varphi$ so atomic variables are left unchanged in the Gödel $\mathbf{0}_U$-transform. Hence any classical formula φ expressed in the language of $\neg$, $\vee$, and $\Rightarrow$ (excluding the meet $\wedge$) would be unchanged by the Gödel 0-transform.

Corollary 3 *For any formula φ in the language of $\neg$, $\vee$, and $\Rightarrow$ along with 0 and 1, φ is a classical tautology iff φ is a weak partition tautology iff $\neg\neg\varphi$ is a partition tautology.*

For instance, the Gödel $\mathbf{0}_U$-transform of excluded middle $\sigma \vee \neg\sigma$ is the same formula, $\sigma \vee \neg\sigma$, which is a weak partition tautology, and $\neg\neg(\sigma \vee \neg\sigma)$ is a partition tautology.

The lemma generalizes to any π in the following form.

Proposition 10 *φ is a classical tautology iff $\overset{\pi}{\neg}\overset{\pi}{\neg}\varphi^g_\pi$ is a partition tautology.*

Proof: For any fixed partition π on a universe set U, the interpretation of the Gödel π-transform φ^g_π is in the upper interval $[\pi, \mathbf{1}_U] \subseteq \Pi(U)$. The key to the generalization is the standard result [7] that the upper interval $[\pi, \mathbf{1}_U]$ can be represented as the product of the sets $\Pi(B)$ where B is a non-singleton block of π:

$$[\pi, \mathbf{1}_U] \cong \prod \{\Pi(B) : B \in \pi,\ B \text{ non-singleton}\}.$$

Once we establish that the Gödel π-transform φ^g_π can be obtained, using the isomorphism, by computing the Gödel 0-transform φ^g "component-wise" in $\Pi(B)$, then we can apply the lemma component-wise to obtain the result.

Given a partition π on U, any interpretation of an atomic φ as a partition on U can be cut down to each non-singleton block $B \in \pi$ to yield a partition on B. Then $\varphi^g_\pi = \varphi \vee \pi$ has a block $B \in \pi$ iff $\varphi^g_0 = \varphi^g$ is equal to the zero $\mathbf{0}_B$ of $\Pi(B)$. Proceeding by induction over the complexity of φ, if $\varphi = \sigma \vee \tau$, then a block of $\varphi^g_\pi = \sigma^g_\pi \vee \tau^g_\pi$ is B iff B is a block of both σ^g_π and τ^g_π iff $\sigma^g = \mathbf{0}_B = \tau^g$ in $\Pi(B)$ iff $\varphi^g = \sigma^g \vee \tau^g = \mathbf{0}_B$ in $\Pi(B)$. If $\varphi = \sigma \Rightarrow \tau$, then $\varphi^g_\pi = \sigma^g_\pi \Rightarrow \tau^g_\pi$ has a block $B \in \pi$ iff σ^g_π does not have the block B and τ^g_π has the block B iff σ^g is not equal to $\mathbf{0}_B$ and τ^g is equal to $\mathbf{0}_B$ in $\Pi(B)$ iff $\varphi^g = \sigma^g \Rightarrow \tau^g = \mathbf{0}_B$ in $\Pi(B)$. If $\varphi = \sigma \wedge \tau$, then $\varphi^g_\pi = \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma^g_\pi \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\tau^g_\pi$ has a block $B \in \pi$ iff both σ^g_π and τ^g_π have a block B iff $\sigma^g = \mathbf{0}_B = \tau^g$ in $\Pi(B)$ iff $\varphi^g = \neg\neg\sigma^g \wedge \neg\neg\tau^g = \mathbf{0}_B$ in $\Pi(B)$.

Hence applying the lemma component-wise, φ is a classical tautology iff φ^g never evaluates to $\mathbf{0}_B$ in $\Pi(B)$ iff B is never a block of φ^g_π iff every block $B \in \pi$ is discretized in $\overset{\pi}{\neg}\overset{\pi}{\neg}\varphi^g_\pi$, i.e., $\overset{\pi}{\neg}\overset{\pi}{\neg}\varphi^g_\pi$ is a partition tautology. □

Thus the Gödel π-transform of excluded middle $\varphi = \sigma \vee (\sigma \Rightarrow 0)$ is $\varphi^g_\pi = (\sigma \vee \pi) \vee (\sigma \Rightarrow \pi)$ and $\overset{\pi}{\neg}\overset{\pi}{\neg}[(\sigma \vee \pi) \vee (\sigma \Rightarrow \pi)]$ is a partition tautology. Note that the single π-negation transform, the double π-negation transform, and the Gödel π-transform all gave different formulas starting with the classical excluded middle tautology.

[7] Since the partition lattice is conventionally written upside down, the usual result is stated in terms of the interval below π [28, p. 192].

2.4 Some partition tautologies

One source of some interesting results on partitions was Oystein Ore's early work on equivalence relations [41]. He defined two partitions to be *associable* if every block in their meet was a block of one or the other (or both).

Lemma 8 *The following statements are equivalent:*

1. σ and τ are associable;
2. no block of one partition just partially overlaps with a block of the other, i.e., for any $C \in \sigma$ and $D \in \tau$, if $C \cap D \neq \emptyset$, then $C \subseteq D$ or $D \subseteq C$;
3. $\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau) = \operatorname{indit}(\sigma \wedge \tau)$, i.e., $\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau) = \operatorname{dit}(\sigma \wedge \tau)$.

Proof: If $C \cap D \neq \emptyset$, $C - D \neq \emptyset$, and $D - C \neq \emptyset$, then a block in the meet will strictly contain C and D so it is not a block in σ or in τ so not #2 implies not #1, i.e., #1 implies #2 And if #2 holds, and $C \cap D \neq \emptyset$, then say $C \subseteq D$. Then for any other $C' \cap D \neq \emptyset$, we cannot have $D \subseteq C'$ so $C' \subseteq D$. Thus D is the exact union of the $C \in \sigma$ that overlap it so it is a block in the meet. And if $C \cap D \neq \emptyset$ and $D \subseteq C$, then by symmetry C is the exact union of $D \in \tau$ that overlap it so it is a block in the meet. Hence #1 and #2 are equivalent. Since $\operatorname{indit}(\sigma \wedge \tau) = \overline{\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau)}$, if #3 is not true, then there exists $(u, u') \in \operatorname{indit}(\sigma)$ and $(u', u'') \in \operatorname{indit}(\tau)$ so that (u, u'') is in the closure but not in the union. But that implies that there is a $C \in \sigma$ and $D \in \tau$ with $u \in C - D$, $u' \in C \cap D$, and $u'' \in D - C$ which negates #2 so #2 implies #3. Suppose #3 holds. For any $C \cap D \neq \emptyset$, there is a block $M \in \sigma \wedge \tau$ with $C, D \subseteq M$. But if $C \nsubseteq D$ and $D \nsubseteq C$, then there is $u \in C - D$, $u' \in C \cap D$, and $u'' \in D - C$. Then $(u, u') \in \operatorname{indit}(\sigma)$ and $(u', u'') \in \operatorname{indit}(\tau)$ and the union of the inditsets is an inditset, so (u, u'') must be included in the union but it is included in neither of the inditsets in the union. Hence $C \subseteq D$ or $D \subseteq C$ so #3 implies #2. $\square$

Although Ore didn't define the π-regular partitions (not having the partition implication), the partitions in the Boolean core $\mathcal{B}[\pi, \mathbf{1}_U]$ are all associable since their block-wise operation follows from their 'truth table' (see Table 4.2). Ore showed that any partition joined with the meet of two associable partitions will distribute across the meet. Hence we have the following corollary for any partitions φ, σ, τ, and π.

Lemma 9 (Ore's associability theorem) $\varphi \vee \left(\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\tau\right) = \left(\varphi \vee \overset{\pi}{\neg}\sigma\right) \wedge \left(\varphi \vee \overset{\pi}{\neg}\tau\right)$.

Moreover, if we assume that φ is in the upper segment $[\pi, \mathbf{1}_U]$, then the other distributivity law holds. In proving results about π-regular partitions, it is often useful to think in terms of B-slots as pictured in Figure 2.1.

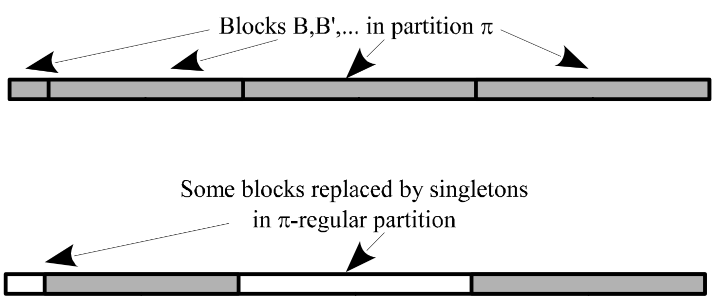

Figure 2.1: The B-slots in a π-regular partition.

Lemma 10 ("Dual" to Ore's theorem) *If $\pi \precsim \varphi$, then* $\varphi \wedge \left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau\right) = \left(\varphi \wedge \overset{\pi}{\neg}\sigma\right) \vee \left(\varphi \wedge \overset{\pi}{\neg}\tau\right)$.

Proof: If $\pi \precsim \varphi$, then every block $J \in \varphi$ is contained in a block B of π. If the result on the left-hand side for $\left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau\right)$ for that B-slot is $\mathbf{1}_B$ (i.e., the discretized version of B), then J stays a block in $\varphi \wedge \left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau\right)$. If the result of $\left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau\right)$ for that B is $\mathbf{0}_B$, then B is a block in the meet $\varphi \wedge \left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau\right)$. On the right-hand side, if the result of $\left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau\right)$ concerning that B-slot was $\mathbf{1}_B$, then either $\overset{\pi}{\neg}\sigma$ or $\overset{\pi}{\neg}\tau$ (or both) was $\mathbf{1}_B$ then either $\left(\varphi \wedge \overset{\pi}{\neg}\sigma\right)$ or $\left(\varphi \wedge \overset{\pi}{\neg}\tau\right)$ is J in part of the B-slot (and the other is J or B) so the join is J. If the B-slot in $\left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau\right)$ was $\mathbf{0}_B$, then both $\left(\varphi \wedge \overset{\pi}{\neg}\sigma\right)$ and $\left(\varphi \wedge \overset{\pi}{\neg}\tau\right)$ are B and thus so is the meet. □

Corollary 4 *Any $\varphi \in [\pi, \mathbf{1}_U]$ and any π-regular partition are associable.*

Proof: The meet $\varphi \wedge \overset{\pi}{\neg}\sigma$ has in the B-slot either J (along with other $J' \in \varphi$) or B so its blocks are always a block of one of the partitions in the meet. □

Proposition 11 (Distributivity over the Boolean core) *If $\pi \preceq \varphi$,*

$$\varphi \vee \left(\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\tau\right) = \left(\varphi \vee \overset{\pi}{\neg}\sigma\right) \wedge \left(\varphi \vee \overset{\pi}{\neg}\tau\right)$$
$$\varphi \wedge \left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau\right) = \left(\varphi \wedge \overset{\pi}{\neg}\sigma\right) \vee \left(\varphi \wedge \overset{\pi}{\neg}\tau\right). \square$$

F. William Lawvere has developed a number of interesting results in the context of co-Heyting algebras ([34]; [35]) we might look at the corresponding results in partition algebras. Since he was working with co-Heyting algebras (analogous to the algebra of equivalence relations), we need to dualize some of the terminology for the algebra of partitions. Thus the "difference from 1" (e.g., $(\neg\sigma)^d = (\sigma \Rightarrow 0)^d = \sigma^d \not\Leftarrow 0^d = 0^d - \sigma^d$ in the algebra of equivalence relations where $0^d = \widehat{1}$ is the top or "one" of that algebra) has the dual "implication to $\mathbf{0}_U$," i.e., $\neg\sigma = \sigma \Rightarrow \mathbf{0}_U$, in the partition algebra. Since the "implication to zero" negation is so trivial, we will also relativize the negation using an arbitrary π in place of $\mathbf{0}_U$.

Lawvere defined the "boundary" of an element as the meet with its negation, so dualizing we can define the π-*coboundary* of a partition σ as the join $\partial^\pi\sigma = \sigma \vee \overset{\pi}{\neg}\sigma$ with its π-negation. Thinking in terms of the B-slot in $\partial^\pi\sigma = \sigma \vee \overset{\pi}{\neg}\sigma$, if there is a $C \in \sigma$ with $B \subseteq C$, then the B-slot is $\mathbf{1}_B$ and otherwise the B-slot is the restriction $\sigma \upharpoonright B = \{C \cap B \neq \emptyset : C \in \sigma\}$. Lawvere's boundary is nowhere dense in the sense that its double negation is zero, and in our dual case, the π-coboundary is π-dense in the sense that its π-double-negation transform is $\mathbf{1}_U$:

$$\overset{\pi}{\neg}\overset{\pi}{\neg}\partial^\pi\sigma = \overset{\pi}{\neg}\overset{\pi}{\neg}\left(\sigma \vee \overset{\pi}{\neg}\sigma\right) = \mathbf{1}_U.$$

Lawvere defines the "core" of an element as its double negation and then proves that each element is equal to its boundary joined with its core. In the dual case of partitions, we have the π-version of that result.

Proposition 12 (Lawvere's boundary + core law for partitions) $\partial^\pi\sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma = \sigma \vee \pi$.

Proof: This is easily proved from Ore's associability theorem using some basic identities such as the "law of contradiction" in $\boldsymbol{B}_\pi$: $\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma = \pi$ as well as $\sigma \preceq \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ so that $\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma = \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$. Then using Ore's theorem:

$$\sigma \vee \pi = \sigma \vee \left(\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma\right) = \left(\sigma \vee \overset{\pi}{\neg}\sigma\right) \wedge \left(\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma\right) = \partial^\pi\sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma. \square$$

Restricting to $\sigma \in [\pi, 1]$, any σ that refines π (so that $\sigma \vee \pi = \sigma$) can be reconstructed from its π-double-negation $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ by taking the meet with its π-coboundary $\partial^\pi \sigma = \sigma \vee \overset{\pi}{\neg}\sigma$.

Corollary 5 *If $\sigma \in [\pi, 1]$, then $\sigma = \partial^\pi \sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$.* $\square$

Lawvere showed that the Leibniz $(fg)' = f(g') + (f')g$ holds in the co-Heyting algebra of closed subsets by replacing the derivative by the boundary. In our case, the dual Leibniz rule holds using the π-coboundary.

Proposition 13 (co-Leibniz rule for partitions) $\partial^\pi(\sigma \vee \tau) = (\partial^\pi \sigma \vee \tau) \wedge (\sigma \vee \partial^\pi \tau)$.

Proof: Analyzing using B-slots, $\partial^\pi(\sigma \vee \tau) = (\sigma \vee \tau) \vee \overset{\pi}{\neg}(\sigma \vee \tau)$ so the blocks of $\sigma \vee \tau$ are the non-empty intersections, and the contents of the B-slot are $\mathbf{1}_B$ if there is a $C \in \sigma$ and $D \in \tau$ such that $B \subseteq C \cap D$, and otherwise are the restriction $(\sigma \vee \tau) \upharpoonright B = \{C \cap D \cap B \neq \emptyset : C \in \sigma, D \in \tau\}$. On the RHS, the B-slot contents of $\partial^\pi \sigma \vee \tau = \sigma \vee \tau \vee \overset{\pi}{\neg}\sigma$ are $\mathbf{1}_B$ if there is a $C \in \sigma$ with $B \subseteq C$ and otherwise $(\sigma \vee \tau) \upharpoonright B$. The B-slot contents of $\sigma \vee \partial^\pi \tau = \sigma \vee \tau \vee \overset{\pi}{\neg}\tau$ are $\mathbf{1}_B$ if there is a $D \in \tau$ with $B \subseteq D$ and otherwise $(\sigma \vee \tau) \upharpoonright B$. Hence the B-slot contents of the meet $(\partial^\pi \sigma \vee \tau) \wedge (\sigma \vee \partial^\pi \tau)$ are $\mathbf{1}_B$ if there is both a $C \in \sigma$ and a $D \in \tau$ with $B \subseteq C, D$ and is otherwise $(\sigma \vee \tau) \upharpoonright B$. Hence both sides have the same B-slot content and thus are equal. $\square$

2.5 Partition logic via the RST-closure space $\wp(U \times U)$

2.5.1 The RST-closure space $\wp(U \times U)$

In the Boolean logic of subsets, the standard semantics is the powerset Boolean algebra $\wp(U)$ and the analogous structure for partition logic would seem to be the algebra of partitions $\Pi(U)$. But partition logic is more complicated and has a richer theory. The partition algebra $\Pi(U)$ is isomorphic to the algebra of open subsets $\mathcal{O}(U \times U)$ where an open subset of $U \times U$ (a partition relation) is the complement of the Reflexive-Symmetric-Transitive or RST closure of any subset of $U \times U$. The powerset Boolean algebra on $U \times U$ (i.e., the BA of all binary relations on U) equipped with the RST-closure operation will be called the *RST-closure space of* U, $\wp(U \times U)$.

The RST-closure is a closure operation in the standard sense [4, p. 111]: for any $V, W \subseteq U \times U$,

- $V \subseteq \overline{V}$,
- $\overline{V} = \overline{\overline{V}}$ (idempotent) and
- If $V \subseteq W$, then $\overline{V} \subseteq \overline{W}$ (monotone).

It follows that the *closed subsets*, i.e., $V = \overline{V}$, form a complete lattice, the complete lattice of equivalence relations on U in our case. The complements of the closed subsets of $\wp(U \times U)$ are the open subsets which are the partition relations (ditsets of partitions) and give the isomorphism $\Pi(U) \cong \mathcal{O}(U \times U)$. Arbitrary intersections of closed subsets are closed and thus arbitrary unions of open subsets are open. But the RST-closure is not topological in that sense that a union of closed subsets is not necessarily closed, and thus an intersection of open subsets is not necessarily open. Just as every subset V has a closure (which is an equivalence relation, the intersection of all closed subsets containing V), so every subset has an interior $\operatorname{int}(V) = \overline{(V^c)}^c$, the complement of the RST-closure of the complement of V, which is the ditset of a partition on U which might be denoted. $\lambda(\operatorname{int}(V))$, read as "the partition whose ditset is $\operatorname{int}(V)$."

The adjunctive definition of the partition implication:

$$\operatorname{dit}(\tau) \cap \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi) \text{ iff } \operatorname{dit}(\tau) \subseteq \operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi))$$

was formulated in $\wp(U \times U)$. It hints that the deeper analysis of partition logic might be better formulated in that closure space rather than $\Pi(U)$.

By working in a powerset BA, we also have the luxury of using illustrations using Venn diagrams. Each partition π on U partitions the space into $\operatorname{dit}(\pi)$ and $\operatorname{indit}(\pi)$. The largest consequence of working with $\wp(U \times U)$ is the logical definition of information as distinctions so the information given in a partition π is given by its ditset $\operatorname{dit}(\pi)$. For finite U, a probability measure on $U \times U$ is obtained by using the product measure from U where the probability measure on U could be equiprobable points or point probabilities. Then the probability assigned to $\operatorname{dit}(\pi)$ is the logical entropy $h(\pi)$. The development of logical entropy and its relation to Shannon entropy provides new logical foundations for information theory that has been developed elsewhere [23]. Hence the focus here is on partition logic itself.

The "structure theorem" for those two complementary subsets gives:

$$\operatorname{indit}(\pi) = \cup_{B\in\pi} B \times B \text{ and } \operatorname{dit}(\pi) = \cup_{B,B'\in\pi, B\neq B'} B \times B'$$

where the union for $\operatorname{dit}(\pi)$ are interpreted to include both $B \times B'$ and $B' \times B$. Given two partitions π and σ, the two partitions on $U \times U$ intersect to give $2^2 = 4$ atomic areas. The unions in the following propositions are interpreted as including the permutations of the factors in the Cartesian products.

Proposition 14 *Structure theorem for 4-part partition of closure space $U \times U$ given by any σ and π:*

1. $\bigcup_{B\in\pi, C\in\sigma} (B - (B \cap C)) \times (C - (B \cap C)) = \operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)$;
2. $\bigcup_{B\in\pi, C\in\sigma} (B - (B \cap C)) \times (B \cap C) = (\operatorname{indit}(\pi) \cap \operatorname{dit}(\sigma))$;
3. $\bigcup_{B\in\pi, C\in\sigma} (B \cap C) \times (C - (B \cap C)) = (\operatorname{indit}(\sigma) \cap \operatorname{dit}(\pi))$;and
4. $\bigcup_{B\in\pi, C\in\sigma} (B \cap C) \times (B \cap C) = \operatorname{indit}(\sigma) \cap \operatorname{indit}(\pi)$.

Proof: Part 1: The union is disjoint since each summand will differ by a B or a C. Assume $(u, u') \in (B - (B \cap C)) \times (C - (B \cap C))$ so $u \in B - (B \cap C) = B - C$ and $u' \in C - B$. Since $u' \in C - B$, it must be in a different block of π than B so $(u, u') \in \operatorname{dit}(\pi)$. Similarly since $u \in B - C$, it must be in a different block of σ than C so $(u, u') \in \operatorname{dit}(\sigma)$ and thus $(u, u') \in \operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)$. Conversely if $(u, u') \in \operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)$, then u is in some block $B \in \pi$ and u' is in some block $C \in \sigma$. But since (u, u') is a dit of both partitions u cannot be in C and u' cannot be in B so $u \in B - C$ and $u' \in C - B$ and thus $(u, u') \in (B - C) \times (C - B)$.

Part 2: Let $(u, u') \in (B - C) \times (B \cap C)$ so u is in B but not in C while u' is in both so $(u, u') \in \operatorname{indit}(\pi) \cap \operatorname{dit}(\sigma)$. Conversely, if $(u, u') \in \operatorname{indit}(\pi) \cap \operatorname{dit}(\sigma)$ then for $u, u' \in B \in \pi$ and $u' \in C \in \sigma$, then since $(u, u') \in \operatorname{dit}(\sigma)$, u must be in a different block $C' \in \sigma$ so $u \in B - C$ and $u' \in B \cap C$.

Parts 3 and 4 proved in a similar manner. $\square$

These structure theorems extend to the 2^n atomic areas determined by n partitions on U.

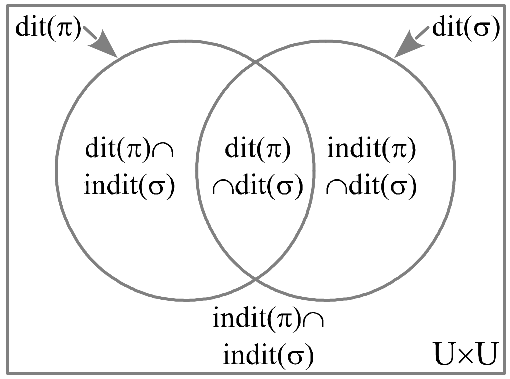

Figure 2.2: Four atomic areas in $U \times U$ Venn diagram for two partitions.

The structure theorem shows how ditsets or inditsets of a partition can be constructed from how the partition 'interacts' with another partition. For instance, the disjoint union of the two atomic areas in Figure 2.2 giving $\operatorname{dit}(\pi)$ is:

$$\begin{aligned}\operatorname{dit}(\pi) &= (\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)) \cup (\operatorname{dit}(\pi) \cap \operatorname{indit}(\sigma)) \\ &= \bigcup_{B \in \pi, C \in \sigma} (B - C) \times (C - B) \bigcup \bigcup_{B \in \pi, C \in \sigma} (B \cap C) \times (C - B).\end{aligned}$$

In logical information theory for a finite U with the product probability distribution on $U \times U$, the logical entropy of a partition is the probability of its ditset, $h(\pi) = \Pr(\operatorname{dit}(\pi))$. Then $\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)$ represents the mutual information (i.e., common distinctions) of π and σ whose probability is $m(\pi, \sigma) = \Pr(\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma))$ and $\operatorname{dit}(\pi) \cap \operatorname{indit}(\sigma) = \operatorname{dit}(\pi) - \operatorname{dit}(\sigma)$ represents the information given by π after we take away the information of σ whose probability is $h(\pi|\sigma)$. Hence the representation of $\operatorname{dit}(\pi)$ as the disjoint union gives the representation of the information in π as the sum of the information in common with σ plus the information in π after the information in σ has been taken away:

$$h(\pi) = m(\pi, \sigma) + h(\pi|\sigma).$$

The structure theorem also allows us to connect properties of partitions with properties of their ditsets. For instance, two partitions π and σ on a finite U are said to be (stochastically) *independent* if with equiprobable points, for all $B \in \pi$ and $C \in \sigma$, $p_{B \cap C} = p_B p_C$, i.e., $\frac{|B \cap C|}{|U|} = \frac{|B|}{|U|} \frac{|C|}{|U|}$ or $|B \cap C| \, |U| = |B| \, |C|$. The equiprobable distribution on U induces the equiprobable distribution on $U \times U$, so we also have the notion of two events $V, W \subseteq U \times U$ being *independent* if $p_{V \cap W} = p_V p_W$, i.e., $\frac{|V \cap W|}{|U \times U|} = \frac{|V|}{|U \times U|} \frac{|W|}{|U \times U|}$ or $|V \cap W| \, |U|^2 = |V| \, |W|$.

Proposition 15 *If π and σ are independent partitions, then* $\operatorname{dit}(\pi)$ *and* $\operatorname{dit}(\sigma)$ *are independent events in $U \times U$ (equiprobable points on finite U).*

Proof: If π and σ are independent partitions, then $|B \cap C| \, |U| = |B| \, |C|$ for all $B \in \pi$ and $C \in \sigma$. Since the unions in the Structure Theorem are disjoint, the unions can be expressed as sums. Thus from Part 1, we have:

$|\text{dit}(\pi) \cap \text{dit}(\sigma)| = \sum_{B\in\pi, C\in\sigma} (|B| - |B\cap C|) \times (|C| - |B \cap C|)$

$$\begin{aligned}
|\text{dit}(\pi) \cap \text{dit}(\sigma)| &= \sum_{B\in\pi, C\in\sigma} (|B| - |B\cap C|)(|C| - |B\cap C|) \\
&= \sum_{B\in\pi, C\in\sigma} \left(|B| - \frac{|B||C|}{|U|}\right)\left(|C| - \frac{|B||C|}{|U|}\right) \\
&= \frac{1}{|U|^2} \sum_{B\in\pi, C\in\sigma} (|B||U| - |B||C|)(|C||U| - |B||C|) \\
&= \frac{1}{|U|^2} \sum_{B\in\pi, C\in\sigma} (|B|(|U| - |C|))(|C|(|U| - |B|)) \\
&= \frac{1}{|U|^2} \left[\sum_{B\in\pi} |B|(|U| - |B|)\right]\left[\sum_{C\in\sigma} |C|(|U| - |C|)\right] \\
&= \frac{1}{|U|^2} |\text{dit}(\pi)| |\text{dit}(\sigma)| . \square
\end{aligned}$$

Furthermore, dividing both sides of the equation by $|U|^2$ gives for independence in the equiprobable case;

$$m(\pi, \sigma) = h(\pi) h(\sigma)$$

as one would expect since logical entropies are the two-draws-from-U probabilities of getting a distinction. Hence in the case of independence, the probability that two draws yield a distinction of both π and σ is the product of the probability of getting a distinction of π times the probability of getting a distinction of σ.

2.5.2 The sixteen binary logical operations

Starting with two subsets $\text{dit}(\sigma)$ and $\text{dit}(\tau)$ in a Venn diagram, there are sixteen possible unions of the four atomic areas (each union defined by whether each atomic area is included or not), so there are sixteen definable areas in the Venn diagram. The interiors of the sixteen definable areas define the ditsets of the sixteen binary logical operations on partitions–and their closures define the sixteen binary logical operations on equivalence relations. The operations have a natural pairing since each definable area has a definable complement so the interiors of those complementary subsets gives a complement-pairing of operations. For instance, the interior $\text{int}[\text{dit}(\sigma) \cap \text{dit}(\tau)]$ is $\text{dit}(\sigma \wedge \tau)$ and the interior of its complement $\text{int}[\text{indit}(\sigma) \cup \text{indit}(\tau)]$ is $\text{dit}(\sigma|\tau)$. Previously, we defined that two partitions φ and φ' are orthogonal if $\neg\varphi \vee \neg\varphi' = \mathbf{1}_U$. The negation $\neg\varphi = \varphi \Rightarrow \mathbf{0}_U$ is either $\mathbf{0}_U$ if $\varphi \neq \mathbf{0}_U$ or $\mathbf{1}_U$ if $\varphi = \mathbf{0}_U$. Hence φ and φ' are orthogonal iff one or both are $\mathbf{0}_U$. There is the partition identity:

$$\neg\varphi \vee \neg\varphi' = \varphi \Rightarrow \neg\varphi' = \varphi' \Rightarrow \neg\varphi$$

which carries the sense that when $\neg\varphi \vee \neg\varphi' = \mathbf{1}_U$ then if one is not $\mathbf{0}_U$, then the other has to be $\mathbf{0}_U$. Since the interior of a definable area and the interior of its complement are disjoint, we know from the Common Dits Theorem that one of them has to be $\mathbf{0}_U$ so the complementary-dual partitions are orthogonal. Table 2.4 gives the complementary duality pairings.

