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Original Paper

Partitions, Objective Indefiniteness, and Quantum Reality: Towards the Objective Indefiniteness Interpretation of quantum mechanics

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Abstract: The purpose of this paper is to show that the mathematics of quantum mechanics (QM) is the vector (Hilbert) space version of the mathematics of partitions at the set level. Since partitions are the math tool to describe indefiniteness and definiteness, this shows how the reality so well described by QM is a non-classical reality featuring the objective indefiniteness of superposition states. The lattice of partitions gives a skeletal model of quantum reality with the partition versions of pure states, non-classical mixed states (i.e., including a superposition state), and the completely distinguished classical state that satisfies the partition logic version of the Principle of Identity of Indistinguishables. Both the key notions of quantum states and quantum observables are respectively the (density) matrix versions of partitions and vector space version of a numerical attributes. The operation of projective measurement given by the Lüders mixture operation is the vector space version of the partition join operation between the partition prefiguring the density matrix of the state being measured and the partition prefiguring the observable being measured. All this adds specific key concepts and structure to Abner Shimony's idea of a literal understanding of QM to form what might be called the "Objective Indefiniteness Interpretation" of QM.

Keywords: Set partitions; objective indefiniteness; superposition states; lattice of partitions; partition logic

1. Introduction: the partition renaissance

1.1. The development of the logic of partitions

The same mathematical notion can be formulated as a partition, an equivalence relation, or a quotient set. That notion, is any formulation, has been woefully underdeveloped. For instance, the lattice operations of join and meet were known in the nineteenth century (Richard Dedekind, Ernst Schröder), and yet it was observed at the start of the twenty-first century that "the only operations on the family of equivalence relations fully studied, understood and deployed are the binary join $\vee$ and meet $\wedge$ operations." [8, p. 445] The Boolean lattice of subsets and the Boolean algebra of subsets (resulting from adding the implication operation on subsets) were intensely developed in the twentieth century, but the implication or any other logical operations on partitions were not defined throughout that century. Yet subsets and quotient sets are dual mathematical concepts. In category theory, where a subset or generally a subobject may be called a "part", the "dual notion (obtained by reversing the arrows) of 'part' is the notion of partition." [32, p. 85] Then the usual Boolean logic of subsets abstractly viewed is "Logic as the Algebra of Parts" [32, p. 193]–but there is no hint of a dual logic of partitions. Only in the twenty-first century was the implication operation defined for partitions along with several algorithms to define all the Boolean or logical operations on partitions–which resulted in the development of the logic of partitions [13]. That started a chain reaction that was even anticipated and might be termed the "partition renaissance."

> Equivalence relations are so ubiquitous in everyday life that we often forget about their proactive existence. Much is still unknown about equivalence relations. Were this situation remedied, the theory of equivalence relations could initiate a chain reaction generating new insights and discoveries in many fields dependent upon it. [8, p. 445]

1.2. Logical information theory as the quantitative version of partition logic

The second wave in this chain reaction was to develop the quantitative version of the logic of partitions. This is what Boole did by presenting probability theory as the quantitative version of subset logic [7]. In view of the duality between subsets and partitions, Gian-Carlo Rota anticipated the nature of the quantitative version of partitions in his writings and MIT lectures; the "lattice of partitions plays for information the role that the Boolean algebra of subsets plays for size or probability" [31, p. 30] or symbolically:

$$\frac{\text{information}}{\text{partitions}} \approx \frac{\text{probability}}{\text{subsets.}}.$$

Since "Probability is a measure on the Boolean algebra of events" that gives quantitatively the "intuitive idea of the size of a set", we may ask by "analogy" for some measure to capture a property for a partition like "what size is to a set." Rota goes on to ask:

> How shall we be led to such a property? We have already an inkling of what it should be: it should be a measure of information provided by a random variable. Is there a candidate for the measure of the amount of information? [42, p. 67]

It turns out that the "analogy" is between the number of elements in a subset ("what size is to a set") and the number of distinctions in a partition–where a "distinction" is a pair of elements in different blocks of the partitions, so they are distinguished by the partition. Just as the normalized size of a subset gives the Boole-Laplace notion of probability of the subset, so the normalized number of distinctions of a partition defines its information content or *logical entropy*. That gives the definition of information as distinctions and provides the most fundamental notion of entropy in information theory ([16]; [36]). Logical entropy is a (probability) measure on a set, so all the compound notions of joint, conditional, and mutual logical entropy follow with the usual Venn diagram relationships. The Shannon entropy (which is not a measure on a set) is a more specialized notion designed for coding theory. There is a non-linear monotonic transform, the dit-bit transform, that transforms all the compound notions of logical entropy into the corresponding formulas for Shannon entropy.