Defined subset $S \subseteq U \times U$	$\operatorname{int}(S)$ is dit set	$\operatorname{int}(S^c)$ is dit set
$U \times U$	$\mathbf{1}_U$	$\mathbf{0}_U$
$\operatorname{dit}(\sigma) \cup \operatorname{dit}(\tau)$	$\sigma \vee \tau$	$\sigma \overline{\vee} \tau = \neg\sigma \wedge \neg\tau$
$\operatorname{indit}(\sigma) \cup \operatorname{dit}(\tau)$	$\sigma \Rightarrow \tau$	$\tau \not\Leftarrow \sigma = \sigma \wedge \neg\tau$
$\operatorname{dit}(\tau)$	τ	$\neg\tau$
$\operatorname{dit}(\sigma) \cup \operatorname{indit}(\tau)$	$\tau \Rightarrow \sigma$	$\sigma \not\Leftarrow \tau = \tau \wedge \neg\sigma$
$\operatorname{dit}(\sigma)$	σ	$\neg\sigma$
$(\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau)) \cup (\operatorname{indit}(\sigma) \cap \operatorname{indit}(\tau))$	$(\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma)$	$(\sigma \vee \tau) \wedge (\sigma \vert \tau)$
$\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau)$	$\sigma \wedge \tau$	$\sigma \vert \tau$

Table 2.4: Complementary-dual pairing of sixteen binary logical operations.

In the standard Venn diagram for two subsets $\operatorname{dit}(\sigma)$ and $\operatorname{dit}(\tau)$, there are four atomic areas and $2^4 = 16$ definable subsets. Those subsets can be arranged as the vertices of the 4-hypercube graph Q_4 where the edges are the Hasse diagram inclusions, i.e., an edge indicates an inclusion with no other subsets in between. Since the interior operator is monotonic, we can take the interiors of all the subsets so the vertices can be labeled with the corresponding partitions and the edges indicate refinement relations. In Figured 2.3, the equivalence $(\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma)$ and the inequivalence (or symmetric difference) have been abbreviated respectively as $\sigma \Longleftrightarrow \tau$ and $\sigma \Delta \tau$.

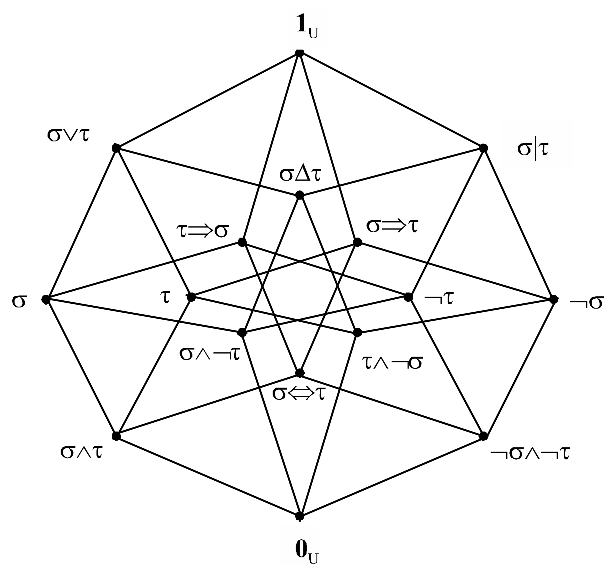

Figure 2.3: Hypercube graph representation of the sixteen binary logical operations on partitions.

With each vertex in the graph, there is an opposite vertex (representing the complementary subset) and the corresponding partitions are the complementary duals. For instance, the dual of $\sigma \vee \tau$ is $\neg\sigma \wedge \neg\tau$ and the dual of $\sigma \Delta \tau$ is $\sigma \Longleftrightarrow \tau$. Furthermore, it might be noticed that whenever two partitions have a common immediate predecessor, then since $\operatorname{int}(S \cap T) = \operatorname{int}(\operatorname{int}(S) \cap \operatorname{int}(T))$, then that common predecessor is their meet. For instance, $(\sigma \vee \tau) \wedge (\sigma|\tau) = \sigma \Delta \tau$ and $\sigma \wedge (\sigma \Delta \tau) = \sigma \wedge \neg\tau$. The same relation for common successors does not hold for the join (unless we dualize to equivalence relations). For instance, $\sigma|\tau$ is an immediate common successor of $\neg\sigma$ and $\neg\tau$ but is not necessarily their join.

Due to the rather 'severe' nature of the $\mathbf{0}_U$-negation $\sigma \Rightarrow \mathbf{0}_U$, we have seen that more interesting structure emerges by relativizing the operations to some fixed π. In the usual Venn diagram for three areas, $\operatorname{dit}(\pi)$, $\operatorname{dit}(\sigma)$, and $\operatorname{dit}(\tau)$, there are eight atomic areas. But to work in the segment $[\pi, \mathbf{1}_U]$, we focus on the areas that include dit (π). There are four atomic areas outside of dit (π) as in Figure 2.4 that could be unioned with dit (π).

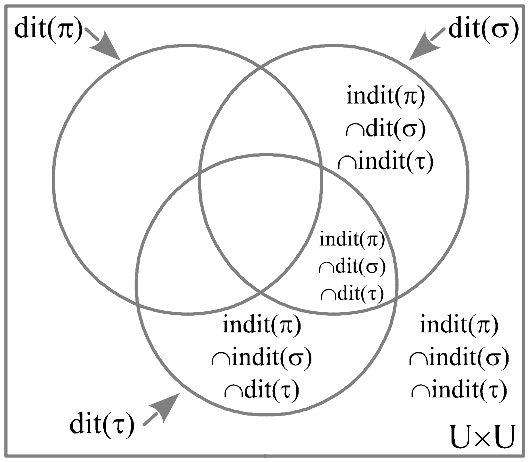

Figure 2.4: The four atomic areas outside of dit (π).

But since we are working in the BA $\wp(U \times U)$, we can distribute the union of dit (π) with these four areas to get the four π-relativized 'atomic' areas:

- $\operatorname{dit}(\pi) \cup (\operatorname{indit}(\pi) \cap \operatorname{dit}(\sigma) \cap \operatorname{indit}(\tau)) = \operatorname{dit}(\pi) \cup (\operatorname{dit}(\sigma) \cap \operatorname{indit}(\tau))$;
- $\operatorname{dit}(\pi) \cup (\operatorname{indit}(\pi) \cap \operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau)) = \operatorname{dit}(\pi) \cup (\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau))$;
- $\operatorname{dit}(\pi) \cup (\operatorname{indit}(\pi) \cap \operatorname{indit}(\sigma) \cap \operatorname{dit}(\tau)) = \operatorname{dit}(\pi) \cup (\operatorname{indit}(\sigma) \cap \operatorname{dit}(\tau))$; and
- $\operatorname{dit}(\pi) \cup (\operatorname{indit}(\pi) \cap \operatorname{indit}(\sigma) \cap \operatorname{indit}(\tau)) = \operatorname{dit}(\pi) \cup (\operatorname{indit}(\sigma) \cap \operatorname{indit}(\tau))$.

There again sixteen definable unions of these four areas and the interiors of those sixteen areas give the sixteen π-relativized 'binary' logical operations on partition–which are actually ternary operations of a special form. Moreover, each of the areas has a complement in $U \times U$ with dit (π) added back in to give the π-complementary-dual π-relativized partition operations in Table 2.5. In Table 2.5, $S = \operatorname{dit}(\sigma)$, $T = \operatorname{dit}(\tau)$, $P = \operatorname{dit}(\pi)$, and ^{c} is complement.

$V \subseteq U \times U$	$\operatorname{int}(V \cup P)$ is ditset	$\operatorname{int}(V^c \cup P)$ is ditset
$V = U \times U$	$\mathbf{1}_U$	π
$V = S \cup T$	$\sigma \vee \tau \vee \pi$	$\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\tau$
$V = S^c \cup T$	$\sigma \Rightarrow (\tau \vee \pi)$	$(\sigma \vee \pi) \wedge \overset{\pi}{\neg}\tau$
$V = T$	$\tau \vee \pi$	$\overset{\pi}{\neg}\tau$
$V = S \cup T^c$	$\tau \Rightarrow (\sigma \vee \pi)$	$\overset{\pi}{\neg}\sigma \wedge (\tau \vee \pi)$
$V = S$	$\sigma \vee \pi$	$\overset{\pi}{\neg}\sigma$
$V = (S \cap T) \cup (S^c \cap T^c)$	$(\sigma \Rightarrow (\tau \vee \pi)) \wedge (\tau \Rightarrow (\sigma \vee \pi))$	$(\sigma \vee \tau \vee \pi) \wedge (\sigma\vert_\pi \tau)$
$V = S \cap T$	$(\sigma \vee \pi) \wedge (\tau \vee \pi)$	$\sigma\vert_\pi \tau$

Table 2.5: π-complementary-duals of π-relativized partitions.

Figure 2.5 is the hypercube graph drawn by replacing each partition in Figure 2.3 by its π-relativized version and the same relationships hold.

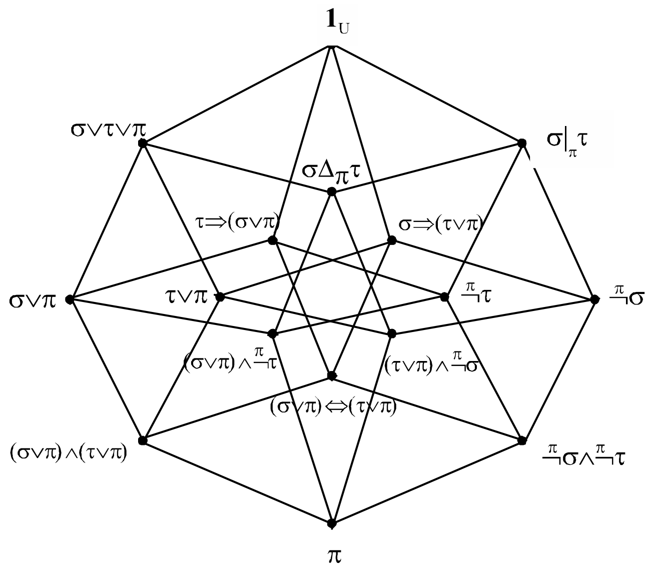

Figure 2.5: Hypercube graph for sixteen π-relativized binary logical operations.

It is then immediate that the meet of each partition with its π-complementary dual is π as indicated in Table 2.5.

Meet of π-complementary-duals $= \pi$
$\mathbf{1}_U \wedge \pi = \pi$
$(\sigma \vee \tau \vee \pi) \wedge \left(\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\tau\right) = \pi$
$(\sigma \Rightarrow (\tau \vee \pi)) \wedge \left((\sigma \vee \pi) \wedge \overset{\pi}{\neg}\tau\right) = \pi$
$(\tau \vee \pi) \wedge \left(\overset{\pi}{\neg}\tau\right) = \pi$
$(\tau \Rightarrow (\sigma \vee \pi)) \wedge \left(\overset{\pi}{\neg}\sigma \wedge (\tau \vee \pi)\right) = \pi$
$(\sigma \vee \pi) \wedge \left(\overset{\pi}{\neg}\sigma\right) = \pi$
$(\sigma \Rightarrow (\tau \vee \pi)) \wedge (\tau \Rightarrow (\sigma \vee \pi)) \wedge (\sigma \vee \tau \vee \pi) \wedge (\sigma\vert_\pi\tau) = \pi$
$((\sigma \vee \pi) \wedge (\tau \vee \pi)) \wedge (\sigma\vert_\pi\tau) = \pi$

Table 2.5: Meet of π-complementary dual partitions.

One immediate result is that the π-negation of the LHS formulas in Table 2.5 are $\pi \Rightarrow \pi = \mathbf{1}_U$, so the π-negation of all the LHS formulas are partition tautologies, e.g., $\overset{\pi}{\neg}\left[(\sigma \vee \pi) \wedge \left(\overset{\pi}{\neg}\sigma\right)\right] = \mathbf{1}_U$.

This basic theorem is essentially a π-generalization of the Common-Dits Theorem.

Theorem 11 *For any $V \subseteq U \times U$, each block $B \in \pi$ is contained in a block of* $\operatorname{int}(V \cup \operatorname{dit}(\pi))$ *or in a block of* $\operatorname{int}(V^c \cup \operatorname{dit}(\pi))$.

Proof: Taking σ as the partition with the ditset $\operatorname{int}(V \cup \operatorname{dit}(\pi))$, i.e., $\sigma = \lambda(\operatorname{int}(V \cup \operatorname{dit}(\pi)))$, and $\tau = \lambda(\operatorname{int}(V^c \cup \operatorname{dit}(\pi)))$, we have $\operatorname{dit}(\pi) \subseteq \operatorname{dit}(\sigma) \subseteq V \cup \operatorname{dit}(\pi)$ and $\operatorname{dit}(\pi) \subseteq \operatorname{dit}(\tau) \subseteq V^c \cup \operatorname{dit}(\pi)$ so that $\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau) = \operatorname{dit}(\pi)$. Complementing gives $\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau) = \operatorname{indit}(\pi)$. Then for any $B \in \pi$, consider the two cut-down partitions on B, $\sigma \upharpoonright B$ and $\tau \upharpoonright B$. Then $(B \times B) \cap (\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau)) = B \times B$ and $(B \times B) \cap \operatorname{indit}(\sigma) = \operatorname{indit}(\sigma \upharpoonright B)$ and $(B \times B) \cap \operatorname{indit}(\tau) = \operatorname{indit}(\tau \upharpoonright B)$. Thus

$$\operatorname{indit}(\tau \upharpoonright B) \cup \operatorname{indit}(\tau \upharpoonright B) = B \times B$$

so by the equivalence relation version of the Common-Dits Theorem, either $\operatorname{indit}(\sigma \upharpoonright B) = B \times B$ or $\operatorname{indit}(\tau \upharpoonright B) = B \times B$ so $B \subseteq C$ for some $C \in \sigma$ or $B \subseteq D$ for some $D \in \tau$. □

Of course, a block $B \in \pi$ is contained in any π-regular partition $\overset{\pi}{\neg}\sigma$ or its π-negation $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ in B_π but the theorem is more general for any φ (taking $V = \operatorname{dit}(\varphi)$) and its π-complementary-dual $\lambda(\operatorname{int}(\operatorname{indit}(\varphi) \cup \operatorname{dit}(\pi)))$.

Corollary 6 *For any $V \in U \times U$, for $\varphi = \lambda(\operatorname{int}(V \cup \operatorname{dit}(\pi)))$ and $\varphi' = \lambda(\operatorname{int}(V^c \cup \operatorname{dit}(\pi)))$, φ and φ' are π-orthogonal.*

Proof: Now φ and φ' are π-orthogonal if $\overset{\pi}{\neg}\varphi \vee \overset{\pi}{\neg}\varphi' = \mathbf{1}_U$. Since B is contained in a block of φ or φ', the B-slot in $\overset{\pi}{\neg}\varphi$ or $\overset{\pi}{\neg}\varphi'$ will be $\mathbf{1}_B$, so the join is $\mathbf{1}_U$. □

Corollary 7 *The pairs of π-complementary dual operations on any σ and τ are π-orthogonal.* □

It should be noted that for different V's, $\lambda(\operatorname{int}(V \cup \operatorname{dit}(\pi)))$ can be the same partition with different π-orthogonals $\lambda(\operatorname{int}(V^c \cup \operatorname{dit}(\pi)))$. For instance with $\pi = \mathbf{0}_U$, for $V = \operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau)$, $\sigma \wedge \tau = \lambda(\operatorname{int}(V))$ and for $V' = \operatorname{int}[\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau)] = \operatorname{dit}(\sigma \wedge \tau)$, then we also have $\sigma \wedge \tau = \lambda(\operatorname{int}(V'))$. But taking complements,

$$\begin{gathered}\lambda(\operatorname{int}(V^c)) = \lambda(\operatorname{int}(\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau))) = \sigma|\tau \text{ and} \\ \lambda(\operatorname{int}(V'^c)) = \lambda(\operatorname{int}(\operatorname{indit}(\sigma \wedge \tau))) = \neg(\sigma \wedge \tau) = (\sigma \wedge \tau) \Rightarrow \mathbf{0}_U\end{gathered}$$

which are not necessarily the same, although there is the ordering relationship:

$$\neg\sigma \vee \neg\tau \precsim \sigma|\tau \precsim \neg(\sigma \wedge \tau).$$

2.5.3 The π-orthogonal algebra $\mathcal{A}_\pi = [\pi, \mathbf{1}_U]$

We have seen a number of suggestive analogies between the upper segment $[\pi, 1_U]$ of $\Pi(U)$ and the Heyting algebra of open subsets of a topological space. All the sixteen binary logical operations on partitions can be defined relative to π, and in particular, negation is then π-negation. In a Heyting algebra and in the upper segment $[\pi, 1_U]$, the negated elements form a Boolean algebra. In the Heyting algebra of open subsets, there is a natural notion of disjointness, namely the intersection of open subsets is the null set, and the negation of an open subset is the largest open subset disjoint from it. Similarly, in the upper segment $[\pi, 1_U]$, there is the notion of π-orthogonality between partitions and the π-negation of a partition is the largest (most refined) partition π-orthogonal to it. All of these similarities suggest that the upper segment $[\pi, 1_U]$ is an algebraic object worthy of study in its own right. Roughly speaking, it is to partitions (i.e., open subsets of $\wp(U \times U)$) what the Heyting algebra is to open subsets of a topological space. Hence we will call the upper segments $[\pi, 1_U]$, *π-orthogonal algebras*, and denote them as $\mathcal{A}_\pi$.

The π-orthogonal algebras are of independent interest since they are not Heyting algebras as we have seen for a number of reasons such as they are not distributive as lattices and the π-negation is not a (relative) pseudo-complementation. Moreover, there is not inclusion either way between partition tautologies and Heyting algebra validities. However one way to study the π-orthogonal algebras is to look at the similarities and dissimilarities with Heyting algebras of open subsets. Hence we start with a number of corollaries of previous results that show similarities.

Corollary 8 *If φ and φ' are π-orthogonal, then $\varphi \vee \pi$ and $\varphi' \vee \pi$ are π-orthogonal and $(\varphi \vee \pi) \wedge (\varphi' \vee \pi) = \pi$. In particular, if $\pi \precsim \varphi, \varphi'$, then $\varphi \wedge \varphi' = \pi$.*

Proof: Since $\varphi \vee \pi$ and $\varphi' \vee \pi$ refine π and each B must be contained in a block of φ or φ', B must equal a block of $\varphi \vee \pi$ or $\varphi' \vee \pi$, so the B-slot in the join is always $\mathbf{1}_B$ and in the meet is $\mathbf{0}_B = B$, i.e., $(\varphi \vee \pi) \wedge (\varphi' \vee \pi) = \pi$. □

For an example using the partition notation that concatenated letters (with no commas in between) are in the same block, let $\varphi = \{abc, def\}$, $\varphi' = \{abe, cdf\}$ and $\pi = \{ab, cd, ef\}$. Then $\overset{\pi}{\neg}\varphi = \{a, b, cd, e, f\}$ and $\overset{\pi}{\neg}\varphi' = \{a, b, c, d, ef\}$ so φ and φ' are π-orthogonal but $\varphi \wedge \varphi' = \mathbf{0}_U$. But $\varphi \vee \pi = \{ab, c, d, ef\}$ and $\varphi' \vee \pi = \{ab, cd, e, f\}$, so they are still π-orthogonal and $(\varphi \vee \pi) \wedge (\varphi' \vee \pi) = \pi$.

Corollary 9 *For any partition φ, $\overset{\pi}{\neg}\varphi$ is the maximal (i.e., most refined) π-orthogonal partition to φ.*

Proof: Any φ and $\overset{\pi}{\neg}\varphi$ are π-orthogonal since $\overset{\pi}{\neg}\varphi \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\varphi = \mathbf{1}_U$, i.e., the law of excluded middle in the BA $\mathcal{B}[\pi, \mathbf{1}_U]$. If φ' is any other π-orthogonal partition to φ, then $\varphi' \Rightarrow \overset{\pi}{\neg}\varphi = \mathbf{1}_U$ so $\varphi' \precsim \overset{\pi}{\neg}\varphi$. $\square$

It is tempting to think that $\varphi \wedge \varphi' = \pi$ is equivalent to π-orthogonality in analogy with disjointness in a Heyting algebra. But that is not so since for $\varphi = \{ab, c\}$, $\varphi' = \{a, bc\}$, and $\pi = \mathbf{0}_U$, then $\varphi \wedge \varphi' = \mathbf{0}_U$ but φ and φ' are not orthogonal. The meet-related characterization of π-orthogonality is just a little more subtle.

Corollary 10 *If $\pi \precsim \sigma, \tau$ and σ and τ are π-orthogonal, then σ and τ are associable.*

Proof: Since $\pi \precsim \sigma, \tau$, for any $B \in \pi$, then exactly B must be a block in either σ or τ so each block in the meet $\sigma \wedge \tau = \pi$ must be a block of σ or τ, so they are associable. $\square$

The converse does not hold since each partition is trivially associable with itself.

Corollary 11 *For $\pi \precsim \sigma, \tau$; σ and τ are π-orthogonal iff* $\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau) = \operatorname{dit}(\pi)$.

Proof: If $\pi \precsim \sigma, \tau$ and σ and τ are π-orthogonal, then σ and τ are associable so by a previous lemma, $\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau) = \operatorname{dit}(\sigma \wedge \tau) = \operatorname{dit}(\pi)$. Conversely, if $\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau) = \operatorname{dit}(\pi)$, then $\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau) = \operatorname{indit}(\pi)$. Thus $\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau) \cup \operatorname{dit}(\pi) = U \times U$ so $\operatorname{indit}(\tau)^c = \operatorname{dit}(\tau) \subseteq \operatorname{indit}(\sigma) \cup \operatorname{dit}(\pi)$ and thus taking interiors,

$$\operatorname{int}[\operatorname{dit}(\tau)] = \operatorname{dit}(\tau) \subseteq \operatorname{int}[\operatorname{indit}(\sigma) \cup \operatorname{dit}(\pi)] = \operatorname{dit}(\sigma \Rightarrow \pi)$$

so $\tau \precsim \overset{\pi}{\neg}\sigma$. Then $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \precsim \overset{\pi}{\neg}\tau$ and

$$1_U = \overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \precsim \overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau$$

so $1_U = \overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau$, i.e., σ and τ are π-orthogonal. $\square$

Lemma 12 *If $\overset{\pi}{\neg}\sigma \preceq \psi$, then $\psi = \overset{\pi}{\neg}\sigma \vee \left(\psi \wedge \overset{\pi}{\neg}\sigma'\right)$ for any π-orthogonal σ'.*

Proof: If $\exists C \in \sigma$ with $B \subseteq C$, then $\left(\overset{\pi}{\neg}\sigma\right)_B = 1_B$ and thus $\psi_B = 1_B$. Otherwise, $\overset{\pi}{\neg}\sigma' = 1_B$ and $\left(\psi \wedge \overset{\pi}{\neg}\sigma'\right)_B = \psi_B$ so for each $B \in \pi$, the expression gives ψ_B. $\square$

Lemma 13 *For any $\pi \preceq \psi$, and any π-orthogonal σ, τ, $\psi = \left(\psi \wedge \overset{\pi}{\neg}\sigma\right) \vee \left(\psi \wedge \overset{\pi}{\neg}\tau\right)$.*

Proof: Since $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau = \mathbf{1}_U$, the B-slot is $\mathbf{1}_B$ in one or the other, and for that one, say, $\left(\psi \wedge \overset{\pi}{\neg}\sigma\right)_B = \psi_B$. The other is either also ψ_B or B, and in either case, the join gives ψ_B. $\square$

Lemma 14 *For any $\pi \preceq \psi$, and any σ, $\psi = \left(\psi \vee \overset{\pi}{\neg}\sigma\right) \wedge \left(\psi \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma\right)$.*

Proof: By Ore's theorem, $\psi = \psi \vee \pi = \psi \vee \left(\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma\right) = \left(\psi \vee \overset{\pi}{\neg}\sigma\right) \wedge \left(\psi \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma\right)$. $\square$

Lemma 15 *If σ, σ' are π-orthogonal and τ, τ' are also π-orthogonal and all refine π, then $\sigma \vee \tau$ and $\sigma' \wedge \tau'$ are π-orthogonal.*

Proof: We have $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\sigma' = \mathbf{1}_U$ and the same for τ and τ'. What we need to show is that $\overset{\pi}{\neg}(\sigma \vee \tau) \vee \overset{\pi}{\neg}(\sigma' \wedge \tau') = \mathbf{1}_U$. Thus we assume that B is not discretized in $\overset{\pi}{\neg}(\sigma \vee \tau)$ which means that B was further distinguished or broken up in σ or in τ. But that means that there is a $C' \in \sigma'$ or a $D' \in \tau'$ such that $C' = B = D'$. Since both σ' and τ' refine π, the block B would be reconstructed whole in $\sigma' \wedge \tau'$ and thus B would be discretized in $\overset{\pi}{\neg}(\sigma' \wedge \tau')$. □

Lemma 16 *If $\pi \preceq \sigma, \tau, \tau'$ and τ, τ' are π-orthogonal, then $\sigma \Rightarrow \tau$ and $\sigma \wedge \tau'$ are π-orthogonal.*

Proof: We have $\overset{\pi}{\neg}\tau \vee \overset{\pi}{\neg}\tau' = \mathbf{1}_U$ and $\pi \preceq \sigma$ which implies that $\pi \preceq (\sigma \Rightarrow \tau), (\sigma \wedge \tau')$. We need to show $\overset{\pi}{\neg}(\sigma \Rightarrow \tau) \vee \overset{\pi}{\neg}(\sigma \wedge \tau') = 1$. Thus we assume that $B \in \pi$ is not discretized in $\overset{\pi}{\neg}(\sigma \Rightarrow \tau)$ which means either 1) $D \in \tau$ was distinguished further from B so B is a block in τ' and is thus discretized in $\overset{\pi}{\neg}(\sigma \wedge \tau')$, or 2) that $D = B$ but was discretized in $\sigma \Rightarrow \tau$ which means that C was also equal to B. But then $C = B$ means that B survives as a block in $\sigma \wedge \tau'$ and thus B is discretized in $\overset{\pi}{\neg}(\sigma \wedge \tau')$. □

Lemma 17 *If $\pi \preceq \varphi, \varphi'$ are π-orthogonal, i.e., $\overset{\pi}{\neg}\varphi \vee \overset{\pi}{\neg}\varphi' = \mathbf{1}_U$, then for any $\psi \in [\pi, \varphi]$ and $\psi' \in [\pi, \varphi']$, ψ, ψ' are also π-orthogonal.*

Proof: Since $\psi \preceq \varphi$, $\overset{\pi}{\neg}\varphi \preceq \overset{\pi}{\neg}\psi$ and similarly for ψ' so $\mathbf{1}_U = \overset{\pi}{\neg}\varphi \vee \overset{\pi}{\neg}\varphi' \preceq \overset{\pi}{\neg}\psi \vee \overset{\pi}{\neg}\psi'$. □

Corollary 12 *Given π-orthogonal $\pi \precsim \varphi, \varphi'$ and any $\pi \preceq \psi$, $\varphi \wedge \psi$ and $\varphi' \wedge \psi$ are π–orthogonal.* □

Corollary 13 *If φ, φ' are π-orthogonal, then $\varphi \Rightarrow \overset{\pi}{\neg}\varphi'$ and $\varphi' \Rightarrow \overset{\pi}{\neg}\varphi$ are partition tautologies.*

Proof: Being π-orthogonal means $\overset{\pi}{\neg}\varphi \vee \overset{\pi}{\neg}\varphi' = \mathbf{1}_U$ which in turn means that any $B \in \pi$ is contained in a block of φ or a block of φ'. If B is contained in a block of φ', then the B-slot in $\overset{\pi}{\neg}\varphi'$ is $\mathbf{1}_B$ and thus also in $\varphi \Rightarrow \overset{\pi}{\neg}\varphi'$. If B was not contained in a block of φ', then the B-slot in $\overset{\pi}{\neg}\varphi'$ is $\mathbf{0}_B = B$ but B then has to be contained in a block of φ so the B-slot in $\varphi \Rightarrow \overset{\pi}{\neg}\varphi'$ is again $\mathbf{1}_B$. Thus $\varphi \Rightarrow \overset{\pi}{\neg}\varphi'$ is a partition tautology and similarly for $\varphi' \Rightarrow \overset{\pi}{\neg}\varphi$ by symmetry. □

Corollary 14 *If φ and φ' are π-orthogonal and φ is a partition tautology, then so is $\overset{\pi}{\neg}\varphi'$.*

Proof: Modus ponens $\left[\varphi \wedge \left(\varphi \Rightarrow \overset{\pi}{\neg}\varphi'\right)\right] \Rightarrow \overset{\pi}{\neg}\varphi'$ is a partition tautology so if φ is, then so is $\overset{\pi}{\neg}\varphi'$. □

The proof technique of using the B-component or B-slot of a partition $\sigma \in [\pi, \mathbf{1}_U]$ is based on the representation of that upper segment as the product of the partition algebras $\Pi(B)$ for non-singleton $B \in \pi$:

$$[\pi, \mathbf{1}_U] \cong \prod \{\Pi(B) : B \in \pi, B \text{ non-singleton}\} \text{ [28, p. 192].}$$

Each $\sigma \in [\pi, \mathbf{1}_U]$ corresponds to the product of the restrictions $\sigma \restriction B \in \Pi(B)$, and given an element of that product, σ is reconstructed by taking the union of all blocks in the chosen partitions in $\Pi(B)$ plus the singleton blocks. The π-regular partitions are the ones constructed with only $\mathbf{0}_B$ or $\mathbf{1}_B$ from each $\Pi(B)$, which make up the Boolean core $\mathcal{B}[\pi, \mathbf{1}_U]$ of the segment $[\pi, \mathbf{1}_U]$. The π-orthogonal pairs $\sigma, \tau \in [\pi, \mathbf{1}_U]$ are the pairs where for every (non-singleton) $B \in \pi$, $\sigma \restriction B = \mathbf{0}_B$ or $\tau \restriction B = \mathbf{0}_B$. The construction of $\overset{\pi}{\neg}\sigma$ means replacing each $\sigma \restriction B = \mathbf{0}_B$ by $\mathbf{1}_B$ and each $\sigma \restriction B \neq \mathbf{0}_B$ by $\mathbf{0}_B$.

3 The dual structure on the algebra of partitions

3.1 Co-negation on partitions

The previous results about the 16 logical operations on partitions were restated as complementary-dual operations on equivalence relations. This produced no new operations on partitions (or partition relations or ditsets); it produced just a complementary viewpoint. Our topic now is the genuinely dual structure on the algebra of partitions $\Pi(U)$ that does define new operations on partitions. The structure on $\Pi(U)$ covered above based on negation and implication will be called the *primal structure* and the structure on $\Pi(U)$ based on co-negation and co-implication or difference is the *dual structure*.

In the analogy with Heyting algebras, this dual structure gives the partition version of co-negation and co-implication associated with co-Heyting algebras–so that the partition algebra is akin to a bi-Heyting or Heyting-Brouwer algebra except that it is more complicated by virtue of being non-distributive.

The previously developed partition negation $\neg\sigma = \sigma \Rightarrow \mathbf{0}_U$ might be associated with the "truth-value" of "Same as $\mathbf{0}_U$?" applied to σ. In other words, if $\sigma = \mathbf{0}_U$, then $\neg\sigma = \mathbf{1}_U$ since σ is then the "Same as $\mathbf{0}_U$", and otherwise when $\sigma \neq \mathbf{0}_U$, then $\neg\sigma = \mathbf{0}_U$. The dual co-negation is associated with the degree of "Difference from $\mathbf{1}_U$?" applied to σ. The indiscrete partition $\mathbf{0}_U$ has only one block U, while the discrete partition $\mathbf{1}_U$ has the maximum number of blocks $|U|$, so the co-negation "Difference from $\mathbf{1}_U$," denoted $\mathbf{1}_U - \sigma = -\sigma$ is a more varied measure than the negation $\sigma \Rightarrow \mathbf{0}_U = \neg\sigma$.

A partition is said to be *modular* if there is at most one block that is not a singleton and there is not just one singleton, so $\mathbf{1}_U$ and $\mathbf{0}_U$ are both modular partitions. The set-of-blocks definition of the *co-negation* $\mathbf{1}_U - \sigma = -\sigma$ is the partition where all the singleton blocks of σ are collected together in one block and all the non-singleton blocks of σ are atomized into singletons. Thus the co-negated partitions are the same as the modular partitions.

In the context of co-Heyting algebras, Lawvere [35, p. 279] described the non-σ negation as the smallest element whose join with σ is the top element–just as the negation $\neg\sigma$ is the largest element whose meet with σ is the bottom element. Both the negation $\neg\sigma$ and co-negation $-\sigma$ satisfy that requirement even though the lattice of partitions $\Pi(U)$ (being non-distributive) is neither a Heyting nor a co-Heyting algebra.

For instance, if $\sigma = \{a, b, cd, ef\}$, then the co-negation is the modular partition $-\sigma = \{ab, c, d, e, f\}$ and the double co-negation is the modular partition $- - \sigma = \{a, b, cdef\}$ with $- - \sigma \precsim \sigma$. The co-negation of $\mathbf{1}_U$ is $\mathbf{1}_U - \mathbf{1}_U = \mathbf{0}_U$, the co-negation of $\mathbf{0}_U$ is $\mathbf{1}_U - \mathbf{0}_U = \mathbf{1}_U$, and for any $\sigma \neq \mathbf{0}_U, \mathbf{1}_U$, the co-negation $-\sigma$ is between the top and bottom of $\Pi(U)$ with the singletons in $-\sigma$ indicating the degree of σ's "difference from $\mathbf{1}_U$." That is, $\mathbf{1}_U - \mathbf{1}_U = -\mathbf{1}_U = \mathbf{0}_U$ indicates $\sigma = \mathbf{1}_U$ has no difference from $\mathbf{1}_U$ (i.e., the "difference from $\mathbf{1}_U$" is maximally false) while $\mathbf{1}_U - \mathbf{0}_U = \mathbf{1}_U$ indicates $\sigma = \mathbf{0}_U$ has complete difference from $\mathbf{1}_U$ (i.e., the "difference from $\mathbf{1}_U$" is maximally true), and otherwise $\mathbf{1}_U - \sigma = -\sigma$ is in between. Any σ with no singletons or a single singleton is also treated as completely different from $\mathbf{1}_U$ so $-\sigma = \mathbf{1}_U$.

To see in more detail how the dualization works between negation and co-negation as well as to set up the relativized cases of π-negation and π-co-negation, consider the following characterizations of the two negations. For the negation $\neg\sigma$, i.e., "Same as 0_U", the definition is given in Table 3.1.

If $(\sigma \vee \mathbf{0}_U)_U \neq U$, then $(\neg\sigma)_U = (\sigma \Rightarrow \mathbf{0}_U)_U = U$
If $(\sigma \vee \mathbf{0}_U)_U = U$, then $(\neg\sigma)_U = (\sigma \Rightarrow \mathbf{0}_U)_U = \mathbf{1}_U \upharpoonright U$

Table 3.1: Definition of the negation $\sigma \Rightarrow \mathbf{0}_U = \neg\sigma$.

Then the dual version is given in Table 3.2 for co-negation, i.e., "Difference from $\mathbf{1}_U$".

If $(\sigma \wedge \mathbf{1}_U)_{\{u\}} \neq \{u\}$,then $(-\sigma)_{\{u\}} = (\mathbf{1}_U - \sigma)_{\{u\}} = \{u\}$
If $(\sigma \wedge \mathbf{1}_U)_{\{u\}} = \{u\}$, then $(-\sigma)_{\{u\}} = (\mathbf{1}_U - \sigma)_{\{u\}} = \mathbf{0}_U \upharpoonright \{u\}$

Table 3.2: Definition of the co-negation $\mathbf{1}_U - \sigma = -\sigma$.

In Table 3.2, the blocks are singletons $\{u\} \in \mathbf{1}_U$, $(\sigma \wedge \mathbf{1}_U)_{\{u\}}$ stands for the block of $\sigma \wedge \mathbf{1}_U$ containing $\{u\}$ as a subset, and $\mathbf{0}_U \upharpoonright \{u\}$ stands for the non-singleton block including u. Hence if $(\sigma \wedge \mathbf{1}_U)_{\{u\}} = \{u\}$, then $\{u\}$ was a singleton block of σ so $(\mathbf{1}_U - \sigma)_{\{u\}}$, the block of the the co-negation containing u, is the non-singleton block represented by $\mathbf{0}_U \upharpoonright \{u\}$, and if $(\sigma \wedge \mathbf{1}_U)_{\{u\}} \neq \{u\}$, then $\{u\}$ was part of a bigger block in σ so it is turned into a singleton, i.e., the block of $(\mathbf{1}_U - \sigma)_{\{u\}}$ containing u is the singleton $\{u\}$.

The *co-regular* partitions are the co-negated elements of $\Pi(U)$, i.e., the modular partitions. We form an algebra $\mathcal{M}[\Pi(U)] = \{-\sigma : \sigma \in \Pi(U)\}$ of the modular partitions. Each modular partition $(\neq \mathbf{1}_U)$ has the form $\{..., M, ...\}$ of one non-singleton block M, called the *modular block*, and otherwise two or more singleton blocks or no singleton blocks when $M = U$. The co-negation in the algebra just interchanges the modular block and the singletons, i.e., the modular block is atomized into singletons and the singletons are collected together to form the modular block of the co-negation. The top of the algebra is $\mathbf{1}_U$ and the bottom $\mathbf{0}_U$. The coatoms in $\mathcal{M}[\Pi(U)]$ are the partitions of the form $\{..., \{u, u'\}, ...\}$ and the atoms are their co-negations $\{\{u\}, U - \{u, u'\}, \{u'\}\}$. Given two co-regular partitions $\sigma = \{..., M, ...\}$ and $\tau = \{..., M', ...\}$, their join is the partition $\{..., M \cap M', ...\}$ which would be $\mathbf{1}_U$ if $M \cap M'$ was a singleton or empty, and that is the join $\sigma \vee \tau$ in $\Pi(U)$. If M and M' overlap, their meet in the algebra is the modular partition $\{..., M \cup M', ...\}$ which is the meet in $\Pi(U)$, but is otherwise not the meet $(..., M, M', ...)$ in $\Pi(U)$. In general, the modular meet $\{..., M \cup M', ...\}$ in the algebra is the double co-negation $- - (\sigma \wedge \tau) = \{..., M \cup M', ...\}$.

Oystein Ore [41] showed that every partition π on U generates a Boolean algebra $\mathcal{B}(\pi)$, a complete subalgebra of $\wp(U)$, whose subsets are just the unions of blocks of π. In the case of $\pi = \mathbf{1}_U$, $\mathcal{B}(\mathbf{1}_U) = \wp(U)$. To relate $\mathcal{M}[\Pi(U)]$ to $\mathcal{B}(\mathbf{1}_U)$, we have to (as before) make a special treatment of the singletons. Since for $|U| = 2$, $\Pi(U)$ is already a Boolean algebra under the partition operations, we assume $|U| \geq 3$. The problem caused by singletons is that partitions of the form $\mu_u = \{\{u\}, U - \{u\}\}$ have a single non-singleton but are not the co-negation of any partitions due to the single singleton. Hence we need to make a number of changes to $\mathcal{M}[\Pi(U)]$ to create $\mathcal{M}[\Pi(U)]^*$ in order to relate it to $\mathcal{B}(\mathbf{1}_U) = \wp(U)$. We make the following changes:

- introduce all the $\mu_u = \{\{u\}, U - \{u\}\}$ partitions as the atoms in $\mathcal{M}[\Pi(U)]^*$;
- introduce a new layer of coatoms between previous coatoms $\{..., \{u, u'\}, ...\}$, the new coatoms being $|U|$ copies of the discrete partition but, for each $u \in U$, with one singleton $\left\{..., \{u\}^*, ...\right\}$ marked to act as the 'modular block' M which is then treated as the co-negation $-\mu_u = -\{\{u\}, U - \{u\}\}$ so that $-\left\{..., \{u\}^*, ...\right\} = \{\{u\}, U - \{u\}\}$;
- in the refinement ordering $\precsim$, $\{..., M, ...\} \precsim \left\{..., \{u\}^*, ...\right\}$ iff $u \in M$; and
- when $M \cap M'$ is a singleton $\{u\}$, then the join is $\{..., M \cap M', ...\} = \left\{..., \{u\}^*, ...\right\}$, otherwise when $M \cap M' = \emptyset$, then $\{..., M \cap M', ...\}$ is the top $\mathbf{1}_U$.

The partition operations in $\mathcal{M}[\Pi(U)]^*$ correspond to the Boolean operations in $\mathcal{B}(1_U) = \wp(U)$ as follows:

- Join: $(..., M, ...) \vee (...M', ...) = (..., M \cap M', ...)$ and if $M \cap M' = \{u\}$ then it is $\left(..., \{u\}^*, ...\right)$. If $M \cap M' = \emptyset$, then the join is $\mathbf{1}_U$.
- Meet: The modular-meet is $- - (..., M, ...) \wedge (...M', ...) = (..., M \cup M', ...)$.
- Co-negation: $-(..., M, ...) = (..., M^c, ...)$ where the singletons are the elements of M and $M^c = U - M$.

These changes form a new algebra $\mathcal{M}[\Pi(U)]^*$ which is a Boolean algebra anti-isomorphic to $\mathcal{B}(\mathbf{1}_U) = \wp(U)$ under the one-to-one correspondence: $\{..., M, ...\} \longleftrightarrow M \in \mathcal{B}(\mathbf{1}_U) = \wp(U)$:

$$\mathcal{M}\left[\Pi\left(U\right)\right]^{*op} \cong \mathcal{B}\left(\mathbf{1}_U\right) = \wp\left(U\right).$$

The Boolean algebra $\mathcal{M}\left[\Pi\left(U\right)\right]^{*}$ is not a sublattice of $\Pi\left(U\right)$ due to modular-meet operation (double co-negation of the ordinary meet), not to mention the added coatoms $\left(..., \{u\}^{*}, ...\right)$. Thus we cannot infer that Boolean tautologies in $\mathcal{M}\left[\Pi\left(U\right)\right]^{*}$ will be partition tautologies in $\Pi\left(U\right)$–as we could do for the Boolean cores in the primal structure (formed by $\vee$, $\wedge$, $\neg$, $\mathbf{0}_U$, and $\mathbf{1}_U$).

The Boolean algebra $\mathcal{B}\left(\mathbf{1}_U\right)$ is formed by taking all unions of the blocks of $\mathbf{1}_U$, i.e., all subsets $M \in \wp\left(U\right)$. The added coatoms $\left\{..., \{u\}^{*}, ...\right\}$ correspond to the singletons (the atoms of $\mathcal{B}\left(\mathbf{1}_U\right)$), the atomic partitions $\mu_u = \left\{\{u\}, U - \{u\}\right\}$ correspond to the coatoms of $\mathcal{B}\left(\mathbf{1}_U\right)$, and in general $\{..., M, ...\} \longleftrightarrow M \in \mathcal{B}\left(\mathbf{1}_U\right)$ to form the anti-isomorphism. The refinement ordering between modular partitions $(..., M, ...) \precsim (..., M', ...)$ iff $M' \subseteq M$ is why it is an order-reversing isomorphism.

For an example as to why $\mathcal{M}\left[\Pi\left(U\right)\right]$ is not a Boolean algebra without the added atoms and coatoms, consider the following example. All the partitions $\pi = \{a, d, bc\}$, $\sigma = \{b, d, ac\}$, and $\tau = \{ab, c, d\}$ are modular but $\pi \wedge \sigma = \{d, abc\}$ and $\pi \wedge \tau = \{d, abc\}$ which are not modular. And the $\sigma \vee \ \tau = \mathbf{1}_U$ so:

$$\pi \wedge (\sigma \vee \tau) = \pi = \{a, d, bc\} \neq \{d, abc\} = (\pi \wedge \sigma) \vee (\pi \wedge \tau)$$

and thus distributivity fails. But when we add the coatoms $\left\{..., \{u\}^{*}, ...\right\}$ as the co-negations of the atomic partitions μ_u, then $\sigma \vee \tau = \{b, d, ac\} \vee \{ab, c, d\} = \{a^{*}, b, c, d\}$ (since $M \cap M' = \{a\}$). And then

$$\begin{aligned} \pi \wedge (\sigma \vee \tau) &= \{a, d, bc\} \wedge \{a^{*}, b, c, d\} = \{abc, d\} \\ &= \{d, abc\} \vee \{d, abc\} = (\pi \wedge \sigma) \vee (\pi \wedge \tau) \end{aligned}$$

so distributivity is not violated. The corresponding operations in $\wp\left(U\right)$ under the anti-isomorphism are:

$$\begin{gathered} \{b, c\} \cup (\{a, c\} \cap \{a, b\}) = \{b, c\} \cup \{a\} = \{a, b, c\} \text{ and} \\ (\{b, c\} \cup \{a, c\}) \cap (\{b, c\} \cup \{a, b\}) = \{a, b, c\} \cap \{a, b, c\} = \{a.b.c\} \end{gathered}$$

which are the elements in the modular block of $\{abc, d\}$.

3.2 Some tautologies for co-negation

We have extensively studied the tautologies for the negation $\sigma \Rightarrow \mathbf{0}_U = \neg\sigma$ and the π-relative negation or implication $\overset{\pi}{\neg}\sigma = \sigma \Rightarrow \pi$. Here only some of the new tautologies for co-negation will be mentioned.

Perhaps the most obvious difference is that the law of excluded middle holds for co-negation so it is a partition tautology:

$$\sigma \vee -\sigma = \mathbf{1}_U.$$

The proof is simply that every block of σ that is not a singleton will be atomized into singletons in $-\sigma$ so the join is all singletons, i.e., $\mathbf{1}_U$.

In the context of co-Heyting algebras ([35], [34]), Lawvere calls $- - \sigma$ the *core of* σ and $\sigma \wedge -\sigma$ the *boundary* $\partial\sigma$ *of* σ and then notes that σ is reconstructed as the join of its core and boundary, and this holds for co-negation in partition algebras as well:

$$- - \sigma \vee \partial\sigma = \sigma.$$

In the case of the example $\sigma = \{a, b, cd, ef\}$, we have $-\sigma = \{ab, c, d, e, f\}$ and $- - \sigma = \{a, b, cdef\}$ so that the boundary is $\partial\sigma = \sigma \wedge -\sigma = \{ab, cd, ef\}$ and $\sigma = -\sigma \vee \partial\sigma$.

The proofs requires us to keep track of a few generic types of blocks for any partition σ:

- the Original Singletons of σ, i.e., $os(\sigma)$;
- the Original Non-Singletons of σ, i.e., $ons(\sigma)$;
- the Modular block obtained by joining the Original Singletons of σ, i.e., $mos(\sigma)$ (unless $|os(\sigma)| = 1$);
- the Singletons obtained by atomizing the Original Non-Singletons of σ or $sons(\sigma)$; and
- the Modular block formed by the union of the Original Non-Singletons of σ or $mons(\sigma)$.

Then we can characterize some of the partitions formed by co-negation as follows:

$$\begin{gathered}\sigma = \{os(\sigma), ons(\sigma)\} \\ -\sigma = \{mos(\sigma), sons(\sigma)\}\end{gathered}$$

Complications arise when there is a single singleton, i.e., $|os(\sigma)| = 1$. Then $os(\sigma) = mos(\sigma)$ so that $-\sigma = \mathbf{1}_U$. Then the treatment of the double co-negation splits into two cases:

$$- - \sigma = \begin{cases} \{os(\sigma), mons(\sigma)\} \text{ if } |os(\sigma)| \neq 1 \\ \mathbf{0}_U \text{ if } |os(\sigma)| = 1 \end{cases}.$$

Note that in either case, the triple co-negation is the same as the single co-negation:

$$- - -\sigma = -\sigma.$$

And in either case, the boundary of σ is:

$$\partial\sigma = \sigma \wedge -\sigma = \{mos(\sigma), ons(\sigma)\}.$$

The proof the law of excluded middle for co-negation is then:

$$\sigma \vee -\sigma = \{os(\sigma), ons(\sigma)\} \vee \{mos(\sigma), sons(\sigma)\} = \{os(\sigma), sons(\sigma)\} = \mathbf{1}_U.$$

Then the proof of Lawvere's core plus boundary result, $- - \sigma \vee \partial\sigma = \sigma$, consists in observing that:

$$- - \sigma \vee \partial\sigma = \{os(\sigma), mons(\sigma)\} \vee \{mos(\sigma), ons(\sigma)\} = \{os(\sigma), ons(s)\} = \sigma$$

or if $mos(\sigma) = os(\sigma)$, i.e., $|os(\sigma)| = 1$, then:

$$- - \sigma \vee \partial\sigma = \mathbf{0}_U \vee \{mos(\sigma), ons(\sigma)\} = \{os(\sigma), ons(s)\} = \sigma.$$

Furthermore, the boundary $\partial(- - \sigma)$ of $- - \sigma$ is the same as the boundary $\partial(-\sigma)$ of $-\sigma$:

$$\partial(- - \sigma) = - - \sigma \wedge - - -\sigma = - - \sigma \wedge -\sigma = \partial(-\sigma) = \{mos(\sigma), mons(\sigma)\}.$$

Thus $\partial(-\sigma)$ is refined by $\partial\sigma$ so $\partial(-\sigma) \Rightarrow \partial\sigma$ is a partition tautology. For negation, $\sigma \precsim \neg\neg\sigma$ so we might expect $- - \sigma \precsim \sigma$ to hold in the dual structure which means that $- - \sigma \Rightarrow \sigma$ is a partition tautology. The proof is simply the observation that $- -\sigma = \{os(\sigma), mons(\sigma)\} \precsim \{os(\sigma), ons(\sigma)\} = \sigma$ or $- - \sigma = \mathbf{0}_U \precsim \sigma$. Then $\sigma = - - \sigma$ if and only if $mons(\sigma) = ons(\sigma)$ and $mos(\sigma) \neq os(\sigma)$, so that σ has only one non-singleton block and not just one singleton block, i.e., when σ is modular.

These calculations for co-negation might be simplified if we translate them into ditsets. The generic structure of any partition σ is $\{os(\sigma), ons(\sigma)\}$. The ditset $\operatorname{dit}(\sigma)$ is made up of three disjoint sets of distinctions which are numbered as follows:

1. $\cup_{\substack{C,C' \in os(\sigma) \\ C \neq C'}} C \times C'$ which might be symbolized as $\operatorname{dit}(os(\sigma), os(\sigma))$;

2. $\cup_{\substack{C\in os(\sigma)\\ C'\in ons(\sigma)}} C\times C' \cup \cup_{\substack{C\in os(\sigma)\\ C'\in ons(\sigma)}} C'\times C$ which might be symbolized as $\operatorname{dit}(os(\sigma), ons(\sigma))$; and

3. $\cup_{\substack{C,C'\in ons(\sigma)\\ C\neq C'}} C\times C'$ which might be symbolized as $\operatorname{dit}(ons(\sigma), ons(\sigma))$.

These three sets of distinctions are disjoint and their union is $\operatorname{dit}(\sigma)$ which might be symbolized as $\sigma = 1+2+3$. The only remaining distinctions are those that would be added when all the non-singletons blocks of σ are atomized into singletons and they form the fourth set of distinctions disjoint from the other three sets:

4. $\cup_{C\in ons(\sigma)} C\times C - \Delta$.

Together these four disjoint set of distinction exhaust all possible distinction so their union is $\mathbf{1}_U$. When σ is co-negated to form $-\sigma = \{mos(\sigma), sons(\sigma)\}$, then the set 1 is eliminated and the set 4 is added on while the sets 2 and 3 remain so the ditset $\operatorname{dit}(-\sigma)$ of $-\sigma$ is the union of the sets 2, 3, and 4 which is symbolized as: $-\sigma = 2+3+4$. The double co-negation $--\sigma$ of σ has the form $--\sigma = \{os(\sigma), mons(\sigma)\}$, so set 1 is added back and the sets 3 and 4 are eliminated which is symbolized as $--\sigma = 1+2$. All the partitions that can be formed from σ by the lattice operations and co-negation will have two possible sets of blocks $os(\sigma)$ or $mos(\sigma)$ for the singletons and three possible sets of blocks $ons(\sigma)$, $sons(\sigma)$, and $mons(\sigma)$ so there are six possible partitions that can be thus formed, and they are illustrated in Figure 3.1.

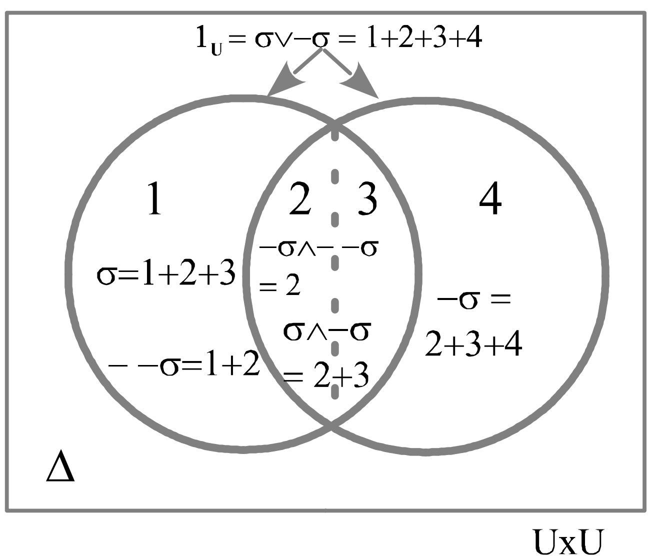

Figure 3.1: The Venn diagram for the six partitions formed from σ by $\vee$, $\wedge$, and $-$.

Then the core + boundary result is simply that the union of the core ditsets $1+2$ and the boundary ditsets $2+3$ is the ditset $1+2+3$ for σ.
Table 7.3 lists the six partitions.

$\sigma \vee -\sigma = \mathbf{1}_U$
$\sigma = 1+2+3$
$-\sigma = 2+3+4$
$--\sigma = 1+2$
$\sigma \wedge -\sigma = 2+3$
$-\sigma \wedge --\sigma = 2$

Table 3.3: The six partitions formed from σ by $\vee$, $\wedge$, and $-$.

Recall that a partition σ is modular if it has at most one non-singleton and not just one singleton. All co-negated partitions are modular and a partition $\sigma \neq \mathbf{0}_U$ is modular if and only if it equals its double co-negation, $\sigma = - - \sigma$. By comparing the formulas in Table 7.3, we see that a partition is modular if and only if the set of distinctions #3 is empty and #1 is not empty.

Since the refinement of partitions is just inclusion between ditsets, we immediately have the refinements: $- - \sigma \precsim \sigma$, $\sigma \wedge -\sigma \precsim \sigma, -\sigma$ (which also follows from the meet being the greatest lower bound), and $-\sigma \wedge - - \sigma$ is refined by the other five partitions. Since the ditset for the join of two partitions is the union of the ditsets, we can represent the joins by just 'adding the numbers.' For instance, that gives immediately; $-\sigma \vee - - \sigma = 1 + 2 + 3 + 4 = \mathbf{1}_U = \sigma \vee -\sigma$ and $- - \sigma \vee (\sigma \wedge -\sigma) = 1 + 2 + 3 = \sigma$. The ditset of the meet of two partitions is the interior of the intersection of the two ditsets–which is the largest ditset contained in that intersection. The intersection of two ditsets of partitions is a set of distinctions but not necessarily the ditset of a partition. But in the case of the six partitions of Table 7.3, all the intersections, which will always be just a sum of the numbered sets, are the ditsets of one of the six functions so they are closed under meets. The co-negation of the double co-negation $- - \sigma = \{os(\sigma), mons(\sigma)\}$ is $\{mos(\sigma), sons(\sigma)\}$ so in terms of the numbered sets, the triple co-negation eliminates set 1 and adds the sets 3 and 4 so it is the same as the single co-negation $-\sigma$. The co-negation $-(\sigma \wedge -\sigma)$ of $\sigma \wedge -\sigma = \{mos(\sigma), ons(\sigma)\}$ is $\mathbf{1}_U$ as will be proven just below using the strong DeMorgan law $-(\sigma \wedge \pi) = -\sigma \vee -\pi$. And by the same law, the co-negation $-(-\sigma \wedge - - \sigma) = - - \sigma \vee - - - \sigma = - - \sigma \vee -\sigma = \mathbf{1}_U$. Thus the six partitions are also closed under co-negation.

With two partitions, the first thing to check is DeMorgan's laws for co-negation. For negation, the 'weak' DeMorgan law $\neg(\sigma \vee \pi) = \neg\sigma \wedge \neg\pi$ holds but the 'strong' law $\neg(\sigma \wedge \pi) = \neg\sigma \vee \neg\pi$ fails. But for co-negation, the weak law fails as shown by the counterexample of $\sigma = \{a, b, cd, ef\}$ and $\pi = \{a, bc, def\}$. Then $\sigma \vee \pi = \{a, b, c, d, ef\}$ and $-(\pi \vee \sigma) = \{abcd, e, f\}$. On the RHS of the weak law, $-\sigma = \{ab, c, d, e, f\}$ and $-\pi = \mathbf{1}_U$ so:

$$-\sigma \wedge -\pi = -\sigma = \{ab, c, d, e, f\} \neq \{abcd, e, f\} = -(\pi \vee \sigma).$$

To analyze the strong DeMorgan law, we need to develop some more of the standard forms for co-negation with two partitions.

$$\begin{aligned}
\sigma \wedge \pi &= \{os(\sigma) \cap os(\pi), ons(\sigma \wedge \pi)\} \\
-(\sigma \wedge \pi) &= \{m[os(\sigma) \cap os(\pi)], sons(\sigma \wedge \pi)\} \\
-\sigma \vee -\pi &= \{mos(\sigma) \cap mos(\pi), sons(\sigma) \cup sons(\pi)\}.
\end{aligned}$$

Then to prove the strong law, we first note that $m(os(\sigma) \cap os(\pi)) = mos(\sigma) \cap mos(\pi)$ since the only singletons in $\sigma \wedge \pi$ are the ones common to both partitions. Then to show $sons(\sigma \wedge \pi) = sons(\sigma) \cup sons(\pi)$, we note that the meet operation only expands any block so non-singleton blocks remain non-singleton and thus the singletons $sons(\sigma \wedge \pi)$ made from the non-singleton blocks of $\sigma \vee \pi$ with be the same as $sons(\sigma) \cup sons(\pi)$ which completes the proof of the strong DeMorgan law for co-negation:

$$-(\sigma \wedge \pi) = -\sigma \vee -\pi.$$

Thus co-negation reverses the situation for negation with respect to the two DeMorgan laws.