1.3. The math of quantum mechanics as the vector (Hilbert) space version of the math of partitions

The third wave in the partition renaissance is the development of the partition interpretation of quantum mechanics. For almost a century (from the mid-1920s to today), controversy has raged about the nature of the reality described by the mathematical formalism of quantum mechanics, i.e., standard von Neumann/Dirac quantum mechanics (QM). This paper proposes a new approach to help answer that question. The mathematical formalism of QM (not the specific physics expressed in that formalism) is the vector space (i.e., Hilbert space) version of the mathematical, indeed logical, machinery used to describe indistinctions and distinctions, i.e., the mathematics of partitions of a set (or, equivalently, equivalence relations on a set or quotient sets).

The two dual lattices of subsets and partitions can be used to represent two different notions of "becoming" or "creation" using the old Greek notions of substance (or matter) and form [1]. Each notion of 'becoming' can be represented as moving from the bottom to the top of the corresponding lattice.

In the case of the subset lattice, the bottom is the null set (no substance) and creation takes place by the introduction of new elements. Each of the new elements is fully formed in terms of what predicates apply to it or in terms of what subsets it belongs to.

In the case of the partition lattice, the bottom is all the elements (substance) of the universe 'blobbed' together in one state without any distinctions. Then 'becoming' takes place by the making of more and more distinctions, i.e., by the substance being increasingly in-formed by information-as-distinctions. As we will see, it is this notion of becoming, i.e., moving from an indefinite state to a more definite state by the making of distinctions, that is key to understand the quantum level notion of measurement. Indeed, that *partition notion of becoming* is key to understanding the whole realm of quantum reality 'beneath' the classical level of fully distinguished states.

These two modes of becoming are illustrated in Figure 1 for the universe set of $U = \{a, b, c\}$.

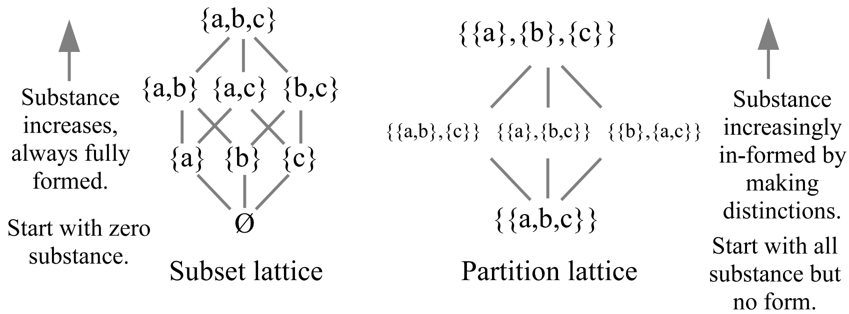

Figure 1: Two modes of "becoming"

If we push these two modes to the universe as a whole, they would give two "creation stories."

Subset creation story: In the Beginning was the Void (no substance) and then fully definite elements ("Its") were created until the full universe U was created.

Partition creation story: In the Beginning was the Blob–all substance with no form (i.e., perfect symmetry [40])–and then, in a Big Bang, distinctions ("Dits") were created (i.e., symmetries were broken) as the substance was increasingly in-formed to reach the universe U of fully distinguished elements.

Needless to say, it is the partition version of creation that is a very skeletal version of current cosmological theory.

Our contention is that the mode of becoming described by quantum mechanics, when stripped down to a 'bare-bones' or skeletal model, is the partition modes which represented by the partition lattice. The classical level is represented by the top partition where everything is fully distinguished; the rest of the lattice, like the underwater part of an iceberg, represents 'underworld' quantum reality that is not directly perceived in the familiar macroscopic classical world.

All three of these developments, the logic of partitions, information theory based on logical entropy, and the partitional interpretation of QM are based on the central analytical concepts variously formulated as indistinctions vs. distinctions, indistinguishability vs. distinguishability, and identity vs. difference.

Since the quantum level reality is not the type described by classical physics, our first task is to recall the pertinent characteristics of that classical vision of reality.