In the co-Heyting context, Lawvere notes that the Leibniz law (e.g., for derivatives of products) holds which in our case would be: $\partial(\sigma \wedge \pi) = (\partial\sigma \wedge \pi) \vee (\sigma \wedge \partial\pi)$. The simple proof of Leibniz's law in a co-Heyting algebra uses distributivity so one might expect it to be false for co-negation, and indeed a counterexample is $\pi = \{a, b, c, def\}$ so $-\pi = \{abc, d, e, f\}$ and for $\sigma = \{a, b, d, cef\}$, then $-\sigma = \{abd, c, e, f\}$. Hence $-\sigma \vee -\pi = \{ab, c, d, e, f\}$ and $\sigma \wedge \pi = \{a, b, cdef\}$. Then the LHS is: $\partial(\pi \wedge \sigma) = \{ab, cdef\}$. And $\partial\sigma = \sigma \wedge -\sigma = \{abd, cef\}$ so $\partial\sigma \wedge \pi = \{U\} = \mathbf{0}_U$ and $\partial\pi = \{abc, def\}$ so $\partial\pi \wedge \sigma = \{U\} = \mathbf{0}_U$ so the RHS is $\{U\} = \mathbf{0}_U$ and thus the Leibniz law fails for co-negation.

3.3 Difference operation on partitions: Relativizing co-negation to π

In the primal structure, the negation $\neg\sigma = \sigma \Rightarrow \mathbf{0}_U$ was the partition representing the extent to which "σ is the same as $\mathbf{0}_U$", and the implication $\sigma \Rightarrow \pi$, obtained by relativizing the negation to π, is the partition representing the extent to which "σ is refined by π" indicated in the Table 3.4. When $\pi = \mathbf{0}_U$, then "σ is refined by π" is the same as "σ is the same as $\mathbf{0}_U$" since the only partition that $\mathbf{0}_U$ refines is itself (refinement being a partial order). The notation $(\sigma \vee \pi)_B$ represents any block of the join $\sigma \vee \pi$ containing part or all of B.

#1.If $(\sigma \vee \pi)_B \neq B$, then $\left(\overset{\pi}{\neg}\sigma\right)_B = (\sigma \Rightarrow \pi)_B = B$	
#2.If $(\sigma \vee \pi)_B = B$, then $\left(\overset{\pi}{\neg}\sigma\right)_B = (\sigma \Rightarrow \pi)_B = \mathbf{1}_U \upharpoonright B$	

Table 3.4: Definition of the implication $\sigma \Rightarrow \pi = \overset{\pi}{\neg}\sigma$ as negation relativized to π.

In the dual structure, the co-negation $-\sigma = \mathbf{1}_U - \sigma$ is the partition representing the extent to which "σ is different from $\mathbf{1}_U$", and the co-implication or difference operation $\overset{\pi}{-}\sigma = \pi - \sigma$, obtained by relativizing the co-negation to π, is the partition representing the extent to which "σ is not refined by π" indicated in Table 3.5 (where the notation $(\sigma \wedge \pi)_B$ represents the block of the meet $\sigma \wedge \pi$ containing the block $B \in \pi$).

#1.If $(\sigma \wedge \pi)_B \neq B$, then $\left(\overset{\pi}{-}\sigma\right)_B = (\pi - \sigma)_B = B$	
#2.If $(\sigma \wedge \pi)_B = B$, then $\left(\overset{\pi}{-}\sigma\right)_B = (\pi - \sigma)_B = \mathbf{0}_U \upharpoonright B$	

Table 3.5: Definition of the difference operation $\pi - \sigma = \overset{\pi}{-}\sigma$ as co-negation relativized to π.

The dual structure defined by π lives in the lower segment $[\mathbf{0}_U, \pi]$ of partitions refined by π. If we treat the blocks $B \in \pi$ as points in a universe set, then the lower segment is isomorphic to $\Pi(\pi) \cong [\mathbf{0}_U, \pi]$ [28, p. 192] where π as a set of blocks plays the role of the universe set of points. Hence the difference $\pi - \sigma$ will be essentially like $\mathbf{1}_\pi - \sigma$ with the blocks $B \in \pi$ playing the role of the singletons $\{B\} \in \mathbf{1}_\pi$. The two rules for determining $\pi - \sigma$ can be restated as follows.

1. If $(\sigma \wedge \pi)_B \neq B$, then the block of $\sigma \wedge \pi$ containing B is larger than B (analogous to a block of σ being a non-singleton in the case of $\mathbf{1}_U - \sigma$), so that block of $\sigma \wedge \pi$ is 'atomized' into the blocks B of π contained in the 'non-singleton' block of $\sigma \wedge \pi$ (analogous to a non-singleton block of σ being atomized into singletons in $\mathbf{1}_U - \sigma$ except that the blocks of π are playing the role of the singletons). That is, if $(\sigma \wedge \pi)_B \neq B$, then the block of $\sigma \wedge \pi$ containing B is larger than B so B stays the same block in $\pi - \sigma$.

2. If $(\sigma \wedge \pi)_B = B$, then B is an exact union of blocks $C \in \sigma$ (and if $\sigma \precsim \pi$, then $B = C$), and then the B-slot $\left(\overset{\pi}{-}\sigma\right)_B = (\pi - \sigma)_B$ in $\overset{\pi}{-}\sigma = \pi - \sigma$ is part of the union of such blocks from π, to form the 'π-modular' block of $\pi - \sigma$, analogous to the modular block when $\pi = \mathbf{1}_U$. That is, if $(\sigma \wedge \pi)_B = B$, then B is an exact union of blocks $C \in \sigma$, then B joins into the π-modular block of $\pi - \sigma$.

A partition σ is said to be π-*modular* if it is refined by π,has at most one block consisting of a union of blocks of π, and does not just have one singleton block B, where the π-*modular block* (when $\sigma \neq \pi$) is the block consisting of the union of two or more blocks of π. Let $\mathcal{M}[\Pi(\pi)]$ be the π-modular partitions of $[\mathbf{0}_U, \pi]$ with the refinement ordering. As in the case of $\mathbf{1}_U - \sigma$, the partitions that are not differences $\pi - \sigma$ are the ones of the form $\mu_B = \{B, \cup_{B' \in \pi; B \neq B'} B'\}$. And, as before, if we start with the set $\mathcal{M}[\Pi(\pi)]$ of all π-modular partitions and then add the new atoms μ_B and the artificial coatoms $\{..., B^*, ...\}$ and make the other corresponding changes so that $\{..., B^*, ...\} = \pi - \mu_B = \overset{\pi}{-}\mu_B$, then we obtain a Boolean algebra anti-isomorphic to $\mathcal{B}(\pi)$.

The *difference, co-implication,* or *π-co-negation* operation $\pi - \sigma = \overset{\pi}{-}\sigma$ creates a π-modular partition in the lower segment $[\mathbf{0}_U, \pi]$ of $\Pi(U)$. If σ is not in that lower segment, then the π-co-negation of $\sigma \wedge \pi$ is the same as the π-co-negation of σ since only $\sigma \wedge \pi$ occurs in rules #1 and #2. In other words, $\pi - \sigma = \pi - (\sigma \wedge \pi)$. For instance, if $\pi = \{a, b, cd, ef, gh\}$ and $\sigma = \{abgh, cd, ef\}$, then $\pi - \sigma = \overset{\pi}{-}\sigma = \{a, b, cdef, gh\}$ since $\{abgh\}$ is a union of blocks of π it is 'atomized' into the respective blocks of π as $\{a, b, gh\}$ and since $\{cd, ef\}$ are two singletons as blocks of π, they are combined into the π-modular block $cdef$ of $\pi - \sigma$. And $\sigma \wedge \pi = \{abgh, cd, ef\}$ so that $\pi - (\sigma \wedge \pi) = \overset{\pi}{-}(\sigma \wedge \pi) = \{a, b, cdef, gh\} = \pi - \sigma$.

For any $\sigma \in [\mathbf{0}_U, \pi]$, the difference operation is like co-negation in the form $\mathbf{1}_\pi - \sigma$ (thinking of the $B \in \pi$ as if they were singletons) so the results for $1_U - \sigma = -\sigma$ can be applied *mutatis mutandis*. Thus for $\sigma \in [\mathbf{0}_U, \pi]$, the law of excluded middle in $[\mathbf{0}_U, \pi]$ takes the form of the partition tautology: $\sigma \vee \overset{\pi}{-}\sigma = \pi$.

The ditset analysis, developed above for co-negation, extends to π-co-negation. The π-co-negated partition is in the lower segment $[\mathbf{0}_U, \pi]$ so its ditset will always be contained in $\operatorname{dit}(\pi)$. Any $\sigma \in [\mathbf{0}_U, \pi]$ will have the form of some original π-singletons $o\pi s(\sigma)$ in the form $B \in \pi$ and some original π-non-singletons $o\pi ns(\sigma)$ in the form $B \cup B' \cup ...$ so it has the form:

$$\sigma = \{o\pi s(\sigma), o\pi ns(\sigma)\}.$$

Then we apply the #1 and #2 rules to get:

$$\pi - \sigma = \overset{\pi}{-}\sigma = \{mo\pi s(\sigma), so\pi ns(\sigma)\}.$$

$1 = \operatorname{dit}(o\pi s(\sigma), o\pi s(\sigma)) = \cup_{\substack{B,B' \in o\pi s(\sigma) \\ B \neq B'}} B \times B'$
$2 = \operatorname{dit}(o\pi s(\sigma), o\pi ns(\sigma)) = \cup_{\substack{B \in o\pi s(\sigma) \\ B' \subseteq C' \in o\pi ns(\sigma)}} B \times B' \cup ...$
$3 = \operatorname{dit}(o\pi ns(\sigma), o\pi ns(\sigma)) = \cup_{\substack{B \subseteq C \in o\pi ns(\sigma) \\ B' \subseteq C' \in o\pi ns(\sigma), C \neq C', B \neq B'}} B \times B'$
$4 = \cup_{\substack{B,B' \subseteq C \in o\pi ns(\sigma) \\ B \neq B'}} B \times B'$

Table 3.6: Ditset analysis for π-co-negation.

This is illustrated in Figure 3.2.

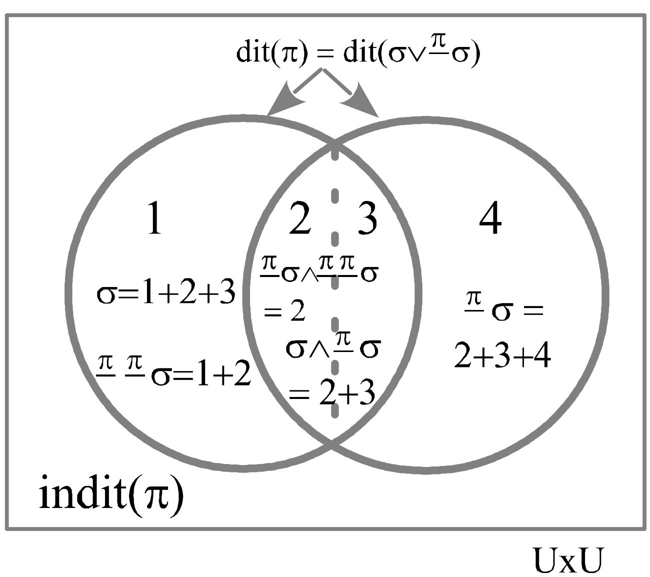

Figure 3.2: Ditset analysis π-co-negations of a partition σ.

Hence we have the Table 3.7 ditset analysis of the π-negated elements that can be formed from a single $\sigma \in [\mathbf{0}_U, \pi]$.

$\sigma \vee \overset{\pi}{-}\sigma = \sigma \vee (\pi - \sigma) = 1 + 2 + 3 + 4 = \pi$
$\sigma = 1 + 2 + 3$
$\overset{\pi}{-}\sigma = \pi - \sigma = 2 + 3 + 4$
$\overset{\pi}{-}\overset{\pi}{-}\sigma = \pi - (\pi - \sigma) = 1 + 2$
$\sigma \wedge \overset{\pi}{-}\sigma = \sigma \wedge (\pi = \sigma) = 2 + 3$
$\overset{\pi}{-}\sigma \wedge \overset{\pi}{-}\overset{\pi}{-}\sigma = 2$

Table 3.7: Formulas generated by π-negation of σ where $\sigma \precsim \pi$.

With π-negation, the partition $\sigma \Rightarrow \pi$ is the partition indicating the extent to which "σ is refined by π" so when it is totally true, we have refinement:

$$\overset{\pi}{\neg}\sigma = \sigma \Rightarrow \pi = \mathbf{1}_U \text{ iff } \sigma \precsim \pi.$$

For π-co-negation, we have the dual result (which also holds in co-Heyting algebras [43]):

$$\overset{\pi}{-}\sigma = \pi - \sigma = \mathbf{0}_U \text{ iff } \pi \precsim \sigma$$

since $\pi - \sigma = \mathbf{0}_U$ means for all $B \in \pi$, $(\sigma \wedge \pi)_B = B$ (the #2 rule above) so every $B \in \pi$ is an exact union of blocks $C \in \sigma$, i.e., $\pi \precsim \sigma$. In terms of the interpretation, when $\pi - \sigma = \mathbf{0}_U$ so "π is not refined by σ" is totally false, i.e., is a 'contradiction', then (by classical double negation), $\pi \precsim \sigma$. In that sense, showing that $\pi - \sigma = \mathbf{0}_U$ is like a proof by contradiction. The partition $\pi - \sigma$ is the partition representing the extent to which "π is not refined by σ" and when "π is not refined by σ" is the contradiction $\mathbf{0}_U$, then π is refined by σ, $\pi \precsim \sigma$.

Since the π-negation takes any σ to a partition $\sigma \Rightarrow \pi = \overset{\pi}{\neg}\sigma$ refining π and the π-co-negation takes any σ to a partition $\pi - \sigma = \overset{\pi}{-}\sigma$ refined by π, the mixed double negations take any σ to the top or bottom depending on the order of the negations:

$$\mathbf{1}_U = (\pi - \sigma) \Rightarrow \pi = \overset{\pi}{\neg}\overset{\pi}{-}\sigma$$
$$\mathbf{0}_U = \pi - (\sigma \Rightarrow \pi) = \overset{\pi}{-}\overset{\pi}{\neg}\sigma.$$

Another partition equation that gives some of the meaning of difference-from-π as subtraction is:

$$\pi \vee \sigma = (\pi - \sigma) \vee \sigma.$$

That is, to form $\pi \vee \sigma$, you could take σ away from π and then join σ back to get the same result. The partition join $\pi \vee \sigma$ is the partition formed by all the non-empty intersections $B \cap C$ for $B \in \pi$ and $C \in \sigma$. The difference $\pi - \sigma$ is formed using two rules. In the #2 rule, if $(\pi \wedge \sigma)_B = B$, then B is an exact union of some blocks $C \in \sigma$ so B would be joined into the π-modular block of $\pi - \sigma$, but then all the intersections of any $C \in \sigma$ with that block would give the same $B \cap C$ as in $\pi \vee \sigma$. In the #1 rule, $(\pi \wedge \sigma)_B \neq B$, then B is part of a larger block in the meet $\pi \wedge \sigma$, and thus B remains as the block B in $\pi - \sigma$ so the intersection $B \cap C$ is the same in both $\pi \vee \sigma$ and $(\pi - \sigma) \vee \sigma$.

A formula in lattice theory (with $\mathbf{0}_U$ and $\mathbf{1}_U$) is dualized by keeping the atomic variables the same, interchanging join and meet, and interchanging $\mathbf{0}_U$ and $\mathbf{1}_U$. Identities dualize into identities, e.g., $\pi \vee (\pi \wedge \sigma) = \pi$ dualizes to $\pi \wedge (\pi \vee \sigma) = \pi$. Dualization of formulas extends to the implication and co-implication by dualizing $\sigma \Rightarrow \pi$ to $\pi - \sigma$ and vice-versa, but the relationship between a formula involving the implications and its dual is more subtle.

This dualization should be distinguished from the complementary duality between partitions, represented by partition relations on $U \times U$, and equivalence relations on $U \times U$. The complementary dual of a statement about partitions is an equivalent statement about equivalence relations. The lattice-theoretic dualization and the extension to implication and co-implication is not that sort of rather trivial complementary duality. The dualization of a partition statement considered now is a different statement about partitions, not just an equivalent restatement in terms of equivalence relations.

A formula that equals $\mathbf{0}_U$ for any partitions substituted for the atomic variables and for any U ($|U| \geq 2$) is a *partition contradiction*, just as a formula that equals $\mathbf{1}_U$ for any substitution is a partition tautology.

The proof that any partition tautology was also a subset (or truth-table) tautology used the isomorphism $\wp(1) \cong \Pi(2)$, and the same proof strategy shows that any partition contradiction is also a subset or truth-table contradiction (i.e., the negation of a tautology). The subset operation that corresponds to the partition difference or co-implication operation $\pi - \sigma$ is the subset difference $S - T = S \cap T^c$ or in terms of propositional variables, $\pi \wedge \neg\sigma$. The set $U = 2 = \{0, 1\}$ allows only two partitions in $\Pi(2)$, namely $\mathbf{0}_U = \{\{0, 1\}\}$ and $\mathbf{1}_U = \{\{0\}, \{1\}\}$. In the isomorphism $\wp(1) \cong \Pi(2)$ between the two-element Boolean algebra and the partition algebra on the two-element set 2, the partition difference operation on $\mathbf{0}_U$ and $\mathbf{1}_U$ are isomorphic to the Boolean operations on zero and one of $\wp(1)$ as indicated in the following truth-table Table 7.8 (the the zeros and ones are interpreted as $\mathbf{0}_U$ and $\mathbf{1}_U$, or as $0, 1 \in U$ as the case may be).

π	σ	$\pi - \sigma$	$\pi \wedge \neg\sigma$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	0	0

Table 3.8: Partition operation $\pi - \sigma$ in $\Pi(2)$ same as Boolean operation $\pi \wedge \neg\sigma$ in $\wp(1)$.

Since the other partition operations and Boolean operations are isomorphic in $\wp(1) \cong \Pi(2)$, any partition contradiction has to hold for $U = 2$, and thus the corresponding Boolean formula is also a contradiction so all partition contradictions are also Boolean contradictions.

The dual to the partition tautology of modus ponens $[\sigma \wedge (\sigma \Rightarrow \pi)] \Rightarrow \pi$ is the formula: $\pi - [\sigma \vee (\pi - \sigma)]$. But $\sigma \vee (\pi - \sigma) = \sigma \vee \pi$ and $\pi \precsim \sigma \vee \pi$ for any π and σ, so we have:

$$\overset{\pi}{\neg}\left(\sigma \wedge \overset{\pi}{\neg}\sigma\right) = [\sigma \wedge (\sigma \Rightarrow \pi)] \Rightarrow \pi = \mathbf{1}_U$$
$$\overset{\pi}{-}\left(\sigma \vee \overset{\pi}{-}\sigma\right) = \pi - [\sigma \vee (\pi - \sigma)] = \mathbf{0}_U$$

which are the π-relativizations of:

$$\neg(\sigma \wedge \neg\sigma) = (\sigma \wedge (\sigma \Rightarrow \mathbf{0}_U)) \Rightarrow \mathbf{0}_U = \mathbf{1}_U$$
$$-(\sigma \vee -\sigma) = \mathbf{1}_U - [\sigma \vee (\mathbf{1}_U - \sigma)] = \mathbf{0}_U.$$

Modus ponens is a case where the dual of a partition tautology is a partition contradiction that is a different statement about partitions. Note that if a partition tautology has its main connective as an implication, then there is a trivial restatement as a contradiction using co-implication since both are equivalent to a certain refinement relation. Thus the modus ponens formula $[\sigma \wedge (\sigma \Rightarrow \pi)] \Rightarrow \pi = \mathbf{1}_U$

is equivalent to $[\sigma \wedge (\sigma \Rightarrow \pi)] \precsim \pi$ which in turn is equivalent to $[\sigma \wedge (\sigma \Rightarrow \pi)] - \pi = \mathbf{0}_U$, but that trivial restatement is not the dualization contradiction: $\pi - [\sigma \vee (\pi - \sigma)] = \mathbf{0}_U$.

There are many promising cases where identities dualize to identities. For instance, in the primal structure, the 'weak' DeMorgan law $\neg(\sigma \vee \tau) = \neg\sigma \wedge \neg\tau$ holds, and its dual, the 'strong' DeMorgan law, holds in the dual structure:

$$\begin{aligned}\neg(\sigma \vee \tau) &= \neg\sigma \wedge \neg\tau \\ -(\sigma \wedge \tau) &= -\sigma \vee -\tau.\end{aligned}$$

If Φ is a formula in the primal structure ($\vee$, $\wedge$, $\Rightarrow$, $\mathbf{0}_U$, and $\mathbf{1}_U$), then let $\varphi(\Phi)$ be the dual formula in the dual structure ($\vee$, $\wedge$, $-$, $\mathbf{0}_U$, and $\mathbf{1}_U$). It is tempting to conjecture that the lattice theoretic dualization extends to implication and co-implication, i.e., to conjecture that: Φ is a partition tautology iff $\varphi(\Phi)$ is a partition contradiction. However, that is not true. There at least two special aspects of implication in the primal structure that are not mirrored in the dual structure so the implication, if Φ is a partition tautology, then $\varphi(\Phi)$ is a partition contradiction, does not hold.

One special aspect is the way that partial refinement is exemplified in the two structures. In the primal structure, and implication $\sigma \Rightarrow \pi$ indicated partial refinement when $B \subseteq C$ and then B is atomized into singletons in $\sigma \Rightarrow \pi$. In the partition tautology $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ (the weak law of excluded middle), there is a set of singletons in $\overset{\pi}{\neg}\sigma$ and a complementary set of singletons in $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ so the join is all singletons, i.e., $\mathbf{1}_U$. But in the dual structure, partial refinement in $\pi - \sigma$ is indicated by $B = \cup C$ being added to the modular block of $\pi - \sigma$. In the dual formula, $\varphi\left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma\right) = \overset{\pi}{-}\sigma \wedge \overset{\pi}{-}\overset{\pi}{-}\sigma$, the modular blocks will be complementary but their meet does not give $\mathbf{0}_U$. Taking the simple case of $\pi = \mathbf{1}_U$, and $\sigma = \{a, b, cd\}$, $\mathbf{1}_U - \sigma = -\sigma = \{ab, c, d\}$, $--\sigma = \{a, b, cd\}$, and $-\sigma \wedge --\sigma = \{ab, cd\} \neq \mathbf{0}_U$.

Another special aspect of the primal structure (due to the Common Dits Theorem), is the 'extremism' of negation, i.e., $\neg\sigma = \sigma \Rightarrow \mathbf{0}_U$ is either $\mathbf{1}_U$ (when $\sigma = \mathbf{0}_U$) or $\mathbf{0}_U$ otherwise. Consider the partition tautology, $\neg\sigma \Rightarrow (\sigma \Rightarrow \pi)$, which is the classical "reductio" that a contradiction implies anything. The reductio is a partition tautology since if $\sigma \neq \mathbf{0}_U$, then $\neg\sigma = \sigma \Rightarrow \mathbf{0}_U = \mathbf{0}_U$ so the whole expression is $\mathbf{0}_U \Rightarrow (\sigma \Rightarrow \pi) = \mathbf{1}_U$, and if $\sigma = \mathbf{0}_U$, then $\sigma \Rightarrow \pi = \mathbf{1}_U$ so the whole expression is also $\mathbf{1}_U$. The dual formula is, $(\pi - \sigma) - (\mathbf{1}_U - \sigma)$, so consider the case where $\pi = \{abe, cd\}$ and $\sigma = \{a, b, c, de\}$ so that $\sigma \wedge \pi = \mathbf{0}_U$ and thus $\pi - \sigma = \pi$ while $\mathbf{1}_U - \sigma = \{abc, d, e\}$. Then $(\pi - \sigma) - (\mathbf{1}_U - \sigma) = \pi - (\mathbf{1}_U - \sigma) = \pi \neq \mathbf{0}_U$ since $\pi \wedge (\mathbf{1}_U - \sigma) = \mathbf{0}_U$.

The opposite conjecture, if a formula Φ in the dual structure is a partition contradiction, then the formula $\varphi^{-1}(\Phi)$ in the primal structure is a partition tautology, is still open.

4 The (quantum) logic of vector-space partitions

4.1 Linearization from sets to vector spaces

The logic of (set) partitions is dual to the usual logic of subsets (usually presented in the special case of propositional logic) in the sense of the category-theoretic duality between partitions and subsets.[8] The usual quantum logic [5] first generalizes from the set notion of a subset to the corresponding vector-space notion of a subspace and then specializes the vector spaces to those used in quantum mechanics. Our purpose is to present the dual quantum logic by first generalizing the set notion of a partition to the corresponding vector-space notion, and then specialize to the (finite-dimensional) vector spaces of quantum mechanics. But what is the vector-space notion corresponding to the set notion of partition?

[8] As Gian-Carlo Rota put it: "categorically speaking, the Boolean σ-algebra of events and the lattice Σ of all Boolean σ-subalgebras are dual notions" [47, p. 65] using the characterization of partitions by Boolean subalgbras [33, p. 43] that goes back to Ore [41]. The category theorist, F. William Lawvere, called subobjects "parts" and then noted that: "The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition." [36, p. 85]

There is a 'semi-algorithmic' procedure to generalize set concepts to the corresponding vector-space notions that we will call *linearization*. Gian-Carlo Rota might call this procedure a "yoga" [46, p. 251]. The basic idea is simple.

Yoga of Linearization: Take a basis set of a vector space V (over a field $\mathbb{K}$) and apply to that basis set any set notion, and then whatever is generated in the vector space is the corresponding vector-space notion.

The set notion of cardinality applied to a basis set generates the notion of dimension. The set notion of a subset applied to a basis set generates the notion of a subspace. The set notion of a partition applied to a basis set generates the notion of a direct-sum decomposition. If the set partition of the basis set is the inverse image of a numerical attribute[9] $f : U \to \mathbb{K}$, then it generates the notion of a (diagonalizable = there is a basis set of eigenvectors) linear operator $F : V \to V$ defined by $Fu = f(u)\,u$ linearly extended to the whole space. For $r \in \mathbb{K}$ in the image $f(U) \subseteq \mathbb{K}$, the inverse image $f^{-1}(r)$ is a constant subset of U. If for $S \subseteq U$, we let rS stand for defining a function having the value r on the elements of S, then the numerical attribute $f : U \to \mathbb{K}$ satisfies $f\left(f^{-1}(r)\right) = rf^{-1}(r)$ and similarly for any subset $S \subseteq f^{-1}(r)$, $f(S) = rS$. Then the equation characterizing the constant sets of f is $f(S) = rS$ for some $r \in \mathbb{K}$, and the vector-space version is the eigenvector equation $Fv = \lambda v$ for some $\lambda \in \mathbb{K}$. The set of constant r-sets of f is the powerset $\wp\left(f^{-1}(r)\right)$ for some $r \in f(U)$, and the corresponding space of eigenvectors of F for some eigenvalue λ is the eigenspace V_λ associated with λ. Starting with a diagonalizable linear operator $F : V \to V$, the numerical attribute $f : U \to \mathbb{K}$ reappears as the eigenvalue function on a basis of eigenvectors. Given two vector spaces V and V' over $\mathbb{K}$ with basis sets U and U', the set notion of the direct or Cartesian product $U \times U'$ generates the notion of the tensor product $V \otimes V'$ generated by the ordered pairs $(u, u') \in U \times U'$ written as $u \otimes u'$. These results are summarized in Table 4.1.[10]

Set concept	Vector-space concept
Universe set U	Basis set of a space V
Cardinality of a set U	Dimension of a space V
Subset of a set U	Subspace of a space V
Partition of a set U	Direct-sum decomposition of a space V
Numerical attribute $f : U \to \mathbb{K}$	Diagonalizable linear op. $F : V \to V$
Value r in image $f(U)$ of f	Eigenvalue λ of F
Constant set S of f	Eigenvector v of F
Set of constant r-sets $\wp\left(f^{-1}(r)\right)$	Eigenspace V_λ of λ
Direct product of sets	Tensor product of spaces
Elements (u, u') of $U \times U'$	Basis vectors $u \otimes u'$ of $V \otimes V'$

Table 4.1: Linearization of set concepts to corresponding vector-space concepts.

The usual notion of logic is the Boolean logic of subsets (ordinarily represented in the special case of propositional logic) and since subsets linearize to subspaces, the usual quantum logic [5] is the logic of subspaces of a vector space specialized to the Hilbert spaces of quantum mechanics. Since set partitions are category-theoretic duals to subsets, we have presented the dual logic of set partitions above, and the corresponding quantum logic is the logic of direct-sum decompositions [19] of a vector space specialized to the Hilbert spaces of quantum mechanics. The general logic of direct-sum decompositions (i.e., vector space partitions) is the subject of this chapter with some specializations to the quantum case.

[9] A attribute or property might be said to hold or not hold of an element in a set, but a numerical attribute, like height, weight, or age, assigns a numerical value to each element of a set. And an attribute can be considered as a numerical attribute with possible values $0, 1 \in 2$.

[10] Since set-concepts can be formulated in vector spaces over $\mathbb{Z}_2$, the vector-space concepts of QM over $\mathbb{C}$ can be scaled back to $\mathbb{Z}_2$ in an appropriate way to create a pedagogical or 'toy' model of QM over $\mathbb{Z}_2$, i.e., quantum mechanics over sets [17].

4.2 Direct-Sum Decompositions

Let V be a finite-dimensional vector space over a field $\mathbb{K}$. Intuitively, a *direct-sum decomposition (DSD)* of V is a set of non-zero subspaces $\{V_i\}_{i\in I}$ of V such that every vector $v \in V$ can be uniquely expressed as a sum $v = \sum_{i\in I} v_i$ where $v_i \in V_i$. We have seen how the notion of a set partition linearizes to the notion of a DSD when a set partition on a set U is usually defined as a set of non-empty subsets or blocks $\{B_i\}_{i\in I}$ such that the blocks are mutually disjoint and jointly exhaustive (i.e., whose union is U). But it might be noted that a set partition could be equivalently defined as a 'delinearized' DSD. That is, a set partition is a set of non-empty subsets $\{B_i\}_{i\in I}$ of U such that every non-empty subset $S \subseteq U$ can be uniquely expressed as the union of subsets of the B_i (e.g., the projections $S \cap B_i$) for $i \in I$. For a spectrum of values $r_1, ..., r_m$, for $f : U \to \mathbb{R}$ with $B_i = f^{-1}(r_i)$, the unique decomposition of $S \subseteq U$ is illustrated in Figure 4.1.

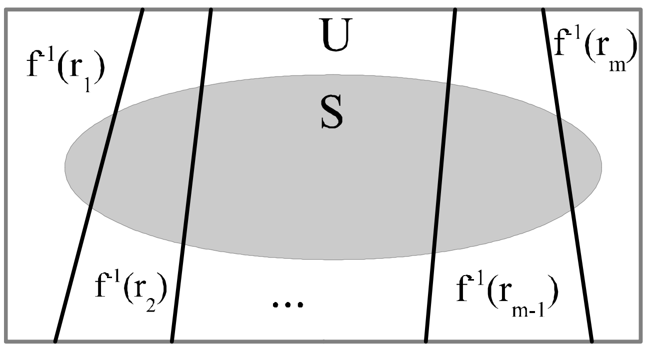

Figure 4.1: Decomposition of $S \subseteq U$ in terms of the constant sets of $f : U \to \mathbb{R}$.

Every self-adjoint (or Hermitian) operator in a Hilbert space, or, more generally, any diagonalizable operator $V \to V$ determines the DSD of its eigenspaces V_i consisting of the eigenvectors v_i for an eigenvector λ_i. But the notion of a DSD makes sense independently of diagonalizable linear operators and their eigenspaces. For instance, in the pedagogical model of "QM over sets" [17], the vector space is $\mathbb{Z}_2^n$, and the only linear operators $\mathbb{Z}_2^n \to \mathbb{Z}_2^n$ are the projection operators to a subspace and their DSDs of eigenspaces are always binary corresponding to the two possible eigenvalues of $1, 0 \in \mathbb{Z}_2$. But there can be DSDs of any cardinality ($\leq n$) determined by (say) real-valued numerical attributes on a basis set. Let $U = \{u_1, ..., u_n\}$ be a basis set for $\mathbb{Z}_2^n$ (so each vector in $\mathbb{Z}_2^n$ 'is' just a subset $S \subseteq U$) and let $f : U \to \mathbb{R}$ be a numerical attribute on U (where the values of the attribute are not in the base field $\mathbb{Z}_2$). Then there is the DSD of subspaces $\wp\left(f^{-1}(r)\right)$ of the constant sets (or 'eigenvectors') $f(S) = rS$ for $S \subseteq U$ and r a value (or 'eigenvalue') in the image (or 'spectrum') $f(U) \subseteq \mathbb{R}$. As the base field is increased up to the complex numbers, then all real-valued observables can be "internalized" as self-adjoint operators taking values in the base field.

Let $DSD(V)$ be the set of DSDs on the vector space V. It is also convenient to think of DSDs as just vector-space partitions with the subspaces V_i as the blocks in the partition. Then it is easily seen that there is a natural refinement partial order on the vector-space partitions on a vector space V, That is, if $\pi = \{V_i\}_{i\in I}$ and $\sigma = \{W_j\}_{j\in J}$, then σ is *refined* by π, $\sigma \precsim \pi$, if for every $V_i \in \pi$, there is a $W_j \in \sigma$ such that $V_i \subseteq W_j$–so that $DSD(V)$ becomes a partially ordered set. The partially ordered set has a bottom element, namely the *indiscrete* DSD $\mathbf{0} = \{V\}$ (nicknames 'the blob')–but no unique top element since each different basis set will determine a different DSD of one-dimensional subspaces generated by the basis set.

4.3 Compatibility of DSDs

For our proofs, a more rigorous definition of a DSD or vector-space partition is:

Definition 18 *Let V be a finite dimensional vector space over a field $\mathbb{K}$. A direct sum decomposition (DSD) of V is a set of subspaces $\{V_i\}_{i\in I}$ such that $V_i \cap \sum_{i'\neq i} V_{i'} = \{0\}$ (the zero space) for $i \in I$ and which span the space, i.e., in terms of direct sums, $\oplus_{i\in I} V_i = V$.*

To fix notation, the following are arbitrary DSDs: $\pi = \{V_i\}_{i\in I}$, $\sigma = \{W_j\}_{j\in J}$, and $\tau = \{X_k\}_{k\in K}$ of V.

In the logic of (set) partitions, we had the luxury of assuming a fixed universe set U, but with vector-space partitions, there are many different basis sets so we have to deal with the problem of incompatibility (or non-commutativity) of DSDs. When the DSDs are generated by linear operators, this is the non-commutativity of operators, but we are dealing with DSD's without assuming any corresponding operators. Intuitively, a diagonalizable linear operator is determined by a DSD plus an eigenvalue in the base field for each distinct subspace in the DSD. But the most important thing about eigenvalues is not their actual values but when they are the same or different, and that information is conveyed by the DSDs. Hence we need to define compatibility and incompatibility between DSDs to correspond to the usual distinction between commutativity and non-commutativity of operators.

Given two DSDs $\pi = \{V_i\}_{i\in I}$ and $\sigma = \{W_j\}_{j\in J}$, their *proto-join* $\pi \vee \sigma$ is the set of non-zero subspaces $\{V_i \cap W_j | V_i \cap W_j \neq \{0\}\}_{(i,j)\in I\times J}$. The proto-join $\pi\vee\sigma$ is not necessarily a DSD. The space spanned by the proto-join is denoted $\mathcal{SE}$. If the two DSDs π and σ were defined as the eigenspace DSDs of two diagonalizable operators, then the space $\mathcal{SE}$ spanned by the proto-join would be the space spanned by the simultaneous eigenvectors of the two operators (and hence the notation $\mathcal{SE}$).

Proposition 16 *Let $F, G : V \to V$ be two diagonalizable linear operators on a finite dimensional vector space V. Then $\mathcal{SE}$ is the kernel of the commutator: $\mathcal{SE} = \ker([F,G]) = \ker(FG - GF)$.*

Proof: Let v be any simultaneous eigenvector of the operators, i.e., $Fv = \lambda v$ and $Gv = \mu v$. Then $[F,G](v) = (FG - GF)(v) = (\lambda\mu - \mu\lambda)v = 0$ so the space $\mathcal{SE}$ spanned by the simultaneous eigenvectors is contained in the kernel $\ker([F,G])$, i.e., $\mathcal{SE} \subseteq \ker([F,G])$. Conversely, if we restrict the two operators to the subspace $\ker([F,G])$, then the restricted operators commute on that subspace. Then it is a standard theorem of linear algebra [29, p. 177] that the subspace $\ker([F,G])$ is spanned by simultaneous eigenvectors of the two restricted operators. But if a vector is a simultaneous eigenvector for the two operators restricted to a subspace, they are the same for the operators on the whole space V, since the two conditions $Fv = \lambda v$ and $Gv = \mu v$ only involves a vector in the subspace. Hence $\ker([F,G]) \subseteq \mathcal{SE}$. □

The operators commute if the commutator $[F,G]$ is the zero operator, i.e., $[F,G]v = FG - GF = 0$ for any $v \in V$, so the kernel of the commutator is the whole space V. The commutativity definitions for two DSDs π and σ without using operators are:

Definition 19 *π, σ commute or are compatible, denoted $\pi \leftrightarrow \sigma$, if their proto-join $\pi \vee \sigma$ spans the whole space, i.e., $\mathcal{SE} = V$.*

When two DSDs π and σ are compatible, $\pi \leftrightarrow \sigma$, their proto-join is the *join* DSD*:*

$$\pi \vee \sigma = \{V_i \cap W_j | V_i \cap W_j \neq \{0\}\}_{(i,j)\in I\times J}$$

Join of DSDs when $\pi \leftrightarrow \sigma$.

When the proto-join $\pi \vee \sigma$ is the join so that $\pi \vee \sigma$ is a DSD, then it is clear from the refinement partial ordering of DSDs that $\pi \vee \sigma$ is the join in the sense of the least upper bound on π and σ. Since the subspaces in the join of any DSD π with the indiscrete DSD $\mathbf{0}$ are just the subspaces of π, their proto-join is a join and:

$$\pi \vee \mathbf{0} = \pi.$$

Moreover this means that the indiscrete DSD is compatible with all DSDs, i.e., for any DSD π:

$$0 \leftrightarrow \pi.$$

A set of partitions on a set U might be said to be complete if their join is the discrete partition $\mathbf{1}_U$ whose blocks are singletons. Similarly, in a vector space, a set of compatible (or commuting) DSDs is said to be *complete* is their join is a DSD of one dimensional subspaces. In terms of operators, this is Dirac's Complete Set of Commuting Operators or CSCO [12]. Then the simultaneous 'eigenvector' determined by each one-dimensional intersection can be characterized by the sequence of 'eigenvalues' it had in each DSD of the complete set.

Compatibility defines a binary relation on $DSD(V)$ that is clearly reflexive and symmetric. The relation cannot be transitive since $\mathbf{0}$ is compatible with all DSDs and then $\pi \leftrightarrow \mathbf{0} \leftrightarrow \sigma$ would imply that all DSDs are compatible (see below for counterexamples)–so compatibility is not an equivalence relation. The join operation preserves compatibility, i.e., if $\pi \leftrightarrow \sigma$, then $\pi \leftrightarrow \pi \vee \sigma \leftrightarrow \sigma$. The important thing is that if a set of three DSDs are mutually compatible, then their three-way join is a DSD. To prove that, we need a Lemma.

Lemma 20 *Let the DSDs $\pi = \{V_i\}_{i\in I}$ and $\sigma = \{W_j\}_{j\in J}$ be compatible so that $\pi \vee \sigma$ is a DSD and thus any $v \in V$ has a unique expression $v = \sum_{(i,j)\in I\times J} v_{ij}$ where $v_{ij} \in V_i \cap W_j$. Let $v_i = \sum_{j\in J} v_{ij}$ so that $v_i \in V_i$ and then $v = \sum_{i\in I} v_i$. If $v \in V_i$, then $v = v_i$.*

Proof: Let $\widehat{v_i} = \sum_{i'\in I, i'\neq i} v_{i'}$ so that $v = v_i + \widehat{v_i}$. Hence if $v \in V_i$, then $v - v_i = \widehat{v_i} \in V_i$. Since $\widehat{v_i} \in \oplus_{i'\in I, i'\neq i} V_{i'}$, $\widehat{v_i} \in V_i \cap \oplus_{i'\in I, i'\neq i} V_{i'}$ so $\widehat{v_i} = 0$ since $\pi = \{V_i\}_{i\in I}$ is a DSD which implies $V_i \cap \oplus_{i'\in I, i'\neq i} V_{i'} = \{0\}$. If $\widehat{v_i} = 0$, then $v = v_i$.$\square$

Theorem 21 *Given three DSDs, $\pi = \{V_i\}_{i\in I}$, $\sigma = \{W_j\}_{j\in J}$, and $\tau = \{X_k\}_{k\in K}$ that are mutually compatible, i.e., $\pi \leftrightarrow \sigma$, $\pi \leftrightarrow \tau$, and $\sigma \leftrightarrow \tau$, then $(\pi \vee \sigma) \leftrightarrow \tau$ and equivalently , $\pi \leftrightarrow \sigma \vee \tau$ and thus $\pi \vee \sigma \vee \tau$ is a DSD.*

Proof: We need to prove $\pi \vee \sigma \leftrightarrow \tau$ or equivalently $\pi \leftrightarrow \sigma \vee \tau$, i.e., that

$$\oplus_{(i,j,k)\in I\times J\times K} ((V_i \cap W_j) \cap X_k) = \oplus_{(i,j,k)\in I\times J\times K} (V_i \cap (W_j \cap X_k)) = V.$$

Consider any nonzero $v \in V$ where since $\pi \leftrightarrow \sigma$, there are $v_{ij} \in V_i \cap W_j$ for each $i \in I$ and $j \in J$ such that $v = \sum_{(i,j)\in I\times J} v_{ij}$. Consider any such nonzero v_{ij}. Now since $\pi \leftrightarrow \tau$, there are $v_{ij,i'k} \in V_{i'} \cap X_k$ for each $i' \in I$ and $k \in K$ such that $v_{ij} = \sum_{(i',k)\in I\times K} v_{ij,i'k}$. But since $v_{ij} \in V_i$, by the Lemma, only $v_{ij,ik}$ is nonzero, so $v_{ij} = \sum_{k\in K} v_{ij,ik}$. Symmetrically, since $\sigma \leftrightarrow \tau$, there are $v_{ij,j'k} \in W_{j'} \cap X_k$ for each $j' \in J$ and $k \in K$ such that $v_{ij} = \sum_{(j',k)\in J\times K} v_{ij,j'k}$. But since $v_{ij} \in W_j$, by the Lemma, only $v_{ij,jk}$ is nonzero, so $v_{ij} = \sum_{k\in K} v_{ij,jk}$. Now since $\{X_k\}_{k\in K}$ is a DSD, there is a unique expression for $v_{ij} = \sum_{k\in K} v_{ijk}$ where $v_{ijk} \in X_k$. Hence by uniqueness: $v_{ijk} = v_{ij,ik} = v_{ij,jk}$. But since $v_{ij,ik} \in V_i$ and $v_{ij,jk} \in W_j$ and $v_{ij,ik} = v_{ijk} = v_{ij,jk}$, we have $v_{ijk} \in V_i \cap W_j \cap X_k$. Thus $v = \sum_{(i,j)\in I\times J} v_{ij} = \sum_{(i,j)\in I\times J} \sum_{k\in K} v_{ijk} = \sum_{(i,j,k)\in I\times J\times K} v_{ijk}$. Since v was arbitrary, $\oplus_{(i,j,k)\in I\times J\times K} (V_i \cap W_j \cap X_k) = V$, so $\pi \vee \sigma \vee \tau$ is a DSD.$\square$

Definition 22 *π, σ are incompatible if $\mathcal{SE} \neq V$; and*

Definition 23 *π, σ are conjugate if $\mathcal{SE} = \{\emptyset\}$.*

4.4 Examples of compatibility, incompatibility, and conjugacy

Example of incompatibility (or non-commutativity). To illustrate these concepts, it suffices to take the simplest case of a vector space over $\mathbb{Z}_2$. Consider the $2^3 = 8$-element vector space $\mathbb{Z}_2^3$ with a standard computational basis $\{e_1, e_2, e_3\}$ where the vectors may be represented as column vectors;

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

An alternative basis is:

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } a_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The a_i form a basis since $a_1 + a_3 = e_3$ (remember that $1 + 1 = 0$ in $\mathbb{Z}_2$), $a_2 + a_3 = e_1$, and $a_1 + a_2 + a_3 = e_2$. To define a DSD, all we need is a numerical attribute defined on a basis set. Consider $f : U = \{e_1, e_2, e_3\} \to \mathbb{R}$ defined by: $f(e_1) = 1$ and $f(e_2) = f(e_3) = 17$ (the values don't matter; all that matters is when they are the same or different). Let $g : U' = \{a_1, a_2, a_3\} \to \mathbb{R}$ be defined by: $g(a_1) = g(a_2) = 2$ and $g(a_3) = 5$. Then f defines a DSD π with two blocks: $f^{-1}(1) = \{e_1\}$ which generates the subspace $\left[f^{-1}(1)\right] = \{0, e_1\}$ (where the square brackets $[S]$ indicate the subspace generated by the vectors in S) and $f^{-1}(17) = \{e_2, e_3\}$ with generates the subspace $\left[f^{-1}(17)\right] = \{0, e_2, e_3, e_2 + e_3\}$. Note that in terms of subset of the basis set $\{e_1, e_2, e_3\}$, the subspaces generated by the inverse images are just the powersets: $\wp\left(f^{-1}(1)\right) = \{\emptyset, \{e_1\}\}$ and $\wp\left(f^{-1}(17)\right) = \{\emptyset, \{e_2\}, \{e_3\}, \{e_2, e_3\}\}$. The other numerical attribute g defines another DSD σ with two blocks: $g^{-1}(2) = \{a_1, a_2\}$ which generates the subspace $\left[g^{-1}(2)\right] = \{0, a_1, a_2, a_1 + a_2\}$ and $g^{-1}(5) = \{a_3\}$ which generates the subspace $\left[g^{-1}(5)\right] = \{0, a_3\}$. Then the direct-sums are the whole space: $\left[f^{-1}(1)\right] \oplus \left[f^{-1}(17)\right] = \mathbb{Z}_2^3 = \left[g^{-1}(2)\right] \oplus \left[g^{-1}(5)\right]$. To compute the proto-join of the two DSDs, we need to express the vectors spaces in the same (computational) basis so: $\left[g^{-1}(2)\right] = \{0, a_1, a_2, a_1 + a_2\} = \{0, e_1 + e_2, e_2 + e_3, e_1 + e_3\}$ and $\left[g^{-1}(5)\right] = \{0, a_3\} = \{0, e_1 + e_2 + e_3\}$. Since there are two blocks in each DSD π and σ, there are four possible non-zero intersections in the proto-join $\pi \vee \sigma$, but the only non-zero intersection is the subspace generated by the intersection of $\left[f^{-1}(17)\right]$ and $\left[g^{-1}(2)\right]$:

$$\left[f^{-1}(17)\right] \cap \left[g^{-1}(2)\right] = \{0, e_2, e_3, e_2 + e_3\} \cap \{0, e_1 + e_2, e_2 + e_3, e_1 + e_3\} = \{0, e_2 + e_3\}.$$

Since the simultaneous 'eigenvector' space $\mathcal{SE} = \{0, e_2 + e_3\}$ is neither $V = \mathbb{Z}_2^3$ nor the zero space, π and σ are incompatible but not conjugate.

Example of compatibility. Let U and f be the same as above but consider the basis: $b_1 = e_1$, $b_2 = e_2$, and $b_3 = e_2 + e_3$. Let $g : U'' = \{b_1, b_2, b_3\} \to \mathbb{R}$ where $g(b_1) = 1$, $g(b_2) = 2$, and $g(b_3) = 3$ so the three subspaces in the DSD σ are $\left[g^{-1}(1)\right] = \{0, b_1\} = \{0, e_1\}$, $\left[g^{-1}(2)\right] = \{0, b_2\} = \{0, e_2\}$, and $\left[g^{-1}(3)\right] = \{0, b_3\} = \{0, e_2 + e_3\}$. There are six subspaces determined by the intersections of the two subspaces given by f and the three given by g, but the only non-zero intersections are:

$$\{0, e_1\}, \{0, e_2\}, \text{ and } \{0, e_2 + e_3\}.$$

These three subspaces span the whole space so $\mathcal{SE} = V$, and π and σ are compatible. Each of these 'eigenspaces' is one-dimensional so the DSDs π and σ given by f and g are a complete set of DSDs and the simultaneous 'eigenvectors' in the one-dimensional blocks can be characterized by the sequence of 'eigenvalues', i.e., $e_1 = |1, 1\rangle$, $e_2 = |17, 2\rangle$, and $e_2 + e_3 = |17, 3\rangle$, when the DSDs are determined by numerical attributes to attach 'eigenvalues' to the subspaces in the DSDs.

Example of conjugacy. For a standard example of conjugacy, we need to work in $\mathbb{Z}_2^n$ where n is even so consider $\mathbb{Z}_2^4$ with the computational basis $U = \{e_1, ..., e_4\}$. Let $\hat{U} = \{\hat{e}_1, ..., \hat{e}_4\}$ where $\hat{e}_i$ is the vector formed by excluding e_i from the sum $\sum_{j=1}^{4} e_j$. These four vectors form a basis set for $\mathbb{Z}_2^4$–while the corresponding set of vectors for n odd do not form a basis. Let $f : U \to \mathbb{R}$ be defined by $f(e_1) = f(e_2) = 1$ and $f(e_3) = f(e_4) = 0$ so the two subspaces in π are $\left[f^{-1}(1)\right] = \{0, e_1, e_2, e_1 + e_2\}$ and $\left[f^{-1}(0)\right] = \{0, e_3, e_4, e_3 + e_4\}$. Let $g : \hat{U} \to \mathbb{R}$ be given by $g(\hat{e}_2) = g(\hat{e}_3) = 1$ and $g(\hat{e}_1) = g(\hat{e}_4) = 1$ so that the two subspaces in σ are:

$$\left[g^{-1}(1)\right] = \{0, \hat{e}_2, \hat{e}_3, \hat{e}_2 + \hat{e}_3\} = \{0, e_1 + e_3 + e_4, e_1 + e_2 + e_4, e_2 + e_3\},$$
$$\left[g^{-1}(0)\right] = \{0, \hat{e}_1, \hat{e}_4, \hat{e}_1 + \hat{e}_4\} = \{0, e_2 + e_3 + e_4, e_1 + e_2 + e_3, e_1 + e_4\}.$$

Then it is easily checked that the four intersections of subspaces in the proto-join $\pi \vee \sigma$ are all the zero space $\{0\}$, e.g., f and g have no simultaneous 'eigenvectors', so π and σ are conjugate.

Since the two numerical attributes in the conjugacy example are characteristic functions taking value in the base field $\mathbb{Z}_2$, they define linear operators F and G on $\mathbb{Z}_2^4$ and thus we can compute their commutator $[F, G] = FG - GF$ as usual once restated in the computational basis. The matrix to convert a $0,1$-vector written in the $\hat{U}$-basis to the same $0,1$-vector written in the U-basis is:

$$C_{U \leftarrow \hat{U}} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

For instance, $e_1 = \hat{e}_2 + \hat{e}_3 + \hat{e}_4$ so that vector in $\hat{U}$-basis is the column vector $[0, 1, 1, 1]^t$ (t represents the transpose) and

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \overset{\text{mod}(2)}{=} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which is e_1 in the U-basis. The inverse to the conversion matrix $C_{U \leftarrow \hat{U}}$ is:

$$C_{\hat{U} \leftarrow U} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

so the conversion matrix is its own inverse. The numerical attribute f now defines a linear operator $F : \mathbb{Z}_2^4 \to \mathbb{Z}_2^4$ given by $Fe_1 = 1e_1$ land $Fe_2 = 1e_2$ as well as $Fe_3 = 0e_3$ and $Fe_4 = 0e_4$ both of which are the zero vector 0.The numerical attribute g also defines a linear operator $G : \mathbb{Z}_2^4 \to \mathbb{Z}_2^4$ given by $G\hat{e}_2 = 1\hat{e}_2$ and $G\hat{e}_3 = 1\hat{e}_3$ as well as $G\hat{e}_1 = 0\hat{e}_1$ and $G\hat{e}_4 = 0\hat{e}_4$ both of which are the vector 0. We need to convert the matrix for the operator G defined in the $\hat{U}$-basis into the U-basis in order to commute the commutator of the two operators. The matrix representation of the G operator in the $\hat{U}$-basis is just the projection operator P_1 to the subspace $\left[g^{-1}(1)\right] = \{0, \hat{e}_2, \hat{e}_3, \hat{e}_2 + \hat{e}_3\}$ associated with the eigenvalue 1. The conversion of that matrix to the U-basis is accomplished by pre- and post- multiplying by the appropriate conversion matrices:

$$\begin{aligned} & C_{U \leftarrow \hat{U}} P_1 C_{\hat{U} \leftarrow U} \\ = & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \\ = & \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix} \overset{\text{mod}(2)}{=} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$

The matrix representing the $F : \mathbb{Z}_2^4 \to \mathbb{Z}_2^4$ in the U-basis is the projection matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the commutator in terms of the operators is $[F, G] = FG - GF$, so it is computed in the U-basis as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \overset{\mathrm{mod}(2)}{=} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The resulting matrix, representing the commutator operator $FG - GF$, has a determinant of 1 so it is a non-singular transformation with a kernel of the zero space $\{0\}$. Hence the definition of π and σ being conjugate agrees with the kernel of the commutator $[F, G] = FG - GF : \mathbb{Z}_2^4 \to \mathbb{Z}_2^4$ being the zero space when the numerical attributes f and g define operators $F, G : \mathbb{Z}_2^4 \to \mathbb{Z}_2^4$. All these examples illustrate general aspects of DSDs with no restriction to Hilbert spaces. In fact, by using vector spaces over $\mathbb{Z}_2$ instead of $\mathbb{C}$, all the examples are from the pedagogical model of "quantum mechanics over sets" [17] since vectors in $\mathbb{Z}_2^n$ can be interpreted as subsets of an n-element set, and the different n-element basis sets allows the introduction of the notions on commutativity and non-commutativity that are important in QM over $\mathbb{C}$.

4.5 The meet of DSDs and properties of refinement

The meet of two set partitions $\pi = \{B_i\}_{i \in I}$ and $\sigma = \{C_j\}_{j \in J}$ was defined as the set partition $\pi \wedge \sigma$ whose blocks are a union of some π-blocks and also a union of some σ-blocks, and those blocks are minimal in that regard. The meet of two vector-space partitions or DSDs is just the vector space version of the set definition

Definition 24 *For any two DSDs $\pi = \{V_i\}_{i \in I}$ and $\sigma = \{W_j\}_{j \in J}$, the meet $\pi \wedge \sigma$ is the DSD whose subspaces are direct sums of subspaces from π and the direct sum of subspaces from σ and are minimal subspaces in that regard. That is, $\{Y_l\}_{l \in L}$ is the meet if there is a set partition $\{I_l\}_{l \in L}$ on I and a set partition $\{J_l\}_{l \in L}$ on J such that for all $l \in L$: $Y_l = \oplus_{i \in I_l} V_i = \oplus_{j \in J_l} W_j$ and that holds for no more refined partitions on the index sets.*

Note that for the blob $\mathbf{0} = \{V\}$, $V = \oplus_{i \in I} V_i = \oplus_{j \in J} W_j$ using the blob set partitions $\{I\}$ and $\{J\}$, but in general the meet $\pi \wedge \sigma$ will use more refined partitions on I and J. Thus $\mathbf{0}$ is always a lower bound on π and σ, and we show below that $\pi \wedge \sigma$ least such lower bound.