2. The metaphysics of classical reality

One way to describe the classical notion is that reality is seen as fully definite.[1] This notion was captured by Leibniz in his Principle of Identity of Indistinguishables (PII) [2, Fourth letter, p. 22] and by Kant in his Principle of Complete Determination (*omnimoda determinatio*).

> Every thing, however, as to its possibility, further stands under the principle of thoroughgoing determination; according to which, among all possible predicates of things, insofar as they are compared with their opposites, one must apply to it. [28, B600]

If reality is fully definite, then for any two distinct entities, there had to be some predicate that would apply to one and not the other, so if two entities were truly indistinguishable, then they had to be identical. Another principal characteristic of this classical metaphysics of reality was Leibniz's Principle of Continuity [3, p. 7] which could also be expressed as "*Natura non facit saltus*" (Nature does not make jumps) [33, Bk. IV, chap. xvi]. Yet another principle was Leibniz's Principle of Sufficient Reason "that nothing happens without a reason why it should be so rather than otherwise" [2, Second letter, p. 7]

In the reality described by QM, *all* these principles are violated.[2] The PII is violated by two elementary particles of the same type (e.g., same mass, charge, spin, etc.) which have the same values for observables. If reality was always fully definite, then there would have to be some natural or artificial property (e.g., coloring one particle blue) to distinguish two numerically distinct like particles. And the infamous quantum jumps in QM violate the Principle of Continuity while the objectively probabilistic outcomes of quantum experiments violate the Principle of Sufficient Reason.

Since all these classical principles are violated in QM, how can we better imagine the reality described by QM?

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1 Another description might be "definite all the way down"–to borrow a metaphor from the old idea of turtles all the way down.

2 It might be noted that Einstein's special and general relativity theories are 'classical' in the sense of these principles.

3. Changing the imagery: Superposition as indefiniteness

One of the biggest roadblocks to imagining reality at the quantum level is the wave imagery–not to mention the description of the whole subject of QM as "wave mechanics." In particular, when two definite or eigen-states of a particle are in a superposition, the wave imagery interprets this as the addition of two definite waves to give another *definite* wave as in Figure 2.

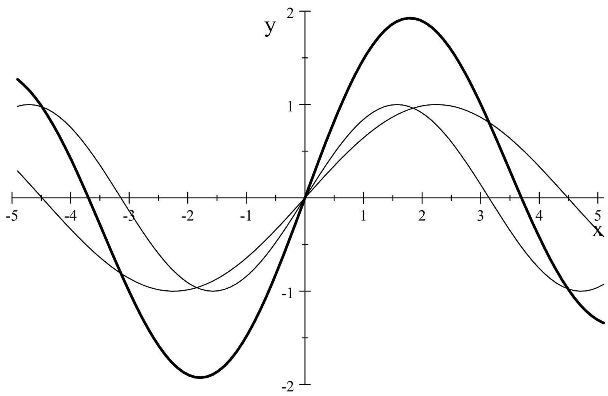

Figure 2: Classical superposition of two definite waves to give a definite wave

It is not the mathematics of waves that is wrong but the wave interpretation. Dirac cautioned against thinking in terms of ordinary waves and their superposition.

> Such analogies have led to the name 'Wave Mechanics' being sometimes given to quantum mechanics. It is important to remember, however, that the superposition that occurs in quantum mechanics is of an essentially different nature from any occurring in the classical theory, as is shown by the fact that the quantum superposition principle demands indeterminacy in the results of observations in order to be capable of a sensible physical interpretation. The analogies are thus liable to be misleading. [11, p. 14]

Vectors over the complex numbers is the natural math to describe waves (i.e., in the polar representation, a complex number is specified by an amplitude and a phase of a 'wave') but a century of looking has not discovered any physical waves to the disappointment of Schrödinger and many others. Vector spaces over the complex numbers are used in QM (among other reasons) because the field of complex numbers is algebraically complete so the math can model a full set of eigenvectors for the observable operators [52, p. 67 fn. 7], not to describe any physical waves. The addition (superposition) of vectors in a vector space over $\mathbb{C}$ can always be mathematically expressed in terms of interference of waves without that implying any physical waves.

> The wave formalism offers a convenient mathematical representation of this latency, for not only can the mathematics of wave effects, like interference and diffraction, be expressed in terms of the addition of vectors (that is, their linear superposition; see [19, Chap. 29.5]), but the converse, also holds. [27, p. 303]

Hence it is not surprising that QM can use the mathematics of waves in the wave-function formulation to describe quantum states–even though that analogy is "misleading."[3]

The whole wave interpretation of QM math is an artifact of using Hilbert space over $\mathbb{C}$; the *real point* is that QM math is really the Hilbert space version of the math of partitions, i.e., the math of indefiniteness and definiteness. There is another equivalent mathematical way to describe quantum states, namely density matrices ([38, p. 102]; [51]), and that mathematical tool shows how to properly interpret superposition. The superposition of two or more definite (or eigen) states should be interpreted as a state that is *indefinite* between those definite states (with certain relative amplitudes).

What is the mathematical concept at the simplest logical level to describe indefiniteness?[4] That is the concept of a partition on a set (or equivalence relation on the set). A partition on a set is a set of nonempty subsets, called "blocks," that are disjoint and jointly exhaustive (i.e., whose union is the whole set). The definiteness comes from the pairs of points in different blocks that are distinguished by the partition and the indefiniteness is given by the pairs of points in the same block which are indistinct or not distinguished according to the partition. Or in the equivalent terms of an equivalence relation on a set, the blocks are the equivalence classes so the pairs of points in different classes are inequivalent or different by that relation and the points in the same class are equivalent or the same in terms of that relation.

> The set can be thought of as being originally fully distinct, while each partition collects together blocks whose distinctions are factored out. Each block represents elements that are associated with an equivalence relation on the set. Then, the elements of a block are indistinct among themselves while different blocks are distinct from each other, given an equivalence relation. [49, p. 1]

In QM, the superposition of two definite states $|a\rangle$ and $|b\rangle$ is given by a linear combination $\alpha|a\rangle + \beta|b\rangle$ where α and β are complex numbers. If we now strip away the complex scalars, the Dirac ket notation for $|a\rangle$ and $|b\rangle$, and the vector space addition, we are left with just the skeletal or scalarless set $\{a, b\}$ which is just a block in a partition

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3 After a century of being mislead by the "wave mechanics," Einstein's statement: "The Lord is subtle, but not malicious," may be too optimistic.

4 The vast literature on metaphysical vagueness (e.g., sorites arguments) does not seem helpful to clarify the notion of "quantum indeterminacy" [34, p. 78]

$\{\{a, b\}, ...\}$ on some 'universe set' $U = \{a, b, ...\}$ or, equivalently, an equivalence class in an equivalence relation on U that, in this case, equates the states $\{a\}$ and $\{b\}$. Thus the alternative quantum way to interpret superposition is *not* the sum of two definite wave states to get another definite wave but the blurring or cohering together of the definite states $\{a\}$ and $\{b\}$ to get an indefinite state $\{a, b\}$.

4. The matrix representation of an equivalence relation

A *binary relation* R on a universe set $U = \{u_1, ..., u_n\}$ is a subset $R \subseteq U \times U$ of the *Cartesian product* $U \times U$, the set of all ordered pairs (u_i, u_j) of elements of U.[5] The Cartesian product $U \times U$ includes the *diagonal* $\Delta = \{(u_i, u_i) : u_i \in U\}$, the set of all self-pairs. An *equivalence relation* $E \subseteq U \times U$ is a binary relation on U that is *reflexive* (i.e., $\Delta \subseteq E$), *symmetric* (i.e., if $(u_i, u_j) \in E$, then $(u_j, u_i) \in E$), and transitive (i.e., if $(u_i, u_j) \in E$ and $(u_j, u_k) \in E$, then $(u_i, u_k) \in E$). Any binary relation R on U can be represented by its $n \times n$ *incidence matrix* $\mathrm{In}(R)$ where the row i and column j entry is: $\mathrm{In}(R)_{ij} = 1$ if $(u_i, u_j) \in R$ and 0 otherwise for $i, j = 1, ..., n$. Every equivalence relation can be paired with the partition formed by its equivalence classes.