Proposition 17 *If $\pi \leftrightarrow \tau$ or $\sigma \leftrightarrow \tau$, then $\pi \wedge \sigma \leftrightarrow \tau$.*

Proof: Suppose that $\pi \leftrightarrow \tau$ so that $\{V_i \cap X_k\}_{(i,k) \in I \times K}$ is a DSD, i.e., $\oplus_{(i,k) \in I \times K} \{V_i \cap X_k\} = V$. The blocks of $\pi \wedge \sigma$ for $Y_l = \oplus_{i \in I_l} V_i = \oplus_{j \in J_l} W_j$. To show that $\pi \wedge \sigma \leftrightarrow \tau$, we need to show that $\{Y_l \cap X_k\}_{(l,k) \in L \times K}$ spans the whole space V, i.e., $\oplus_{(l,k) \in L \times K} Y_l \cap X_k = V$. Now $Y_l \cap X_k = (\oplus_{i \in I_l} V_i) \cap X_k$ and since $\pi \leftrightarrow \tau$, $\{V_i \cap X_k\}_{(i,k) \in I \times K}$ is a DSD, so $V_i = \oplus_{k \in K} V_i \cap X_k$ and then $Y_l \cap X_k = (\oplus_{i \in I_l} V_i) \cap X_k = (\oplus_{(i,k') \in I_l \times K} V_i \cap X_{k'}) \cap X_k$ [31, p. 30]. But the intersection between $(\oplus_{(i,k') \in I_l \times K} V_i \cap X_{k'})$ and X_k is $V_i \cap X_k$ since no non-zero vector in $(\oplus_{(i,k') \in I_l \times K, k' \neq k} V_i \cap X_{k'})$ can be in X_k. Thus $Y_l \cap X_k = \oplus_{i \in I_l} V_i \cap X_k$ and

$$\oplus_{(l,k)\in L\times K} Y_l \cap X_k = \oplus_{(l,k)\in L\times K} \oplus_{i\in I_l} V_i \cap X_k = \oplus_{(i,k)\in I\times K} \{V_i \cap X_k\} = V$$

which is a DSD since $\pi \leftrightarrow \tau$. If $\sigma \leftrightarrow \tau$, the proof is symmetrical. $\square$

"The Blob" absorbs everything it meets (as in the old movie of the same title):

$$\mathbf{0} \wedge \pi = \mathbf{0}.$$

It may be recalled that the refinement partial order on DSDs is defined just like the refinement partial order on partitions but with the subspaces of the DSD replacing the subsets or blocks of the partition. In the partition refinement $\sigma \precsim \pi$, we have for each block $C \in \sigma$, $C = \cup \{B \in \pi : B \subseteq C\}$. The same holds for vector-space partitions.

Lemma 25 *If $\sigma \preceq \pi$, then each subspace $W_j \in \sigma$, $W_j = \oplus \{V_i \in \pi : V_i \subseteq W_j\}$.*

Proof: Consider any nonzero vector $v \in W_j$. Since π is a DSD, $v = \sum_{i\in I} v_i$ where $v_i \in V_i$ so we can divide v into two parts: $v = \sum_{V_i\subseteq W_j} v_i + \sum_{V_{i'}\not\subseteq W_j} v_{i'}$. Now $\sigma \preceq \pi$, so for each $v_{i'} \in V_{i'} \not\subseteq W_j$, there is a $W_{j'}$ such that $v_{i'} \in V_{i'} \subseteq W_{j'}$ so $\sum_{V_{i'}\not\subseteq W_j} v_{i'} \in \sum_{j'\neq j} W_{j'}$. But $\sum_{V_{i'}\not\subseteq W_j} v_{i'} = v - \sum_{V_i\subseteq W_j} v_i \in W_j$ and $W_j \cap \sum_{j'\neq j} W_{j'} = \{0\}$ since σ is a DSD. Thus $v - \sum_{V_i\subseteq W_j} v_i = 0$ so $v \in \oplus \{V_i \in \pi : V_i \subseteq W_j\}$. $\square$

Then $\sigma \preceq \pi$ implies $\pi \leftrightarrow \sigma$ and $\pi \vee \sigma = \pi$ as well as $\pi \wedge \sigma = \sigma$ as expected.

Proposition 18 *For any two DSDs π and σ, if they a common upper bound τ, i.e., $\pi, \sigma \preceq \tau$, then (i) $\pi \leftrightarrow \sigma$, and (ii) the join $\pi \vee \sigma$ is defined which is the least upper bound of π and σ.*

Proof: Since $\pi, \sigma \preceq \tau = \{X_k\}_{k\in K}$, then for each X_k, there is a $V_i \in \pi$ such that $X_k \subseteq V_i$ and there is a $W_j \in \sigma$ such that $X_k \subseteq W_j$ so $X_k \subseteq V_i \cap W_j$. Since the $\{X_k\}_{k\in K}$ span the space so must the nonzero $V_i \cap W_j$ so $\pi \leftrightarrow \sigma$ which proves (i) and makes $\pi \vee \sigma = \{V_i \cap W_j \neq \{0\}\}_{(i,j)\in I\times J}$ into a DSD. To prove (ii), as just shown, for any given X_k, there is a V_i and W_j such that $X_k \subseteq V_i \cap W_j$ so $\pi \vee \sigma$ is the least upper bound of π and σ in the refinement partial order. $\square$

Unlike set partitions on the same set, two DSDs π and σ need not have a common upper bound so $DSD(V)$ is not a join-semilattice.

Lemma 26 *Given a DSD $\pi = \{V_i\}_{i\in I}$, let $X = \oplus_{i\in I_X} V_i$ and $Y = \oplus_{i\in I_Y} V_i$ both be direct sums of some V_i's. If $X \cap Y$ is nonzero, then $X \cap Y = \oplus_{i\in I_X\cap I_Y} V_i$.*

Proof: For any nonzero $v \in X \cap Y$, there is a unique expression $v = \sum_{i\in I_X} v_{i,X}$ where $v_{i,X} \in V_i \subseteq X$ and a unique expression $v = \sum_{i\in I_Y} v_{i,Y}$ where $v_{i,Y} \in V_i \subseteq Y$. Since π is a DSD, there is also a unique expression $v = \sum_{i\in I} v_i$ so, for each nonzero v_i, $v_i = v_{i,X} = v_{i,Y} \in V_i \cap X \cap Y$. Thus for any such i, V_i is a common direct summand to X and Y, so $V_i \subseteq X \cap Y$. Thus every nonzero element $v \in X \cap Y$ is in a direct sum of V_i's for $V_i \subseteq X \cap Y$ and thus $X \cap Y$ is the direct sum of V_i that are common direct summands of X and Y. $\square$

Proposition 19 *The meet $\pi \wedge \sigma$ is the greatest lower bound of π and σ.*

Proof: If $\tau \preceq \pi, \sigma$ then each $X_k = \oplus \{V_i : V_i \subseteq X_k\} = \oplus \{W_j : W_j \subseteq X_k\}$. By the construction of $\pi \wedge \sigma$, there is a set partition $\{I_l\}_{l\in L}$ on I and a set partition $\{J_l\}_{l\in L}$ on J such that each subspace in the meet $\pi \wedge \sigma = \{Y_l\}$ satisfies: $Y_l = \oplus_{i\in I_l} V_i = \oplus_{j\in J_l} W_j$, and where no subsets of I smaller than I_l and subsets of J smaller than J_l have that property. Since each V_i is contained in some X_k, if $i \in I_l$, then $V_i \subseteq Y_l \cap X_k$. Since both Y_l and X_k are direct sums of some V_i, then by the Lemma the nonzero subspace $Y_l \cap X_k$ is also a direct sum of the common direct summand V_i's. Symmetrically, since the same Y_l and X_k are direct sums of some W_j's, then by the Lemma the nonzero subspace $Y_l \cap X_k$ is also a direct sum of the common direct summand W_j's. But since Y_l is the smallest direct sum of both V_i's and W_j's, $Y_l \cap X_k = Y_l$, i.e., $Y_l \subseteq X_k$, and thus $\pi \wedge \sigma$ is the greatest (in the refinement partial ordering) lower bound on π and σ. □

Since the blob $\mathbf{0}$ is a lower bound for all DSDs, the meet of two DSDs always exists, which means that $DSD(V)$ is a meet semi-lattice.

The binary DSDs $\alpha = \{A_1, A_2\}$ are the atoms of the meet-semi-lattice $DSD(V)$, i.e., the DSDs so that there are no DSDs between them and the blob $\mathbf{0}$. A meet-semi-lattice is said to be *atomistic* if every non-blob element is a join of atoms and $DSD(V)$ is atomistic since any non-blob DSD $\pi = \{V_i\}_{i \in I}$ is clearly the join of the atoms $\{V_i, \oplus_{i' \in I, i' \neq i} V_{i'}\} \precsim \pi$.

4.6 The partition lattice determined by a maximal DSD

There is no maximum DSD, only maximal DSDs. Each maximal DSD in the partial ordering is a *discrete* or *completely decomposed* DSD of one-dimensional subspaces (or rays) of V (so the number of blocks is the dimension of V). Each basis set $U = \{u_1, ..., u_n\}$ determines a maximal DSD $\omega = \{[u_i]\}_{i=1}^{n}$ of the one-dimensional subspaces $[u_i]$ generated by the basis elements. A maximal ω determines the segment $[\mathbf{0}, \omega] = \{\pi | \pi \preceq \omega\}$ of $DSD(V)$ between 0 and ω. Fixing a basis set U so that $\omega = \{[u_i]\}_{i=1}^{n}$ then associates a set partition block B with every V_i of the vector space partition or DSD $\pi \precsim \omega$, namely the elements of the basis set U that generate V_i, so that the segment $[\mathbf{0}, \omega]$ with the induced operations of join and meet is isomorphic with the set partition lattice $\Pi(U)$. Or we could think of the one-dimensional subspaces as just points in a set ω so the segment is isomorphic to the partition lattice $\Pi(\omega)$ on the points ω.

$$\Pi(\omega) \cong \{\pi | \pi \preceq \omega\} = [\mathbf{0}, \omega] \subseteq DSD(V)$$

Thus $DSD(V)$ can be viewed as a set of overlapping partition lattices (or partition algebras when more operations are added) where a particular partition algebra is picked out by picking a maximal DSD. This relationship in the quantum logic of DSDs is analogous to the way in which a complete set of one-dimensional subspaces determines a Boolean algebra in the usual quantum logic of subspaces when viewed as a set of overlapping Boolean algebras or a "partial Boolean algebra" [30, p. 193]. Figure 4.2 illustrates the general 'shape' of $DSD(V)$ with the partition lattices determined by picking out a maximal DSD ω.

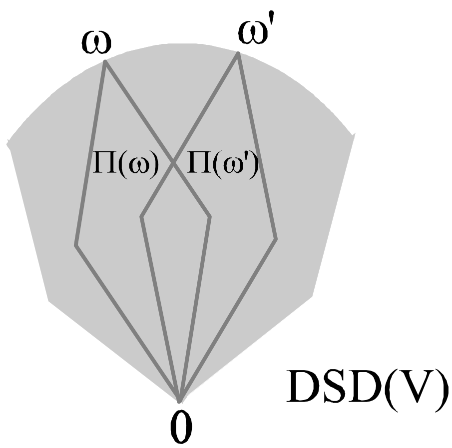

Figure 4.2: General shape of $DSD(V)$ with partition lattices determined by discrete DSDs.

For vector spaces over finite fields, e.g., $\mathbb{Z}_2^n$, the number of DSDs can be computed by the formulas developed in the Appendix. Over a finite field with q element, each discrete DSD determines $(q-1)^n$

bases since there are $q-1$ choices out of each one-dimensional subspace to be its basis element, but for $q=2$, $(q-1)^n = 1$ so the number of discrete DSDs and the number of basis sets are the same in $\mathbb{Z}_2^n$. In $\mathbb{Z}_2^2$, there are only three two-element bases, i.e., three discrete DSDs, and no other DSDs except the indiscrete one 0. Thus fixing a basis ω determines the partition lattice $\Pi(\omega) \cong \wp(1)$ which we saw before was isomorphic to the two-element Boolean algebra $\wp(1)$. This is illustrated in Figure 8.3. The vectors in a vector space over $\mathbb{Z}_2$ can be represented as subsets of a computational basis set $U = \{a,b\}$ where the addition of vectors is the symmetric difference $S+T = (S-T) \cup (T-S)$ of subsets. Thus for example $\{a\} + \{a,b\} = \{b\}$ where over $\mathbb{Z}_2$, the addition of elements is mod (2) so that $\{a\} + \{a\} = \emptyset$. For the sake of brevity in the figures, the zero vector (empty set) is left out of the subspaces in the illustrated DSDs and the elements in a subset are shorted to juxtaposition so that $\{a,b\}$ is written as $\{ab\}$. Thus the DSD $\{\{\emptyset, \{a\}\}, \{\emptyset, \{a,b\}\}\}$ is shorted to $\{\{a\}, \{ab\}\}$ in Figure 4.3.

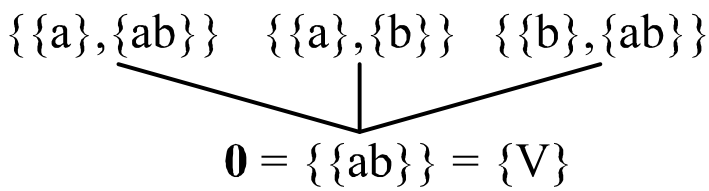

Figure 4.3: The Hasse diagram for $DSD\left(\mathbb{Z}_2^2\right)$.

As n increases, the number of DSDs in $\mathbb{Z}_2^n$ increases rapidly, so for $n=3$, there are 28 basis sets and also 28 DSDs between the 28 discrete DSDs and the one indiscrete ones. Each discrete DSD has three blocks and those three blocks can be combined (i.e., direct summed) in $\binom{3}{2} = 3$ ways to form the three two-block DSDs between the three-block discrete DSD and the one-block blob $\mathbf{0}$. Figure 4.4 illustrates the fragment of $DSD\left(\mathbb{Z}_2^3\right)$ for two discrete DSDs, each with their three intermediate DSDs.

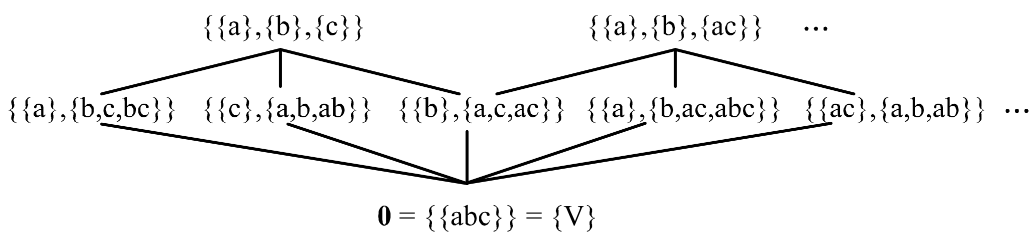

Figure 4.4: The part of $DSD\left(\mathbb{Z}_2^3\right)$ for two discrete DSDs, i.e., two different basis sets.

The fragment of $DSD\left(\mathbb{Z}_2^3\right)$ shown in Figure 4.4 is useful to illustrate the non-transitivity of compatibility. The DSD $\{\{b\}, \{a,c,ac\}\}$ and the DSD $\{\{c\}, \{a,b,ab\}\}$ to its left in Figure 4.4 have proto-join is $\{\{a\}, \{b\}, \{c\}\}$ which spans the whole space so they are compatible. To the right of $\{\{b\}, \{a,c,ac\}\}$ in Figure 8.4 is the DSD $\{\{a\}, \{b,ac,abd\}\}$ and their proto-join is $\{\{a\}, \{b\}, \{ac\}\}$ which also spans the space so they are also compatible. But for the tow DSDs on the left and right of $\{\{b\}, \{a,c,ac\}\}$ in Figure 8.4, their proto-join is $\{\{a\}, \{b\}\}$ which does not span the space so they are not compatible.

Picking a basis set such as $\omega = \{\{a\}, \{b\}, \{c\}\}$ for $\mathbb{Z}_2^3$ then determines a segment $[\mathbf{0}, \omega]$ that is isomorphic to the set partition lattice on a three element set where each ray in the discrete DSD is treat as a point. Thus for this basis, the set partition lattice is illustrated in Figure 4.5 where a block $\{a,b\}$ in the set partition is abbreviated as ab.

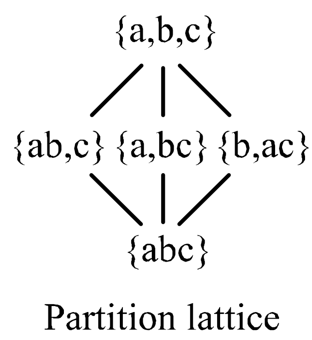

Figure 4.5: Partition lattice isomorphic to segment $[\mathbf{0}, \omega]$ for $\omega = \{\{a\}, \{b\}, \{c\}\}$.

If the basis set $\omega = \{\{ac\}, \{bc\}, \{abc\}\}$ had been chosen, then the segment $[0, \omega]$ would still be isomorphic to the set partition lattice on a three-element set where the three elements are denoted ac, bc, and abc, as illustrated in Figure 4.6.

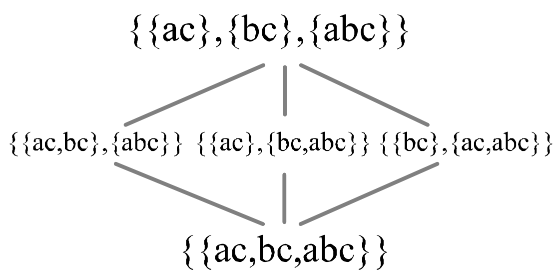

Figure 4.6: The partition lattice determined by another three-element basis set for $\mathbb{Z}_2^3$.

To fix notation, let an arbitrary maximal DSD be $\omega = \{U_z : z \in Z\}$ where each U_z is a one-dimensional subspace or ray and $|Z| = \dim(V)$. For any $\pi \in [\mathbf{0}, \omega]$, $\pi \preceq \omega$ so ω is (by definition) the maximum or *top* DSD in $[\mathbf{0}, \omega]$ and thus might be symbolized as the discrete DSD $\mathbf{1}_\omega = \omega$. Each subspace $V_i \in \pi \preceq \omega$ has $V_i = \oplus\{U_z : U_z \subseteq V_i, z \in Z\}$ so $\mathbf{1}_\omega$ absorbs what it joins and is the unit element for meets within $[\mathbf{0}, \omega]$:

$$\pi \vee \mathbf{1}_\omega = \mathbf{1}_\omega \text{ and } \pi \wedge \mathbf{1}_\omega = \pi.$$

All the DSDs π and σ compatible with ω, i.e., $\pi, \sigma \in [\mathbf{0}, \omega]$, are compatible with each other since they have a common upper bound.

Fixing a maximal DSD ω reduces much of the reasoning in $[\mathbf{0}, \omega]$ to reasoning about sets. For instance, the number of DSDs in the segment $[\mathbf{0}, \omega]$ is the number of partitions on a set with cardinality $\dim(V)$, i.e., $|[\mathbf{0}, \omega]|$ is the Bell number [45] for n, the dimension of V.

Indeed, given any DSD $\pi = \{V_i\}_{i \in I}$, each subspace W_j of $\sigma \in [\mathbf{0}, \pi]$ determines a subset $C_j = \{V_i : V_i \subseteq W_j\}$ so σ defines a set partition $\sigma(\pi) = \{C_j\}_{j \in J}$ on π as a set so $W_j = \oplus C_j$ for $j \in J$. Thus the lower segment $[\mathbf{0}, \pi]$ is isomorphic to the set-based partition lattice (join and meet operations) on that set π, and, in particular, $[\mathbf{0}, \omega]$ is isomorphic to the lattice of set partitions on the set ω. As a partition lattice, $[\mathbf{0}, \pi]$, has the usual properties of partition lattices. Many theorems about set partitions can then be transferred over in an appropriate form to $[\mathbf{0}, \omega]$.

A *distinction* or *dit* of a DSD $\pi = \{V_i\}_{i \in I} \in [\mathbf{0}, \omega]$ for $\omega = \{U_z\}_{z \in Z}$ is a pair $(U_z, U_{z'})$ in distinct subspaces, i.e., $U_z \subseteq V_i$ and $U_{z'} \subseteq V_{i'}$ for some distinct $V_i, V_{i'} \in \pi$. In terms of eigenvectors and eigenvalues of a diagonalizable linear operator $F : V \to V$, a distinction or dits is a pair of eigenvectors of F with different eigenvalues. The common-dits property of non-blob set partitions [**?**, p. 106] carries over to DSDs in $[\mathbf{0}, \omega]$.

Proposition 20 (Common dits) *Any two non-blob DSDs $\pi, \sigma \in [\mathbf{0}, \omega]$ have a dit in common.*

Proof: Since π is not the blob, there are U_z, $U_{z'}$ with $U_z \subseteq V_i$ and $U_{z'} \subseteq V_{i'}$ for $V_i \neq V_{i'}$. If $U_z \subseteq W_j \in \sigma$ and $U_{z'} \subseteq W_{j'} \in \sigma$ for $W_j \neq W_{j'}$ we are finished so assume $U_z \oplus U_{z'} \subseteq W_j$ for some $j \in J$. Since σ is also not the blob, there is a $U_{z''}$ contained in some $W_{j''}$ where $W_{j''} \neq W_j$. Then $U_{z''}$ cannot be in the same subspace of π as U_z and $U_{z'}$ since those two are in different subspaces of π, so either $(U_z, U_{z''})$ or $(U_{z'}, U_{z''})$ is a dit common to π and σ. □

For instance, in the segment $[\mathbf{0}, \omega]$ for $\omega = \{\{a\}, \{b\}, \{ac\}\}$ of Figure 8.4, there are three intermediate DSDs and $\binom{3}{2} = 3$ pairs of intermediate DSDs: $(\{a\}, \{b\})$ is a common dit of the DSDs $\{\{b\}, \{a, c, ac\}\}$ and $\{\{a\}, \{b, ac, abc\}\}$, and $(\{a\}, \{ac\})$ is a common dit of $\{\{a\}, \{b, ac, abc\}\}$ and $\{\{ac\}, \{a, b, ab\}\}$, while $(\{b\}, \{ac\})$ is a common dit of $\{\{b\}, \{a, c, ac\}\}$ and $\{\{ac\}, \{a, b, ab\}\}$.

As is appropriate to anything called a "logic," there should be an implication operation. Given the close connections between set partitions and vector-space partitions or DSDs given a maximal element ω, the DSD implication can be easily defined within $[\mathbf{0}, \omega]$.

Definition 27 *For $\sigma, \pi \in [\mathbf{0}, \omega]$, implication is:*

$$\begin{aligned} \sigma \Rightarrow_\omega \pi = &\{U_z | U_z \subseteq V_i \text{ if } \exists V_i \in \pi \text{ and } W_j \in \sigma, V_i \subseteq W_j\} \\ &\cup \{V_i | V_i \in \pi \text{ and there is no } W_j \in \sigma, V_i \subseteq W_j\}. \end{aligned}$$

Since $V_i = \oplus \{U_z : U_z \subseteq V_i\}$, the implication is a DSD. For instance, for $\mathbb{Z}_2^4$ with $\omega = \{\{a\}, \{b\}, \{c\}, \{d\}\} = \mathbf{1}_\omega$, $\sigma = \{\{a\}, \{b, c, d, bc, bc, cd, bcd\}\}$ and $\pi = \{\{a\}, \{b\}, \{c, d, cd\}\}$, then $\sigma \Rightarrow_\omega \pi = \{\{a\}, \{b\}, \{c\}, \{d\}\} = \mathbf{1}_\omega$ since there is the inclusion of subspaces $\{c, d, cd\} \subseteq \{b, c, d, bc, bc, cd, bcd\}$. But the definition of the implication illustrates *contextuality* in the sense that it may differ depending on which segment is being considered. In the above example, $\omega' = \{\{a\}, \{b\}, \{c\}, \{cd\}\} = \mathbf{1}_{\omega'}$ is also a maximal element in $DSD\left(\mathbb{Z}_2^4\right)$ and both $\pi, \sigma \in [\mathbf{0}, \omega']$, and there is the same inclusion of subspaces so the subspace $\{c, d, cd\}$ is atomized to its one-dimensional components which are now $\{c\}$ and $\{cd\}$ so that:

$$\sigma \Rightarrow_{\omega'} \pi = \{\{a\}, \{b\}, \{c\}, \{cd\}\} = \mathbf{1}_{\omega'} \neq \mathbf{1}_\omega = \{\{a\}, \{b\}, \{c\}, \{d\}\} = \sigma \Rightarrow_\omega \pi.$$

In this case, both the contextualized implications were equal to $\mathbf{1}_\omega$ and $\mathbf{1}_{\omega'}$ which means in either case that $\sigma \precsim \pi$ (refinement is not a contextual notion).

Recall that for set partitions $\pi = \left\{f^{-1}(r)\right\}_{r \in f(u)}$ and $\sigma = \left\{g^{-1}(r))\right\}_{r \in g(U)}$, given by numerical attributes or random variables $f, g : U \to \mathbb{R}$, then $\sigma \Rightarrow \pi = \mathbf{1}_U$ iff $\sigma \precsim \pi$ which meant that f was a sufficient statistic for g. If the value of f was known for a experiment, then the value of g was determined. In the quantum context, F and G would be compatible observables determining DSDs $\pi = \{V_i\}_{i \in I}$ of the eigenspaces of F and $\sigma = \{W_j\}_{j \in J}$ of the eigenspaces of G, and ω as a maximal DSD of simultaneous eigenvectors for F and G. Then $\sigma \Rightarrow_\omega \pi = \mathbf{1}_\omega$, i.e., $\sigma \precsim \pi$, would mean that F was a sufficient observable for G. In any measurement using the measurement basis ω, the eigenvalue returned by an F-measurement determines the eigenvalue of the observable G. In general, $\sigma \Rightarrow_\omega \pi$ is the DSD in $[\mathbf{0}, \omega]$ that gives the extent to which the eigenvalue returned by an F-measurement determines the eigenvalue of G.

The one-dimensional subspaces U_z in the DSD $\sigma \Rightarrow \pi$ give the f eigenvalues, i.e., $U_z \subseteq V_i$, that determine the g eigenvalues. For instance if g had degenerate eigenvalues and F_{π_1},...,F_{π_m} were observables with DSDs $\pi_1, ..., \pi_m$ also in $[\mathbf{0}, \omega]$ (and thus compatible), then $\sigma \Rightarrow \vee_{i=1}^m \pi_i = \mathbf{1}_\omega$ implies that the eigenvalues of F_{π_1},...,F_{π_m} are sufficient to uniquely determine the eigenvalues of g. When $\vee_{i=1}^m \pi_i = \mathbf{1}_\omega$ as well, then the eigenvalues of F_{π_1},...,F_{π_m} are sufficient to uniquely label the rays $U_z \in \omega$.

4.7 Exploiting duality in between the logics of subspaces and DSDs

The set partition operations (e.g., join, meet, and implication) on the partitions on a given universe set U can be represented as subset operations on certain subsets, i.e., ditsets, of $U \times U$. For a set partition $\pi = \{B_1, ..., B_m\}$ on U, a *distinction* or *dit of* π is an ordered pair $(u, u') \in U \times U$ of elements in distinct blocks of π. The ditset $\operatorname{dit}(\pi)$ of π is a binary relation on U (i.e., a subset of $U \times U$), and it is the complement in $U \times U$ of the equivalence relation associated with π. A partition relation on $U \times U$ is defined as the complement of an equivalence relation. The partition relations on $U \times U$ are in one-to-one correspondence with the partitions on U. Given a partition π on U, the ditset $\operatorname{dit}(\pi)$ is the corresponding partition relation, and given a partition relation, the equivalence classes in the complementary equivalence relation give the corresponding partition.

We have seen that the operations on the set partitions (join, meet, and implication) have corresponding operations on ditsets. Since $\sigma \preceq \pi$ iff $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$, the partial order of refinement between partitions is just inclusion between ditsets. In this manner the partition algebra $\Pi(U)$ of partitions on U is represented as the algebra of the ditsets of $U \times U$.

With $\omega = \{U_z\}_{z \in Z}$ fixed and playing the role of U, the above construction can be transferred to vector spaces. The operations on DSDs in $\Pi(\omega)$ (the segment $[\mathbf{0}, \omega]$ endowed with the lattice and implication operations) can be represented as subspace operations on certain subspaces of the tensor product $V \otimes V$ that are direct sums of the subspaces in the maximal DSD $\omega \otimes \omega = \{U_z \otimes U_{z'} | (U_z, U_{z'}) \in \omega \times \omega\}$ of one-dimensional subspaces on $V \otimes V$. The easiest translation uses the fact that a DSDs $\pi = \{V_i\}_{i \in I} \in \Pi(\omega)$ defines a set partition $\pi(\omega) = \{B_i\}_{i \in I}$ on $\omega = \{U_z\}_{z \in Z}$ as a set where: $B_i = \{U_z | U_z \subseteq V_i\}$ and $V_i = \oplus B_i = \oplus \{U_z | U_z \subseteq V_i\}$ for $i \in I$. Then the *ditspace* $\operatorname{Dit}(\pi)$ defined by the DSD π is the following subspace of $V \otimes V$:

$$\operatorname{Dit}(\pi) = \oplus \{U_z \otimes U_{z'} | (U_z, U_{z'}) \in \operatorname{dit}(\pi(\omega))\}.$$

Note that by the Common-Dits Theorem, any two nonzero ditspaces, i.e., ditspaces for non-blob DSDs $\pi, \sigma \in \Pi(\omega)$, have a nonzero intersection. The operations on the ditspaces are those induced by the operations on the ditsets. For $\pi, \sigma \in \prod(\omega)$,

$$\begin{aligned}
\operatorname{Dit}(\pi \vee \sigma) &= \oplus \{U_z \otimes U_{z'} | (U_z, U_{z'}) \in \operatorname{dit}(\pi(\omega) \vee \sigma(\omega))\} \\
\operatorname{Dit}(\pi \wedge \sigma) &= \oplus \{U_z \otimes U_{z'} | (U_z, U_{z'}) \in \operatorname{dit}(\pi(\omega) \wedge \sigma(\omega))\} \\
\operatorname{Dit}(\sigma \Rightarrow \pi) &= \oplus \{U_z \otimes U_{z'} | (U_z, U_{z'}) \in \operatorname{dit}(\sigma(\omega) \Rightarrow \pi(\omega))\}.
\end{aligned}$$

The smallest ditspace is $\operatorname{Dit}(\mathbf{0}) = \{0\}$ and the largest ditspace is $\operatorname{Dit}(\mathbf{1}_\omega) = \oplus \{U_z \otimes U_{z'} | U_z \neq U_{z'}\}$, and the partial ordering is inclusion. Then the partition algebra of DSDs in $\Pi(\omega)$ is represented by the algebra of the ditspaces of $V \otimes V$ for DSDs in $\Pi(\omega)$.