A *partition* π on U is a set of nonempty subsets (the blocks) $\pi = \{B_1, ..., B_m\}$ which are disjoint (i.e., $B_i \cap B_j = \emptyset$ for $i \neq j$) and jointly exhaustive (i.e., $\cup_{j=1}^m B_j = U$). The corresponding equivalence relation is $E(\pi) = \cup_{j=1}^m (B_j \times B_j)$, i.e., the set of all ordered pairs of elements in the same block–so the blocks of the partition are the equivalence classes. In terms of the language of distinctions and indistinctions, the indistinctions of π are the *indit set* $\mathrm{indit}(\pi) = E(\pi) = \cup_{j=1}^m (B_j \times B_j)$. The set of all ordered pairs of elements in different blocks is the set of distinctions or *ditset* $\mathrm{dit}(\pi) = U \times U - \mathrm{indit}(\pi)$, since each ordered pair has to either be in the same block or in different blocks of π. The quotient set determined by π is the set resulting from treating each $B_j \in \pi$ as a single point.

For example, for $U = \{a, b, c\}$, the partition $\pi = \{\{a\}, \{b, c\}\}$ is paired with the equivalence relation whose ordered pairs are $\mathrm{indit}(\pi) = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$ and whose incidence matrix has a 1 for each indistinction:

$$\mathrm{In}(\mathrm{indit}(\pi)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The five 1's correspond to the five ordered pairs in $\mathrm{indit}(\pi)$ and the four 0's correspond to

[5] Since our purpose is conceptual clarity rather than mathematical generality, we always take U to be a finite set.

the four ordered pairs in the ditset $\operatorname{dit}(\pi) = \{(a,b),(a,c),(b,a),(c,a)\}$. The *trace* of an $n \times n$ matrix is the sum of its diagonal elements. If we divide $\operatorname{In}(\operatorname{indit}(\pi))$ by its trace $\operatorname{tr}[\operatorname{In}(\operatorname{indit}(\pi))]$, then we have an example of the density matrix of a partition:

$$\rho(\pi) = \frac{\operatorname{In}(\operatorname{indit}(\pi))}{\operatorname{tr}[\operatorname{In}(\operatorname{indit}(\pi))]} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

We may further assume that the points $u_i \in U$ have a probability $\Pr(u_i) = p_i$ assigned to them to form a discrete probability distribution $p = (p_1, ..., p_n)$ where $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. Then the blocks $B_j \in \pi$ have the probability $\Pr(B_j) = \sum\{p_i : u_i \in B_j\}$. Then we can form the density matrix $\rho(B_j)$ for a block defined by: $\rho(B_j)_{ik} = \frac{\sqrt{p_i p_k}}{\Pr(B_j)}$ if $u_i, u_k \in B_j$ (assuming $\Pr(B_j) \neq 0$) and 0 otherwise. The density matrix $\rho(B_j)$ is the projector $|b_j\rangle\langle b_j|$ formed from the column vector $|b_j\rangle$ whose i^{th} entry is $\sqrt{p_i/\Pr(B_j)}$ if $u_i \in B_j$ and 0 otherwise. Then the density matrix for the whole partition π is just the probabilistic sum of the density matrices or projectors for the blocks:

$$\rho(\pi) = \sum_{j=1}^{m} \Pr(B_j)\,\rho(B_j) = \sum_{j=1}^{m} \Pr(B_j)\,|b_j\rangle\langle b_j|.$$

Then it is easily checked that $\rho(\pi)_{ik} = \sqrt{p_i p_k}$ if $(u_i, u_k) \in \operatorname{indit}(\pi)$ and 0 otherwise. Assuming all the probabilities p_i are positive, the non-zero entries in $\rho(\pi)$ correspond to the indits $\operatorname{indit}(\pi)$ and the zero entries correspond to the ditset $\operatorname{dit}(\pi)$. In the example of $\pi = \{\{a\},\{b,c\}\}$, the density matrix $\rho(\pi) = \frac{\operatorname{In}(\operatorname{indit}(\pi))}{\operatorname{tr}[\operatorname{In}(\operatorname{indit}(\pi))]}$ is the matrix obtained with the uniform probability distribution $\Pr(\{a\}) = \Pr(\{b\}) = \Pr(\{c\}) = \frac{1}{3}$.

The density matrices $\rho(\pi)$ are still set level concepts that allow the translation of set partition (or equivalence relation) concepts into the matrix form that can then be directly compared to the corresponding quantum level density matrix. Those translations, along with some addition facts about the eigenvalues of $\rho(\pi)$ are given in Table 1.