Given the basic (category-theoretic) duality between subsets and partitions, this construction (using ditsets) to represent partition operations as subset operations–with the corresponding vector space version of the construction using ditspaces–has a dual construction to represent subset operations by partition operations. In the set case, instead of working with certain *subsets* (ditsets) of the *product* $U \times U$, the dual set construction works with certain *partitions* on the *coproduct* (disjoint union) $U \uplus U$. And for the vector space version, instead of working with subspaces (ditspaces) of the tensor product $V \otimes V$, the dual vector space construction works with DSDs on the coproduct or direct sum $V \oplus V^*$ (where V^* is a copy of V).

The set partition implication endows a rich structure on the partition algebra $\Pi(U)$ of set partitions on U (always $|U| \geq 2$). For $\pi \in \Pi(U)$, the *π-regular partitions* are the partitions of the form $\sigma \Rightarrow \pi$, which may be symbolized as $\overset{\pi}{\neg}\sigma$, for any $\sigma \in \Pi(U)$. They are all in the segment $[\pi, \mathbf{1}_U]$ and they form a Boolean algebra, the *Boolean core* $\mathcal{B}_\pi$ of $[\pi, \mathbf{1}_U]$, under the partition operations of join, meet, and π-negation, where the π-negation of $\sigma \Rightarrow \pi = \overset{\pi}{\neg}\sigma$ is $(\sigma \Rightarrow \pi) \Rightarrow \pi = \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$. The dual construction uses this Boolean algebra based on partition operations.

We start with the set version of the dual construction and then go over the vector space version in more detail. Given a subset $S \subseteq U$, the *subset corelation* $\Delta(S)$ is the partition on the disjoint union $U \uplus U^*$ (U^* being a copy of U) whose blocks are the pairs $\{u, u^*\}$ for $u \in S$ and singletons $\{u\}$ and $\{u^*\}$ if $u \notin S$. Thus $\Delta(S)$ just encodes in a partition on the disjoint union which $u \in U$ are in the subset $S \subseteq U$. The subset corelations are partitions on the coproduct $U \uplus U$ defined by subsets of U, and they are dual to the relations that are subsets of the product $U \times U$. At the two extremes, $\Delta(U)$ is the bottom partition $\mathbf{0}_{U \uplus U^*}$ on $U \uplus U^*$ consisting of all pairs $\{u, u^*\}$ for $u \in U$, and $\Delta(\emptyset) = \mathbf{1}_{U \uplus U^*}$. The key lemma (see below) is that $(\Delta(S) \Rightarrow \Delta(U)) = \Delta(S^c)$ (analogous to $\sigma \Rightarrow 0_U = \neg\sigma$) so the $\Delta(U)$-negated partitions on $U \uplus U^*$ are the same as the complementary subset corelations $\Delta(S^c)$. Then it can be seen (proof below) that the Boolean core $\mathcal{B}_{\Delta(U)}$ of $[\Delta(U), \mathbf{1}_{U \uplus U^*}]$ is a Boolean algebra using the partition operations of join, meet, and $\Delta(U)$-negation that is isomorphic to the powerset BA $\wp(U)$ under the correspondence $\Delta(S) \Rightarrow \Delta(U) \mapsto S$ for $S \in \wp(U)$:

$$\mathcal{B}_{\Delta(U)} \cong \wp(U).$$

In that manner, the Boolean subset operations on subsets of U are represented by partition operations on certain partitions on $U \uplus U^*$[15, p. 320].

For the vector space version of the dual construction, note that given a maximal DSD $\omega = \{U_z\}_{z \in Z}$, there is the associated powerset BA $\wp(\omega)$ or $\wp(Z)$ depending on whether we take ω or Z as playing the role of U. Choosing the Z option, for each $S \in \wp(Z)$, there is an associated subspace $A(S) = \oplus\{U_z | z \in S\}$ and an associated projection operator $P_S : V \to V$ to that subspace. Each atomic DSD $\{A, A'\}$ in $\Pi(\omega)$ has the form $\{A(S), A(S^c)\}$ (where $S^c = Z - S$ is the complement in Z) with $V = A(Z)$ and $\{0\} = A(\emptyset)$. Thus there is an induced BA structure on the subspaces $\{A(S) | S \in \wp(Z)\}$ and on the projection operators $\{P_S | S \in \wp(Z)\}$ isomorphic to $\wp(Z)$. But how can this BA of certain subspaces of V be represented using the DSD operations of the logic of vector space partitions?

Let $V \oplus V^*$ be the direct sum (coproduct) of V with a copy V^* of itself. Given a maximal element $\omega = \{U_z\}_{z \in Z}$ of V, then the union with the copy $\omega^* = \{U_z^*\}_{z \in Z}$ forms a maximal element $\omega \cup \omega^*$ in the refinement ordering of DSDs in $DSD(V \oplus V^*)$ so we can work in the partition logic $\Pi(\omega \cup \omega^*)$.

Definition 28 *For $S \in \wp(Z)$ with the corresponding subspace $A(S)$, let $\Delta(A(S))$ or just $\Delta(S)$ be the DSD in $\Pi(\omega \cup \omega^*)$, called a subspace corelation, consisting of all the one-dimensional subspaces U_z and U_z^* for $z \notin S$, i.e., $U_z \nsubseteq A(S)$, and $U_z \oplus U_z^*$ for $z \in S$, i.e., $U_z \subseteq A(S)$.*

This is clearly just the subspace version of the subset corelation. Then $\Delta(Z)$ is the bottom $\mathbf{0}_{\omega \cup \omega^*}$ DSD consisting of all the subspaces $U_z \oplus U_z^*$ for $z \in Z$ and $\Delta(\emptyset) = \mathbf{1}_{\omega \cup \omega^*}$.

Lemma 29 $\Delta(S) \Rightarrow \Delta(Z) = \Delta(S^c)$.

Proof: For any $z \in S$, we have $U_z \oplus U_z^*$ in both $\Delta(S)$ and $\Delta(Z)$, so $U_z \oplus U_z^*$ is discretized in $\Delta(S) \Rightarrow \Delta(Z)$ into U_z and U_z^* separately. For any $z \in S^c$, $U_z \oplus U_z^*$ is only in $\Delta(Z)$ so it remains whole in $\Delta(S) \Rightarrow \Delta(Z)$ so that implication DSD is $\Delta(S^c)$. $\square$

Thus the $\Delta(Z)$-negated DSDs $\Delta(S) \Rightarrow \Delta(Z)$ are the subspace corelations in $\Pi(\omega \cup \omega^*)$. The Boolean core $\mathcal{B}_{\Delta(Z)}$ of the segment $[\Delta(Z), \omega \cup \omega^*]$ is a BA with the DSD operations of join, meet, implication, and $\Delta(Z)$-negation in $\Pi(\omega \cup \omega^*)$.

Proposition 21 *$\mathcal{B}_{\Delta(Z)} \cong \wp(Z)$.*

Proof: The isomorphism associates $\Delta(S) \Rightarrow \Delta(Z) \in \mathcal{B}_{\Delta(Z)}$ with $S \in \wp(Z)$. For $S, T \in \wp(Z)$, the union $S \cup T$ is associated with the join $(\Delta(S) \Rightarrow \Delta(Z)) \vee (\Delta(T) \Rightarrow \Delta(Z)) = \Delta(S^c) \vee \Delta(T^c) = \Delta(S^c \cap T^c) = \Delta((S \cup T)^c) = \Delta(S \cup T) \Rightarrow \Delta(Z)$. The other Boolean operations of meet and $\Delta(Z)$-negation go in a similar manner. The null set $\emptyset \in \wp(Z)$ is associated with $\Delta(\emptyset) \Rightarrow \Delta(Z) = \Delta(\emptyset^c) = \Delta(Z)$ which is the bottom of the BA $\mathcal{B}_{\Delta(Z)}$, and $Z \in \wp(Z)$ is associated with $\Delta(Z) \Rightarrow \Delta(Z) = \Delta(Z^c) = \Delta(\emptyset) = \mathbf{1}_{\omega \cup \omega^*}$ which is the top of $\mathcal{B}_{\Delta(Z)}$. If $S \subseteq T$ in $\wp(Z)$, then $T^c \subseteq S^c$ so $\Delta(S) \Rightarrow \Delta(Z) = \Delta(S^c) \preceq \Delta(T^c) = \Delta(T) \Rightarrow \Delta(Z)$ in the refinement ordering of $\Pi(\omega \cup \omega^*)$. $\square$

The treatment of *DSD operations* on V as *subspace* operations on $V \otimes V$, and the dual treatment of *subspace operations* on V as *DSD operations* on $V \oplus V^*$ exhibit the dual relationship between the two logics of DSDs and subspaces.

4.8 DSDs, CSCOs, and CSCDs

For a self-adjoint operator F on a Hilbert space V (or diagonalizable operator on any V), the projections P_{λ_i} can be constructed from the DSD $\pi = \{V_{\lambda_i}\}_{i\in I}$ of eigenspaces for the eigenvalues $\{\lambda_i\}_{i\in I}$, and then the operator can be reconstructed–given the eigenvalues–from the decomposition $F = \sum_{i\in I} \lambda_i P_{\lambda_i}$. A set partition $\pi = \{B, B', ...\}$ on U can always be construed as the inverse-image partition of a numerical attribute $f : U \to \mathbb{R}$ by assigning different values to the different blocks in π. Thus the logic of set partitions is the logic of numerical attributes or random variables which is abstracted from the specific values and only reflects when the values were the same or different. In the same sense, the logic of DSDs of a vector space is the logic of diagonalizable operators on the space which is abstracted from the specific eigenvalues and only reflects when the eigenvalues were the same or different.

Given a state ψ and a self-adjoint operator $F : V \to V$ on a finite dimensional Hilbert space V, the operator determines the DSD $\pi = \{V_{\lambda_i}\}_{i\in I}$ of eigenspaces for the eigenvalues λ_i. The projective measurement operation uses the eigenspace DSD to decompose ψ into the unique parts given by the projections $P_{\lambda_i}(\psi)$ into the eigenspaces V_{λ_i}, where $P_{\lambda_i}(\psi)$ is the outcome of the projective measurement with probability $\Pr(\lambda_i|\psi) = \|P_{\lambda_i}(\psi)\|^2 / \|\psi\|^2$.

The eigenspace DSD $\pi = \{V_{\lambda_i}\}_{i\in I}$ of F is refined by one or more maximal DSDs, i.e., $\pi = \{V_{\lambda_i}\}_{i\in I} \preceq \omega = \{U_z\}_{z\in Z}$. For each such ω, there is a set partition $\pi(\omega) = \{B_{\lambda_i}\}_{i\in I}$ on ω such that $V_{\lambda_i} = \oplus B_{\lambda_i}$. If some of the V_{λ_i} have dimension larger than one ("degeneracy"), then more measurements by commuting operators will be necessary to further decompose down to single dimensional eigenspace. If two self-adjoint operators commute, then their eigenspace DSDs are compatible. Given another self-adjoint operator $G : V \to V$ commuting with F, its eigenspace DSD $\sigma = \{W_{\mu_j}\}_{j\in J}$ (for eigenvalues μ_j of G) is compatible with $\pi = \{V_{\lambda_i}\}_{i\in I}$ and thus has a join DSD $\pi \vee \sigma$ in $DSD(V)$ which is also in $\Pi(\omega)$ for one or more maximal ω each representing an orthonormal basis of simultaneous eigenvectors. The combined measurement by the two commuting operators is just the single measurement using the join DSD $\pi \vee \sigma$.

Dirac's notion of a Complete Set of Commuting Operators (CSCO) $\{F_{\pi_l}\}_{l=1}^m$ [**?**] translates into the language of the quantum logic of DSDs as a *Complete Set of Compatible DSDs (CSCD)* $\{\pi_l\}_{l=1}^m$ whose join $\vee_{l=1}^m \pi_l$ *is* a maximal DSD $\omega = \mathbf{1}_\omega$ in $DSD(V)$ and thus is *the* maximum DSD $\mathbf{1}_\omega$ in $\Pi(\omega)$. As noted above, the eigenvalues of the observables F_{π_l} can then be used to uniquely label the $U_z \in \mathbf{1}_\omega = \omega$.

In partition logic on sets, a *valid formula*, i.e., a *partition tautology*, is a logical formula (using the partition operations of join, meet, and implication) so that when any partitions on the universe set U are substituted for the variables, the result is the discrete partition $\mathbf{1}_U$ on that set. Restated for DSDs, a *DSD tautology* in the partition logic $\Pi(\omega)$ for any maximal ω in $DSD(V)$ for any V is any formula (in the language of join, meet, and implication) so that no matter which DSDs of $\Pi(\omega)$ are substituted for the variables, the result is $\mathbf{1}_\omega$. For instance, *modus ponens* $\sigma \wedge (\sigma \Rightarrow_\omega \pi) \Rightarrow_\omega \pi$ is a DSD tautology in the partition logic $\Pi(\omega)$, so for any DSDs $\pi, \sigma \in \Pi(\omega)$, π is sufficient for $\sigma \wedge (\sigma \Rightarrow_\omega \pi)$. In the Boolean core $\mathcal{B}_\pi$ of $[\pi, \omega]$, the ordinary Boolean tautologies, like the law of excluded middle,

$$(\sigma \Rightarrow_\omega \pi) \vee ((\sigma \Rightarrow_\omega \pi) \Rightarrow_\omega \pi) = \overset{\pi}{\neg}_\omega \sigma \vee \overset{\pi}{\neg}_\omega \overset{\pi}{\neg}_\omega \sigma,$$

hold for any $\pi, \sigma \in \Pi(\omega)$, so they are DSD tautologies in the whole partition logic $\Pi(\omega)$, where that formula is the *weak* law of excluded middle for π-negation. Thus for any DSDs $\pi, \sigma \in \Pi(\omega)$, the DSDs $\sigma \Rightarrow_\omega \pi$ and $(\sigma \Rightarrow_\omega \pi) \Rightarrow_\omega \pi$ form a CSCD since their join is the discrete DSD $\mathbf{1}_\omega$. The law of excluded middle in $\mathcal{B}_\pi$ generalizes to the DSD tautology that is the disjunctive normal form decomposition of $\mathbf{1}_\omega$ for any number of variables. For instance, for any π, σ, and τ in $\Pi(\omega)$, we have the DSD tautology:

$$\left(\overset{\pi}{\neg}_\omega \overset{\pi}{\neg}_\omega \sigma \wedge \overset{\pi}{\neg}_\omega \overset{\pi}{\neg}_\omega \tau\right) \vee \left(\overset{\pi}{\neg}_\omega \overset{\pi}{\neg}_\omega \sigma \wedge \overset{\pi}{\neg}_\omega \tau\right) \vee \left(\overset{\pi}{\neg}_\omega \sigma \wedge \overset{\pi}{\neg}_\omega \overset{\pi}{\neg}_\omega \tau\right) \vee \left(\overset{\pi}{\neg}_\omega \sigma \wedge \overset{\pi}{\neg}_\omega \tau\right)$$

so those four disjuncts form a CSCD. Assigning distinct real numbers to the subspaces of the disjunct DSDs defines commuting self-adjoint operators that form one of Dirac's CSCOs.

4.9 Some concluding thoughts

It is early days in the development of the logic of set partitions and the logic of vector space partitions or direct-sum decompositions. In spite of the category-theoretic duality with subsets, partitions are considerably more complicated and that is reflected in the primal and dual structures in the partition algebras $\Pi(U)$. Compared to Boolean algebras, partition algebras are rather understudied. In fact, there was over a century gap between the definition of the lattice operations of join and meet on partitions in the nineteenth century and the definition of the implication and other logical operations in the twenty-first century so that one could speak of "partition algebras" instead of just partition lattices. Moreover, the study of "partition lattices" ([4]; [28]) using the upside-down lattice of equivalence relations did not promote the comparisons with the Boolean algebras or the Heyting and co-Heyting algebras.

The logic of subspaces specialized to Hilbert spaces, i.e., the usual quantum logic initiated by Birkhoff and Von Neumann [5], has been extensively, if not exhaustively, developed. The logic of the dual concept of direct-sum decompositions was developed here using the sets-to-vector-spaces "Yoga of linearization".[11] The development was in general terms without focusing exclusively on Hilbert spaces. Much of the quantum flavor can be brought out in the general case and even in the special case of vector spaces over $\mathbb{Z}_2$ [17].

The Boolean logic of subsets is usually treated as the logic of propositions by associating with each subset the proposition that a generic element of the universe set is a member of the subset. Subset formulas that always evaluate to the universe set (i.e., subset tautologies) are then associated with the propositional formulas that always evaluate to True, i.e., truth-table tautologies. Boole [7] also developed the quantitative theory that measures subsets, i.e., probability theory. The quantum logic of subspaces is often characterized as the logic of the QM propositions that a state vector is in a subspace.

The logic of set partitions is the logic of numerical attributes or random variables that abstracts away from the numerical values and focuses on the inverse-image partition that retains the information as to whether the values were the same or different (i.e., the distinction between different blocks of the partition). If a proposition is to be associated with a partition, then it is the proposition that the partition distinguishes a pair of different elements from the universe set, i.e., that the pair of elements is a distinction of the partition (two elements in different blocks of the partition). If a partition formula always evaluates to the discrete partition then it is associated with the proposition that the formula always distinguishes any pair of different elements, i.e., the formula is a partition tautology. The quantitative theory that measures the extent to which a partition distinguishes is the information theory of logical entropy where the logical entropy of a partition is the probability that two independent draws from the universe set give a distinction of the partition–just as the probability of a subset is the probability that a single draw from the universe set gives an element of the subset.[12] Linearization yields the logic of direct-sum decompositions as the logic of diagonalizable linear operators (or self-adjoint operators or observables in the QM case) that abstracts away from the numerical values of the eigenvalues and that focuses on the DSD of the eigenspaces that still reflects whether the eigenvalues were the same or different (i.e., the distinction between the different eigenspaces). If a proposition is to be associated with an observable in QM, then it is the proposition that the observable distinguishes between its eigenvectors (by being in different blocks of the DSD of eigenspaces). The quantum logical entropy of an observable applied to a pure state is

[11] See also the use of q-analogues in the Appendix where enumeration formulas for vector spaces with q elements reduce to set formulas by taking $q-1$.

[12] Note the dual role of *elements* of a subsets and *distinctions* of a partition [22].

the probability that two independent projective measurements of the pure state by that observable will yield a distinction of the observable's DSD of eigenspaces, i.e., distinct eigenvalues.

Set concept	Linearization	Quantum version over Hilbert space
Boolean logic of subsets	Logic of subspaces	Orthomodular lattice of closed subspaces
Logic of partitions	Logic of DSDs	Quantum logic of DSDs

Table 4.2: Parallel development of subset and partition logics

The topic initially developed in this book is summarized in the bottom row of Table 4.2. The quantitative or information-theoretic versions of the logics of set and vector-space partitions have been developed elsewhere ([18], [20], [23]).

5 Appendix: Counting direct-sum decompositions of finite vector spaces

5.1 Reviewing q-analogs: Again from sets to vector spaces

The theory of q-analogs shows how many "classical" combinatorial formulas for finite sets can be extended to finite vector spaces where q is the cardinality of the finite base field $GF(q)$, i.e., $q = p^n$, a power of a prime.

The natural number n is replaced by:

$$[n]_q = \frac{q^n-1}{q-1} = 1 + q + q^2 + ... + q^{n-1}$$

so as $q \to 1$, then $[n]_q \to n$ in the passage from vector spaces to sets. The factorial $n!$ is replaced, in the q-analog

$$[n]_q! = [n]_q [n-1]_q ... [1]_q$$

where $[1]_q = [0]_q = 1$.

To obtain the Gaussian binomial coefficients we calculate with ordered bases of a k-dimensional subspace of an n-dimensional vector space over the finite field $GF(q)$ with q elements. There are q^n elements in the space so the first choice for a basis vector has $(q^n - 1)$ (excluding 0) possibilities, and since that vector generated a subspace of dimension q, the choice of the second basis vector is limited to $(q^n - q)$ elements, and so forth. Thus:

$$\begin{gathered}(q^n-1)(q^n-q)\left(q^n-q^2\right)...\left(q^n-q^{k-1}\right)\\ = (q^n-1)\,q^1\left(q^{n-1}-1\right)q^2\left(q^{n-2}-1\right)...q^{k-1}\left(q^{n-k+1}-1\right)\\ = \frac{[n]_q!(q-1)^k}{[n-k]_q!}q^{(1+2+...+(k-1))} = \frac{[n]_q!(q-1)^k}{[n-k]_q!}q^{k(k-1)/2} = \frac{[n]_q!(q-1)^k}{[n-k]_q!}q^{\binom{k}{2}}.\end{gathered}$$

Number of ordered bases for a k-dimensional subspace in an n-dimensional space.

But for a space of dimension k, the number of ordered bases are:

$$\begin{gathered}\left(q^k-1\right)\left(q^k-q\right)\left(q^k-q^2\right)...\left(q^k-q^{k-1}\right)\\ = \left(q^k-1\right)q^1\left(q^{k-1}-1\right)q^2\left(q^{k-2}-1\right)...q^{k-1}\left(q^{k-k+1}-1\right)\\ = [k]_q!\,(q-1)^k\,q^{k(k-1)/2} = [k]_q!\,(q-1)^k\,q^{\binom{k}{2}}\end{gathered}$$

Number of ordered bases for a k-dimensional space.

Thus the number of subspaces of dimension k is the ratio:

$$\binom{n}{k}_q = \frac{[n]_q!(q-1)^k q^{k(k-1)/2}}{[n-k]_q![k]_q!(q-1)^k q^{k(k-1)/2}} = \frac{[n]_q!}{[n-k]_q![k]_q!}$$
Gaussian binomial coefficient

where $\binom{n}{k}_q \to \binom{n}{k}$ as $q \to 1$, i.e., the number of k-dimensional subspaces $\to$ number of k-element subsets. Many classical identities for binomial coefficients generalize to Gaussian binomial coefficients [27].

5.2 The direct formulas for counting partitions of finite sets

Using sophisticated techniques, the direct-sum decompositions of a finite vector space over $GF(q)$ have been enumerated in the sense of giving the exponential generating function for the numbers ([3]; [48]). Our goal is to derive by elementary methods the formulas to enumerate these and some new but related direct-sum decompositions.

Two subspaces of a vector space are said to be *disjoint* if their intersection is the zero subspace 0. A *direct-sum decomposition* (DSD) of a finite-dimensional vector space V over a base field F is a set of (nonzero) pair-wise disjoint subspaces, called *blocks* (as with partitions), $\{V_i\}_{i=1,...,m}$ that span the space. Then each vector $v \in V$ has a unique expression $v = \sum_{i=1}^{m} v_i$ with each $v_i \in V_i$. Since a direct-sum decomposition can be seen as the vector-space version of a set partition, we begin with counting the number of partitions on a set.

Each set partition $\{B_1, ..., B_m\}$ of an n-element set has a "type" or "signature" number partition giving the cardinality of the blocks where they might be presented in nondecreasing order which we can assume to be: $(|B_1|, |B_2|, ..., |B_m|)$ which is a number partition of n. For our purposes, there is another way to present number partitions, the *part-count representation*, where a_k is the number of times the integer k occurs in the number partition (and $a_k = 0$ if k does not appear) so that:

$$a_1 1 + a_2 2 + ... + a_n n = \sum_{k=1}^{n} a_k k = n.$$
Part-count representation of number partitions keeping track of repetitions.

Each set partition $\{B_1, ..., B_m\}$ of an n-element set has a part-count signature $a_1, ..., a_n$, and then there is a "classical" formula for the number of partitions with that signature ([1, p. 215]; [32, p. 427]).

Proposition 22 *The number of set partitions for the given signature: $a_1, ..., a_n$ where $\sum_{k=1}^{n} a_k k = n$ is:*

$$\frac{n!}{a_1!a_2!...a_n!(1!)^{a_1}(2!)^{a_2}...(n!)^{a_n}}.$$
.

Proof: Suppose we count the number of set partitions $\{B_1, ..., B_m\}$ of an n-element set when the blocks have the given cardinalities: $n_j = |B_j|$ for $j = 1, ..., m$ so $\sum_{j=1}^{m} n_j = n$. The first block B_1 can be chosen in $\binom{n}{n_1}$ ways, the second block in $\binom{n-n_1}{n_2}$ ways and so forth, so the total number of ways is:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}...\binom{n-n_1-...-n_{m-1}}{n_m} = \frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-...-n_{m-1})!}{n_m!(n-n_1-...-n_m)!}$$
$$= \frac{n!}{n_1!...n_m!} = \binom{n}{n_1, ..., n_m}$$

the multinomial coefficient. This formula can then be restated in terms of the part-count signature $a_1, ..., a_n$ where $\sum_{k=1}^{n} a_k k = n$ as: $\frac{n!}{(1!)^{a_1}(2!)^{a_2}...(n!)^{a_n}}$. But that overcounts since the a_k blocks of size k can be permuted without changing the partition's signature so one needs to divide by $a_k!$ for $k = 1, ..., n$ which yields the formula for the number of partitions with that signature. □

The *Stirling numbers $S(n, m)$ of the second kind* are the number of partitions of an n-element set with m blocks. Since $\sum_{k=1}^{n} a_k = m$ is the number of blocks, the direct formula (as opposed to a recurrence formula) is:

$$S(n,m) = \sum_{\substack{1a_1+2a_2+...+na_n=n \\ a_1+a_2+...+a_n=m}} \frac{n!}{a_1!a_2!...a_n!(1!)^{a_1}(2!)^{a_2}...(n!)^{a_n}}$$

Direct formula for Stirling numbers of the second kind.

The *Bell numbers* $B(n)$ are the total number of partitions on an n-element set so the direct formula is:

$$B(n) = \sum_{m=1}^{n} S(n,m) = \sum_{1a_1+2a_2+...+na_n=n} \frac{n!}{a_1!a_2!...a_n!(1!)^{a_1}(2!)^{a_2}...(n!)^{a_n}}$$

Direct formula for total number of partitions of an n-element set.

5.3 The direct formulas for counting DSDs of finite vector spaces

Each DSD $\pi = \{V_i\}_{i=1,...,m}$ of a finite vector space of dimension n also determines a number partition of n using the dimensions $n_i = \dim(V_i)$ in place of the set cardinalities, and thus each DSD also has a signature $a_1, ..., a_n$ where the subspaces are ordered by nondecreasing dimension and where $\sum_{k=1}^{n} a_k k = n$ and $\sum_{k=1}^{n} a_k = m$.

Proposition 23 *The number of DSDs of a vector space V of dimension n over $GF(q)$ with the part-count signature $a_1, ..., a_n$ is:*

$$\frac{1}{a_1!a_2!...a_n!} \frac{[n]_q!}{\left([1]_q!\right)^{a_1}...\left([n]_q!\right)^{a_n}} q^{\frac{1}{2}\left(n^2 - \sum_k a_k k^2\right)}$$

Number of DSDs for the given signature $a_1, ..., a_n$ where $\sum_{k=1}^{n} a_k k = n$.

Proof: Reasoning first in terms of the dimensions n_i, we calculate the number of ordered bases in a subspace of dimension n_1 of a vector space of dimension n over the finite field $GF(q)$ with q elements. There are q^n elements in the space so the first choice for a basis vector is $(q^n - 1)$ (excluding 0), and since that vector generated a subspace of dimension q, the choice of the second basis vector is limited to $(q^n - q)$ elements, and so forth. Thus:

$$\begin{aligned} &(q^n - 1)(q^n - q)\left(q^n - q^2\right)...\left(q^n - q^{n_1 - 1}\right) \\ &= (q^n - 1) q^1 \left(q^{n-1} - 1\right) q^2 \left(q^{n-1} - 1\right) ... q^{n_1 - 1} \left(q^{n - n_1 + 1} - 1\right) \\ &= (q^n - 1)\left(q^{n-1} - 1\right) ... \left(q^{n - n_1 - 1} - 1\right) q^{(1+2+...+(n_1 - 1))} \end{aligned}$$

Number of ordered bases for an n_1-dimensional subspace of an n-dimensional space.

If we then divide by the number of ordered bases for an n_1-dimension space:

$$(q^{n_1} - 1)(q^{n_1} - q)...\left(q^{n_1} - q^{n_1 - 1}\right) = (q^{n_1} - 1)\left(q^{n_1 - 1} - 1\right)...(q - 1) q^{(1+2+...+(n_1 - 1))}$$

we could cancel the $q^{n_1(n_1 - 1)}$ terms to obtain the Gaussian binomial coefficient

$$\frac{(q^n - 1)\left(q^{n-1} - 1\right)...\left(q^{n - n_1 - 1} - 1\right) q^{(1+2+...+(n_1 - 1))}}{(q^{n_1} - 1)(q^{n_1 - 1} - 1)...(q - 1) q^{(1+2+...+(n_1 - 1))}} = \binom{n}{n_1}_q = \frac{[n]_q!}{[n - n_1]_q! [n_1]_q!}$$

Number of different n_1-dimensional subspaces of an n-dimensional space.

If instead we continue to develop the numerator by multiplying by the number of ordered bases for an n_2-dimensional space that could be chosen from the remaining space of dimension $n - n_1$ to obtain:

$$\begin{aligned} &(q^n - 1)(q^n - q)\left(q^n - q^2\right)...\left(q^n - q^{n_1 - 1}\right) \times (q^n - q^{n_1})\left(q^n - q^{n_1 + 1}\right)...\left(q^n - q^{n_1 + n_2 - 1}\right) \\ &= (q^n - 1)\left(q^{n-1} - 1\right)...\left(q^{n - n_1 - n_2 + 1} - 1\right) q^{(1+2+...+(n_1 + n_2 - 1))}. \end{aligned}$$

Then dividing by the number of ordered bases of an n_1-dimensional space times the number of ordered bases of an n_2-dimensional space gives the number of different "disjoint" (i.e., only overlap is zero subspace) subspaces of n_1-dimensional and n_2-dimensional subspaces.

$$= \frac{(q^n-1)(q^{n-1}-1)...(q^{n-n_1-n_2+1}-1)q^{(1+2+...+(n_1+n_2-1))}}{(q^{n_1}-1)(q^{n_1-1}-1)...(q-1)q^{(1+2+...+(n_1-1))}\times(q^{n_2}-1)(q^{n_2-1}-1)...(q-1)q^{1+2+...+(n_2-1)}}.$$

Continuing in this fashion we arrive at the number of disjoint subspaces of dimensions $n_1, n_2, ..., n_m$ where $\sum_{i=1}^m n_i = n$:

$$\frac{(q^n-1)(q^{n-1}-1)...(q-1)q^{(1+2+...+(n-1))}}{\prod_{i=1,...,m}(q^{n_i}-1)(q^{n_i-1}-1)...(q-1)q^{(1+2+...+(n_i-1))}} = \frac{[n]_q!q^{n(n-1)/2}}{[n_1]_q!q^{n_1(n_1-1)/2}\times...\times[n_m]_q!q^{n_m(n_m-1)/2}}$$
$$= \frac{[n]_q!}{[n_1]_q!...[n_m]_q!}q^{\frac{1}{2}\left[n(n-1)-\sum_{i=1}^m n_i(n_i-1)\right]}.$$

There may be a number a_k of subspaces with the same dimension, e.g., if $n_j = n_{j+1} = k$, then $a_k = 2$ so the term $[n_j]_q!q^{n_j(n_j-1)/2}\times[n_{j+1}]_q!q^{n_{j+1}(n_{j+1}-1)/2}$ in the denominator could be replaced by $\left([k]_q!\right)^{a_k} q^{a_k k(k-1)/2}$. Hence the previous result could be rewritten in the part-count representation:

$$\frac{[n]_q!}{\left([1]_q!\right)^{a_1}...\left([n]_q!\right)^{a_n}}q^{\frac{1}{2}\left[n(n-1)-\sum_k a_k k(k-1)\right]}.$$

And permuting subspaces of the same dimension k yields a DSD with the same signature, so we need to divide by $a_k!$ to obtain the formula:

$$\frac{[n]_q!}{a_1!...a_n!\left([1]_q!\right)^{a_1}...\left([n]_q!\right)^{a_n}}q^{\frac{1}{2}\left[n(n-1)-\sum_k a_k k(k-1)\right]}.$$

The exponent on the q term can be simplified since $\sum_k a_k k = n$:

$$\begin{aligned}\tfrac{1}{2}\left[n(n-1) - \left(\textstyle\sum_k a_k k(k-1)\right)\right] &= \tfrac{1}{2}\left[n^2 - n - \left(\textstyle\sum_k a_k k^2 - \sum_k a_k k\right)\right] \\ &= \tfrac{1}{2}\left[n^2 - n - \left(\textstyle\sum_k a_k k^2 - n\right)\right] = \tfrac{1}{2}\left(n^2 - \textstyle\sum_k a_k k^2\right).\end{aligned}$$

This yields the final formula for the number of DSDs with the part-count signature $a_1, ..., a_n$:

$$\frac{[n]_q!}{a_1!...a_n!\left([1]_q!\right)^{a_1}...\left([n]_q!\right)^{a_n}}q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}. \ \square$$

Note that the formula is not obtained by a simple substitution of $[k]_q!$ for $k!$ in the set partition formula due to the extra term $q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}$, but that it still reduces to the classical formula for set partitions with that signature as $q \to 1$. This formula leads directly to the vector space version of the Stirling numbers of the second kind to count the DSDs with m parts and to the vector space version of the Bell numbers to count the total number of DSDs.

Before giving those formulas, it should be noted that there is another q-analog formula called "generalized Stirling numbers" (of the second kind)–but it generalizes only one of the recurrence formulas for $S(n,m)$. It does not generalize the *interpretation* "number of set partitions on an n-element set with m parts" to count the vector space partitions (DSDs) of finite vector spaces of dimension n with m parts. The Stirling numbers satisfy the recurrence formula:

$$S(n+1,m) = mS(n,m) + S(n-1,m) \text{ with } S(0,m) = \delta_{0m}.$$

Donald Knuth uses the braces notation for the Stirling numbers, $\left\{ {n \atop m} \right\} = S(n,m)$, and then he defines the "generalized Stirling number" [32, p. 436] $\left\{ {n \atop m} \right\}_q$ by the q-analog recurrence relation:

$$\left\{ {n+1 \atop m} \right\}_q = \left(1+q+...+q^{m-1}\right)\left\{ {n \atop m} \right\}_q + \left\{ {n \atop m-1} \right\}_q; \ \left\{ {0 \atop m} \right\}_q = \delta_{0m}.$$

It is easy to generalize the direct formula for the Stirling numbers and it generalizes the set partition interpretation to vector space partitions:

$$D_q(n,m) = \sum_{\substack{1a_1+2a_2+\ldots+na_n=n \\ a_1+a_2+\ldots+a_n=m}} \frac{[n]_q!}{a_1!\ldots a_n!\left([1]_q!\right)^{a_1}\ldots\left([n]_q!\right)^{a_n}} q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}$$

Number of DSDs of a finite vector space of dimension n over $GF(q)$ with m parts.

The number $D_q(n,m)$ is S_{nm} in [48]. Taking $q \to 1$ yields the Stirling numbers of the second kind, i.e., $D_1(n,m) = S(n,m)$. Knuth's generalized Stirling numbers $\left\{ {n \atop m} \right\}_q$ and $D_q(n,m)$ start off the same, e.g., $\left\{ {0 \atop 0} \right\}_q = 1 = D_q(0,0)$ and $\left\{ {1 \atop 1} \right\}_q = 1 = D_q(1,1)$, but then quickly diverge. For instance, all $\left\{ {n \atop n} \right\}_q = 1$ for all n, whereas the special case of $D_q(n,n)$ is the number of DSDs of 1-dimensional subspaces in a finite vector space of dimension n over $GF(q)$ (see table below for $q = 2$). The formula $D_q(n,n)$ is $M(n)$ in [49, Example 5.5.2(b), pp. 45-6] or [48, Example 2.2, p. 75.].

The number $D_q(n,n)$ of DSDs of 1-dimensional subspaces is closely related to the number of basis sets. The old formula for that number of bases is [37, p. 71]:

$$\begin{aligned} &\tfrac{1}{n!}\left(q^n-1\right)\left(q^n-q\right)\ldots\left(q^n-q^{n-1}\right) \\ &= \tfrac{1}{n!}\left(q^n-1\right)\left(q^{n-1}-1\right)\ldots(q^1-1)q^{(1+2+\ldots+(n-1))} \\ &= \tfrac{1}{n!}[n]_q! q^{\binom{n}{2}}(q-1)^n \end{aligned}$$

since $[k]_q = \frac{q^k-1}{q-1} = 1+q+q^2+\ldots+q^{k-1}$ for $k = 1, \ldots, n$.

In the formula for $D_q(n,n)$, there is only one signature $a_1 = n$ and $a_k = 0$ for $k = 2, \ldots, n$ which immediately gives the formula for the number of DSDs with n 1-dimensional blocks and each 1-dimensional block has $q-1$ choices for a basis vector so the total number of sets of basis vectors is given by the same formula:

$$D_q(n,n)(q-1)^n = \frac{[n]_q!}{a_1!} q^{\frac{1}{2}\left(n^2-a_1 1^2\right)}(q-1)^n = \frac{1}{n!}[n]_q! q^{\binom{n}{2}}(q-1)^n.$$

Note that for $q = 2$, $(q-1)^n = 1$ so $D_2(n,n)$ is the number of different basis sets.

Summing the $D_q(n,m)$ for all m gives the vector space version of the Bell numbers $B(n)$:

$$D_q(n) = \sum_{m=1}^{n} D_q(n,m) = \sum_{1a_1+2a_2+\ldots+na_n=n} \frac{1}{a_1!a_2!\ldots a_n!} \frac{[n]_q!}{\left([1]_q!\right)^{a_1}\ldots\left([n]_q!\right)^{a_n}} q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}$$

Number of DSDs of a vector space of dimension n over $GF(q)$.

Our notation $D_q(n)$ is $D_n(q)$ in Bender and Goldman [3] and $|Q_n|$ in Stanley ([48], [49]). Setting $q = 1$ gives the Bell numbers, i.e., $D_1(n) = B(n)$.

5.4 Counting DSDs with a block containing a designated vector v^*

Set partitions have a property not shared by vector space partitions, i.e., DSDs. Given a designated element u^* of the universe set U, the element is contained in some block of every partition on U. But given a nonzero vector v^* in a space V, it is not necessarily contained in a block of any given DSD of V. Some proofs of formulas use this property of set partitions so the proofs do not generalize to DSDs.

Consider one of the formulas for the Stirling numbers of the second kind:

$$S(n,m) = \sum_{k=0}^{n-1} \binom{n-1}{k} S(k, m-1)$$

Summation formula for $S(n,m)$.

The proof using the designated-element u^* reasoning starts with the fact that any partition of U with $|U| = n$ with m blocks will have one block containing u^* so we then only need to count the number of $m-1$ blocks on the subset disjoint from the block containing u^*. If the block containing u^* had $n-k$ elements, there are $\binom{n-1}{k}$ blocks that could be complementary to an $(n-k)$-element block containing u^* and each of those k-element blocks had $S(k, m-1)$ partitions on it with $m-1$ blocks. Hence the total number of partitions on an n-element set with m blocks is that sum.

This reasoning can be extended to DSDs over finite vector spaces, but it only counts the number of DSDs with a block containing a designated nonzero vector v^* (it doesn't matter which one), not all DSDs. Furthermore, it is not a simple matter of substituting $\binom{n-1}{k}_q$ for $\binom{n-1}{k}$. Each $(n-k)$-element subset has a unique k-element subset disjoint from it (its complement), but the same does not hold in general vector spaces. Thus given a subspace with $(n-k)$-dimensions, we must compute the number of k-dimensional subspaces disjoint from it.

Let V be an n-dimensional vector space over $GF(q)$ and let v^* be a specific nonzero vector in V. In a DSD with an $(n-k)$-dimensional block containing v^*, how many k-dimensional subspaces are there disjoint from the $(n-k)$-dimensional subspace containing v^*? The number of ordered basis sets for a k-dimensional subspace disjoint from the given $(n-k)$-dimensional space is:

$$\begin{aligned}\left(q^n - q^{n-k}\right)\left(q^n - q^{n-k+1}\right)\ldots\left(q^n - q^{n-1}\right) &= \left(q^k - 1\right)q^{n-k}\left(q^{k-1} - 1\right)q^{n-k+1}\ldots(q-1)\,q^{n-1} \\ &= \left(q^k - 1\right)\left(q^{k-1} - 1\right)\ldots(q-1)\,q^{(n-k)+(n-k+1)+\ldots+(n-1)} \\ &= \left(q^k - 1\right)\left(q^{k-1} - 1\right)\ldots(q-1)\,q^{k(n-k)+\frac{1}{2}k(k-1)}\end{aligned}$$

since we use the usual trick to evaluate twice the exponent:

$$\begin{aligned}&(n-k) + (n-k+1) + \ldots + (n-1) \\ &\underline{+(n-1) + (n-2) + \ldots + (n-k)} \\ &= (2n-k-1) + \ldots + (2n-k-1) \\ &= k(2n-k-1) = 2k(k + (n-k)) - k^2 - k = 2k(n-k) + k^2 - k.\end{aligned}$$

Now the number of ordered basis set of a k-dimensional space is:

$$\left(q^k - 1\right)\left(q^{k-1} - 1\right)\ldots(q-1)\,q^{\frac{1}{2}k(k-1)}$$

so dividing by that gives:

$$q^{k(n-k)}$$

The number of k-dimensional subspaces disjoint from any $(n-k)$-dimensional subspace.[13]

Note that taking $q \to 1$ yields the fact that an $(n-k)$-element subset of an n-element set has a unique k-element subset disjoint from it.

Hence in the q-analog formula, the binomial coefficient $\binom{n-1}{k}$ is replaced by the Gaussian binomial coefficient $\binom{n-1}{k}_q$ times $q^{k(n-k)}$. Then the rest of the proof proceeds as usual. Let $D_q^*(n,m)$ denote the number of DSDs of V with m blocks with one block containing any designated v^*. Then we can mimic the proof of the formula $S(n,m) = \sum_{k=0}^{n-1}\binom{n-1}{k}S(k, m-1)$ to derive the following:

Proposition 24 *Given a designated nonzero vector $v^* \in V$, the number of DSDs of V with m blocks one of which contains v^* is:*

$$D_q^*(n,m) = \sum_{k=0}^{n-1}\binom{n-1}{k}_q q^{k(n-k)} D_q(k, m-1). \ \square$$

Note that it is $D_q(k, m-1)$, and not $D_q^*(k, m-1)$, on the right-hand side of the formula. Further note that $D_q^*(n,m)$ is the q-analog of Stirling numbers of second kind $S(n,m)$ in the sense that taking $q = 1$ gives the right-hand side of: $\sum_{k=0}^{n-1}\binom{n-1}{k}S(k, m-1)$ since $D_1(k, m-1) = S(k, m-1)$, and the left-hand side is the same as $S(n,m)$ since *every* set partition of an n-element with m blocks has to have a block containing some designated element u^*. Thus *both* $D_q(n,m)$ and $D_q^*(n,m)$ can be seen as q-analogs of the Stirling numbers $S(n,m)$.

Since the Bell numbers can be obtained from the Stirling numbers of the second time as: $B(n) = \sum_{m=1}^{n} S(n,m)$, there is clearly a similar formula for the Bell numbers:

[13] This was proven using Möbius inversion on the lattice of subspaces by Crapo [11].

$$B(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B(k)$$
Summation formula for $B(n)$.

This formula can also be directly proven using the designated element u^* reasoning, so it can be similarly be extended to computing $D_q^*(n)$, the number of DSDs of V with a block containing a designated nonzero vector v^*.

Proposition 25 *Given any designated nonzero vector $v^* \in V$, the number of DSDs of V with a block containing v^* is:*

$$D_q^*(n) = \sum_{k=0}^{n-1} \binom{n-1}{k}_q q^{k(n-k)} D_q(k). \ \square$$

Note that $D_q^*(n)$ is the q-analog of the Bell numbers $B(n)$ in the sense that taking $q = 1$ yields the classical summation formula for $B(n)$ since $D_1(k) = B(k)$ and every partition has to have a block containing a designated element u^*. Thus *both* $D_q(n)$ and $D_q^*(n)$ can be seen as q-analogs of the Bell numbers $B(n)$.

Furthermore the D^* numbers have the expected relation:

Corollary 15 $D_q^*(n) = \sum_{m=1}^{n} D_q^*(n,m)$. $\square$

Since we also have formulas for the total number of DSDs, we have the number of DSDs where the designated element is not in one of the blocks: $D_q(n,m) - D_q^*(n,m)$ and $D_q(n) - D_q^*(n)$.

5.5 Computing initial values for $q = 2$

For a natural interpretation for the D^*-numbers, consider the pedagogical model of quantum mechanics using finite-dimensional vector spaces V over $GF(2) = \mathbb{Z}_2$, called "quantum mechanics over sets," QM/Sets [17]. The "observables" or attributes are defined by real-valued functions on basis sets. Given a basis set $U = \{u_1, ..., u_n\}$ for $V = \mathbb{Z}_2^n \cong \wp(U)$, a *real-valued attribute* is a function $f : U \to \mathbb{R}$. It determines a set partition $\left\{f^{-1}(r)\right\}_{r \in f(U)}$ on U and a DSD $\left\{\wp\left(f^{-1}(r)\right)\right\}_{r \in f(U)}$ on $\wp(U)$. In full QM, the important thing about an "observable" is not the specific numerical eigenvalues, but its eigenspaces for distinct eigenvalues, and that information is in the DSD of its eigenspaces. The attribute $f : U \to \mathbb{R}$ cannot be internalized as an operator on $\wp(U) \cong \mathbb{Z}_2^n$ (unless its values are $0, 1$), but it nevertheless determines the DSD $\left\{\wp\left(f^{-1}(r)\right)\right\}_{r \in f(U)}$ which is sufficient to pedagogically model many quantum results. Hence a DSD can be thought of an "abstract attribute" (without the eigenvalues) with its blocks serving as "eigenspaces." Then a natural question to ask is given any nonzero vector $v^* \in V = \mathbb{Z}_2^n$, how many "abstract attributes" are there where v^* is an "eigenvector"–and the answer is $D_2^*(n)$. And $D_2^*(n,m)$ is the number of "abstract attributes" with m distinct "eigenvalues" where v^* is an "eigenvector."

In the case of $n = 1, 2, 3$, the DSDs can be enumerated "by hand" to check the formulas, and then the formulas can be used to compute higher values of $D_2(n,m)$ or $D_2(n)$.

All vectors in the n-dimensional vector space $\mathbb{Z}_2^n$ over $GF(2) = \mathbb{Z}_2$ will be expressed in terms of a computational basis $\{a\}, \{b\}, ..., \{c\}$ so any vector S in $\mathbb{Z}_2^n$ would be represented in that basis as a subset $S \subseteq U = \{a, b, ..., c\}$. The addition of subsets (expressed in the same basis) is the symmetric difference: for $S, T \subseteq U$,

$$S + T = (S - T) \cup (T - S) = (S \cup T) - (S \cap T).$$

Since all subspaces contain the zero element which is the empty set $\emptyset$, it will be usually suppressed when listing the elements of a subspace. And subsets like $\{a\}$ or $\{a, b\}$ will be denoted as just a and ab. Thus the subspace $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ is denoted as $\{a, b, ab\}$. A k-dimensional subspace has 2^k elements so only $2^k - 1$ are listed.

For $n=1$, there is only one nonzero subspace $\{a\}$, i.e., $\{\emptyset,\{a\}\}$, and $D_2(1,1)=D_2(1)=1$.

For $n=2$, the whole subspace is $\{a,b,ab\}$ and it has three bases $\{a,b\}$, $\{a,ab\}$, and $\{b,ab\}$. The formula for the number of bases gives $D_2(2,2)=3$. The only $D_2(2,1)=1$ DSD is the whole space.

For $n=3$, the whole space $\{a,b,c,ab,ac,bc,abc\}$ is the only $D_2(3,1)=1$ and indeed for any n and q, $D_q(n,1)=1$. For $n=3$ and $m=3$, $D_2(3,3)$ is the number of (unordered) bases of $\mathbb{Z}_2^3$ (recall $\left\{ {n \atop n} \right\}_q = 1$ for all q). Since we know the signature, i.e., $a_1=3$ and otherwise $a_k=0$, we can easily compute $D_2(3,3)$:

$$\begin{aligned}&\tfrac{1}{a_1!a_2!\ldots a_n!}\tfrac{[n]_q!}{([1]_q!)^{a_1}\ldots([n]_q!)^{a_n}}q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}\\ &=\tfrac{1}{3!}\tfrac{[3]_2!}{([1]_2)^3}2^{\frac{1}{2}\left(3^2-3\right)}=\tfrac{1}{6}\tfrac{7\times 3}{1}2^{\frac{1}{2}(6)}=28=D_2(3,3).\end{aligned}$$

And here they are in Table A.1.

$\{a,b,c\}$	$\{a,b,ac\}$	$\{a,b,bc\}$	$\{a,b,abc\}$
$\{a,c,ab\}$	$\{a,c,bc\}$	$\{a,c,abc\}$	$\{a,ab,ac\}$
$\{a,ab,bc\}$	$\{a,ab,abc\}$	$\{a,ac,bc\}$	$\{a,ac,abc\}$
$\{b,c,ab\}$	$\{b,c,ac\}$	$\{b,c,abc\}$	$\{b,ab,ac\}$
$\{b,ab,bc\}$	$\{b,ab,abc\}$	$\{b,ac,bc\}$	$\{b,bc,abc\}$
$\{c,ab,ac\}$	$\{c,ab,bc\}$	$\{c,ac,bc\}$	$\{c,ac,abc\}$
$\{ab,ac,abc\}$	$\{ab,bc,abc\}$	$\{ac,bc,abc\}$	$\{bc,ab,abc\}$

Table A.1: All bases of $\mathbb{Z}_2^3$.

For $n=3$ and $m=2$, $D_2(3,2)$ is the number of binary DSDs, each of which has the signature $a_1=a_2=1$ so the total number of binary DSDs is:

$$D_2(3,2)=\tfrac{1}{1!1!}\tfrac{[3]_2!}{([1]!)^1([2]!)^1}2^{\frac{1}{2}\left(3^2-1-2^2\right)}=\tfrac{7\times 3}{3}2^{\frac{1}{2}(4)}=7\times 4=28.$$

And here they are in Table A.2:

$\{\{a\},\{b,c,bc\}\}$	$\{\{a\},\{ab,ac,bc\}\}$	$\{\{a\},\{c,ab,abc\}\}$	$\{\{a\},\{b,ac,abc\}\}$
$\{\{b\},\{a,c,ac\}\}$	$\{\{b\},\{ab,ac,bc\}\}$	$\{\{b\},\{c,ab,abc\}\}$	$\{\{b\},\{a,bc,abc\}\}$
$\{\{ab\},\{b,c,bc\}\}$	$\{\{ab\},\{a,bc,abc\}\}$	$\{\{ab\},\{b,ac,abc\}\}$	$\{\{ab\},\{a,c,ac\}\}$
$\{\{c\},\{a,b,ab\}\}$	$\{\{c\},\{ab,ac,bc\}\}$	$\{\{c\},\{a,bc,abc\}\}$	$\{\{c\},\{b,ac,abc\}\}$
$\{\{ac\},\{a,b,ab\}\}$	$\{\{ac\},\{a,bc,abc\}\}$	$\{\{ac\},\{c,ab,abc\}\}$	$\{\{ac\},\{b,c,bc\}\}$
$\{\{bc\},\{a,b,ab\}\}$	$\{\{bc\},\{b,ac,abc\}\}$	$\{\{bc\},\{c,ab,abc\}\}$	$\{\{bc\},\{a,c,ac\}\}$
$\{\{abc\},\{a,b,ab\}\}$	$\{\{abc\},\{b,c,bc\}\}$	$\{\{abc\},\{a,c,ac\}\}$	$\{\{abc\},\{ab,ac,bc\}\}$

Table A.2: All binary DSDs for $\mathbb{Z}_2^3$.

The above table has been arranged to illustrate the result that any given k-dimensional subspaces has $q^{k(n-k)}$ subspaces disjoint from it. For $n=3$ and $k=1$, each row gives the $2^2=4$ subspaces disjoint from any given 1-dimensional subspace represented by $\{a\}$, $\{b\}$,..., $\{abc\}$. For instance, the four subspaces disjoint from the subspace $\{ab\}$ (shorthand for $\{\emptyset,\{a,b\}\}$) are given in the third row since those are the "complementary" subspaces that together with $\{ab\}$ form a DSD.

For $q=2$, the initial values up to $n=6$ of $D_2(n,m)$are given the following Table A.3.

$n\backslash m$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	3				
3	0	1	28	28			
4	0	1	400	1,680	840		
5	0	1	10,416	168,640	277,760	83,328	
6	0	1	525,792	36,053,248	159,989,760	139,991,040	27,998,208

Table A.3: $D_2(n,m)$ with $n,m = 1,2,...,6$.

The seventh row $D_2(7,m)$ for $m = 0,1,...,7$ is: 0, 1, 51116992, 17811244032, 209056841728, 419919790080, 227569434624, and 32509919232 which sum to $D_2(7)$.

The row sums give the values of $D_2(n)$ for $n = 0,1,2,...,7$ in Table A.4.

n	$D_2(n)$
0	1
1	1
2	4
3	57
4	$2,921$
5	$540,145$
6	$364,558,049$
7	$906,918,346,689$

Table A.4: $D_2(n)$ for $n = 0,1,...,7$.

We can also compute the D^* examples of DSDs with a block containing a designated element. For $q = 2$, the $D_2^*(n,m)$ numbers for $n,m = 0,1,...,7$ are given in the following Table A.5.

$n\backslash m$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	2					
3	0	1	16	12				
4	0	1	176	560	224			
5	0	1	3456	40000	53760	13440		
6	0	1	128000	5848832	20951040	15554560	2666496	
7	0	1	9115648	1934195712	17826414592	30398054400	14335082496	1791885312

Table A.5: Number of DSDs $D_2^*(n,m)$ containing any given nonzero vector v^*

For $n = 3$ and $m = 2$, the table says there are $D_2^*(3,2) = 16$ DSDs with 2 blocks one of which contains a given nonzero vector, say $v^* = ab$ which represents $\{a,b\}$, and here they are in Table A.6.

$\{\{ab\},\{b,c,bc\}\}$	$\{\{ab\},\{a,bc,abc\}\}$	$\{\{ab\},\{b,ac,abc\}\}$	$\{\{ab\},\{a,c,ac\}\}$
$\{\{c\},\{a,b,ab\}\}$	$\{\{c\},\{ab,ac,bc\}\}$	$\{\{ac\},\{c,ab,abc\}\}$	$\{\{bc\},\{c,ab,abc\}\}$
$\{\{ac\},\{a,b,ab\}\}$	$\{\{a\},\{ab,ac,bc\}\}$	$\{\{a\},\{c,ab,abc\}\}$	$\{\{abc\},\{a,b,ab\}\}$
$\{\{bc\},\{a,b,ab\}\}$	$\{\{b\},\{ab,ac,bc\}\}$	$\{\{b\},\{c,ab,abc\}\}$	$\{\{abc\},\{ab,ac,bc\}\}$

Table A.6: Two-block DSDs of $\wp(\{a,b,c\})$ with a block containing $ab = \{a,b\}$.

The table also says there are $D_2^*(3,3) = 12$ basis sets containing *any* given nonzero vector which we could take to be $v^* = abc = \{a,b,c\}$, and here they are in Table A.7.

$\{a,b,abc\}$	$\{b,ab,abc\}$	$\{a,c,abc\}$	$\{b,bc,abc\}$
$\{a,ab,abc\}$	$\{a,ac,abc\}$	$\{b,c,abc\}$	$\{c,ac,abc\}$
$\{ab,ac,abc\}$	$\{ab,bc,abc\}$	$\{ac,bc,abc\}$	$\{bc,ab,abc\}$

Table A.7: Three-block DSDs (basis sets) of $\wp(\{a,b,c\})$ with a basis element $abc = \{a,b,c\}$.

Summing the rows in the $D_2^*(n,m)$ Table A.5 gives the values for $D_2^*(n)$ for $n = 0,1,...,7$ in Table A.8.

n	$D_2^*(n)$
0	1
1	1
2	3
3	29
4	961
5	$110,657$
6	$45,148,929$
7	$66,294,748,161$

Table A.8: $D_2^*(n)$ for $n = 0, 1, ..., 7$.

The integer sequence $D_2(n, n)$ for $n = 0, 1, 2, ...$ is known as: A053601"Number of bases of an n-dimensional vector space over $GF(2)$" in the *On-Line Encyclopedia of Integer Sequences* (https://oeis.org/). The sequences defined and tabulated here for $q = 2$ have been added to the *Encyclopedia* as: A270880 $[D_2(n, m)]$, A270881 $[D_2(n)]$, A270882 $[D_2^*(n, m)]$, A270883 $[D_2^*(n)]$.

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