Set concept with probabilities	**Set level density matrix concept**
Partition π with point probs. p	Density matrix $\rho(\pi) = \sum_{j=1}^{m} \Pr(B_j)\,|b_j\rangle\langle b_j|$
Point probabilities $\{p_1, ..., p_n\}$	Values of diagonal entries of $\rho(\pi)$
Trivial indits (u_i, u_i) of π	Diagonal entries of $\rho(\pi)$
Non-trivial indits of π	Non-zero off-diagonal entries of $\rho(\pi)$
Dits of π	Zero entries of $\rho(\pi)$
Block probabilities $\Pr(B_j)$ in π	Eigenvalues $\neq 0$ of $\rho(\pi)$
Disjoint blocks of π	Orthog. eigenvectors of $\rho(\pi)$ for eigenvalues $\neq 0$

Table 1: Density matrix translation of partition concepts at the level of sets

These set level notions of partition, equivalence relation, incidence matrix, and density matrix of an equivalence relation all are the 'skeletal' forms that prefigure notions in QM. The initial idea is that all the singletons $\{a\}$, $\{b\}$, and $\{c\}$ prefigure definite eigenstates and a non-singleton block like $\{b, c\}$ prefigures a superposition state. In the incidence matrix $\text{In}\left(\text{indit}\left(\pi\right)\right)$ for the equivalence relation $\text{indit}\left(\pi\right)$, the non-zero off-diagonal entries represented the blobbing or cohering together of the elements in the same equivalence class. In the density matrix $\rho\left(\pi\right) = \sum_{j=1}^{m} \Pr\left(B_j\right) \left|b_j\right\rangle \left\langle b_j\right|$, the non-singleton B_j contribute the non-zero off-diagonal entries are the "coherences" ([9, p. 303]) that indicate the amplitude of the corresponding diagonal states cohering together in a superposition. The most basic non-classical notion in QM is the notion of a superposition; the particularly non-classical entangled states are a special case.

> [T]he off-diagonal terms of a density matrix ... are often called quantum coherences because they are responsible for the interference effects typical of quantum mechanics that are absent in classical dynamics. [4, p. 177].

This translation of set level concepts for π into density matrix concepts in $\rho\left(\pi\right)$ then prefigures arbitrary density matrices in the mathematics of QM. Then a density matrix ρ is not only a Hermitian matrix but a positive matrix so its spectral decomposition gives a orthonormal set of basis eigenvectors $\{|u_j\rangle\}$ and a set of non-negative eigenvalues $\{\lambda_j\}$ that sum to one. Then the intermediate step of translating set partition concepts (disjoint blocks and block probabilities) into density matrix concepts for $\rho\left(\pi\right) = \sum_{j=1}^{m} \Pr\left(B_j\right) \left|b_j\right\rangle \left\langle b_j\right|$ can be completed with the spectral representation of an arbitrary density as: $\rho = \sum_{j=1}^{m} \lambda_j \left|u_j\right\rangle \left\langle u_j\right|$ in the mathematics of QM. That is one example of how the math of partitions is the skeletal version that prefigures the corresponding Hilbert space math of QM. A set partition π with point probabilities is set level concept that prefigures the Hilbert space notion of a quantum state represented as a density matrix.

5. The lattice of partitions: The skeletal model of quantum reality

Of all the partitions on U, there are two extreme cases. The *discrete partition* $\mathbf{1}_U = \{\{u_1\}, ..., \{u_n\}\}$ on U is the partition where all the blocks are the singletons of the elements of U. The discrete partition $\mathbf{1}_U$ is the only partition with no non-singleton blocks, i.e., no superpositions, so it represents a classical state. The set of indistinctions of $\mathbf{1}_U$ is $\text{indit}\left(\mathbf{1}_U\right) = \Delta$, the diagonal of self-pairs, and the ditset is $\text{dit}\left(\mathbf{1}_U\right) = U \times U - \Delta$, all the ordered pairs except the self-pairs. The density matrix $\rho\left(\mathbf{1}_U\right)$ is the diagonal matrix with the diagonal entries p_1,..., p_n. The point probabilities are also the block probabilities since all the blocks are singletons so they are the eigenvalues of $\rho\left(\mathbf{1}_U\right)$.

At the other extreme is the *indiscrete partition* (nicknamed the "Blob") $\mathbf{0}_U = \{U\} = \{\{u_1, ..., u_n\}\}$ with only one block which is U. The indit set is $\text{indit}\left(\mathbf{0}_U\right) = U \times U$, the universal equivalence relation, and $\text{dit}\left(\mathbf{0}_U\right) = \emptyset$ since there are no distinctions in the Blob. Since the non-zero eigenvalues of $\rho\left(\pi\right)$ are the block probabilities $\Pr\left(B_j\right)$ for the blocks of π (Table 1), the only non-zero eigenvalue of $\rho\left(\mathbf{0}_U\right)$ is 1 (like a pure state density matrix).

Given another partition $\sigma = \{C_1, ..., C_{m'}\}$ on U, the partition $\pi = \{B_1, ..., B_m\}$ *refines* σ, written $\sigma \precsim \pi$, if for any block $B_j \in \pi$, there is a block $C_{j'} \in \sigma$ such that $B_j \subseteq C_{j'}$. Intuitively, the blocks of π are made by chopping up some blocks of σ into smaller blocks. The refinement relation on the partitions on U is a *partial order* which means it is reflexive (i.e., $\sigma \precsim \sigma$), anti-symmetric (i.e., if $\sigma \precsim \pi$ and $\pi \precsim \sigma$, then $\sigma = \pi$) and transitive (i.e., if $\gamma \precsim \sigma$ and $\sigma \precsim \pi$, then $\gamma \precsim \pi$ where $\gamma = \{D_1, ..., D_{m''}\}$ is another partition on U).

It was known in the nineteenth century (e.g., by Dedekind and Schröder) that the Boolean operations of join (least upper bound) and meet (greatest lower bound) can be defined not just for subsets but also for partitions so all the partitions on U form a *lattice* $\Pi(U)$. In fact, all the Boolean operations (particularly implication) can also be defined on partitions so there is a logic of partitions [13] that is dual to the usual Boolean lattice $\wp(U)$ of subsets (usually presented only in the special case of propositional logic where $U = 1 = \{*\}$) since subsets and partition are category-theoretic duals [32, p. 85]. In the Boolean lattice on the powerset $\wp(U)$, the join and meet are set union and intersection respectively, and the partial order is set inclusion. In the partition lattice $\Pi(U)$, the partial order of refinement is the same as set inclusion of ditsets, i.e., $\sigma \precsim \pi$ if and only if $\text{dit}(\sigma) \subseteq \text{dit}(\pi)$.

The main partition operation that we will need is the join where the *join* of π and σ, denoted $\pi \vee \sigma$, is the partition on U whose blocks are all the non-empty intersections $B_j \cap C_{j'}$ for $j = 1, ..., m$ and $j' = 1, ..., m'$. It is easily checked that $\text{dit}(\pi \vee \sigma) = \text{dit}(\pi) \cup \text{dit}(\sigma)$, or, in terms of the corresponding equivalence relations, $\text{indit}(\pi \vee \sigma) = \text{indit}(\pi) \cap \text{indit}(\sigma)$. We will not need the meet but intuitively the meet $\pi \wedge \sigma$ is defined by thinking of the blocks of π and σ as small blobs of mercury so whenever they overlap (i.e., $B_j \cap C_{j'} \neq \emptyset$) then they combine to form a bigger blob. A block of $\pi \wedge \sigma$ is an exact union of π-blocks that is also an exact union of σ-blocks (i.e., no overlaps with the blocks of the other partition) and is minimal in that respect.[6] This join and meet operation on partitions make $\Pi(U)$ into a lattice.[7]

In terms of the set level or skeletal partition concepts prefiguring quantum concepts, the indiscrete partition has a pure state density matrix $\rho(\mathbf{0}_U)$, the other partitions with at least one non-singleton block (representing a superposition) prefigure non-classical mixed states, and the discrete partition prefigures the classical state with a diagonal density matrix $\rho(\mathbf{1}_U)$.

The assumed ontology is one of particles but they are not the particles of classical

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[6] The indiscrete partition $\mathbf{0}_U$ is nicknamed "The Blob" because as in the eponymous Hollywood movie, it absorbs everything it meets, i.e., $\mathbf{0}_U \wedge \pi = \mathbf{0}_U$.

[7] It might be noted that some of the older texts on lattice theory [5] define the "lattice of partitions" as the lattice of equivalence relations with the opposite "unrefinement" partial order, i.e., $\text{indit}(\pi) \subseteq \text{indit}(\sigma)$, so the join and meet are reversed.

(or Bohmian) physics. The elements of U represent possible definite or eigenstates of a particle. The key non-classical idea is a particle in an indefinite superposition state (always relative to some basis). The so-called "particle/wave complementarity" is just the "complementarity" between particles in a definite state versus an indefinite state represented by a "wave function" or, less misleadingly, by a density matrix (with non-zero off-diagonal entries connecting superposed eigenstates).

The skeletal states, classical and mixed or pure, of a particle with four possible eigenstates are given in Figure 3 where $U = \{a, b, c, d\}$ and where we use the shorthand notation that shortens the partition $\{\{a, b\}, \{c, d\}\}$ to $\{ab, cd\}$ and so forth for the other partitions. Figure 3 is a Hasse diagram for the lattice of partition on four elements where lines between partitions are refinement relations and there is no partition intermediate between the partitions connected by a line. And there is no notion of continuous evolution from a block of one partition to a smaller block in a more refined one; it is a discontinuous probabilistic jump.[8]

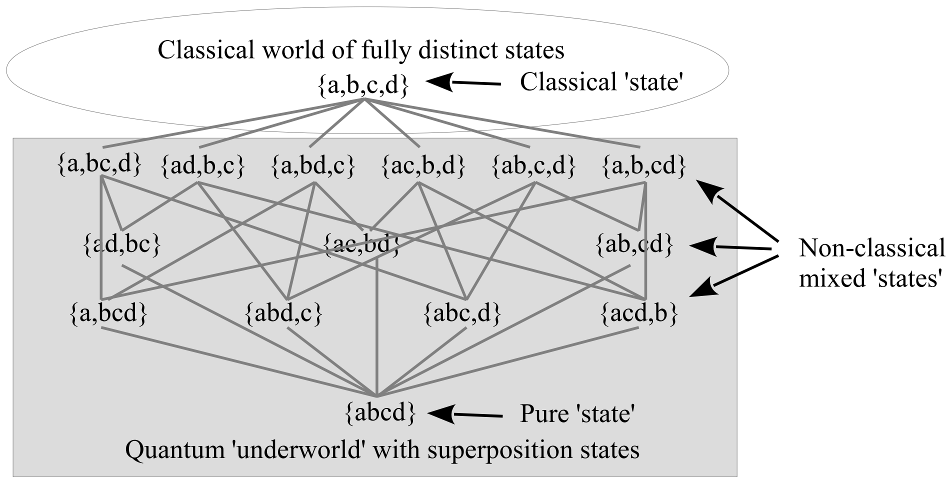

Figure 3: Lattice of partitions as a skeletal view of classical and quantum states

Any non-singleton block prefigures a quantum superposition state where the indistinction or indefiniteness between the superposed states is objective or ontic; it does not just represent a subjective or epistemological indefiniteness. For the discrete partition, all the possible distinctions have been made so it represents a classical statistical mixture between fully definite states with a diagonal density matrix with the diagonal entries being the probabilities of the definite states. "For instance, the statistical mixture describing the state of a classical dice before the outcome of the throw" [4, p. 176] is the 6×6 diagonal matrix with diagonal entries $\frac{1}{6}$.

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8 Superluminal communication would be possible in violation of special relativity if this connection between a level of indefiniteness and classical states was not discontinuous and objectively probabilistic.[24]

One of the characteristics of classical reality was expressed by Leibniz's Principle of Identity of Indistinguishables (PII). Quantifying over properties $F()$, this means that if a and b have all the same properties, then $a = b$, i.e., "$\forall F (F(a) \longleftrightarrow F(b)) \rightarrow a = b$" [23, p. 234]. In terms of partitions, the elements that are not distinguished by the partition are the indistinctions of the partition. Hence the partition π-version of PII is: if $(u, u') \in \operatorname{indit}(\pi)$, then $u = u'$. This holds for $\pi = \mathbf{1}_U$ and *only* for that discrete partition so the correct partition logic version of PII and thus the Criterion for Classicality is:

For all $u, u' \in U$, if $(u, u') \in \operatorname{indit}(\mathbf{1}_U)$, then $u = u'$.
Partition logic version of the Principle of Identity of Indistinguishables.

The view of definite all the way down means that every possible distinction can be made and that is true for only the discrete partition $\mathbf{1}_U$ where $\operatorname{indit}(\mathbf{1}_U) = \Delta$, the diagonal of self-pairs. Any other partition $\pi \neq \mathbf{1}_U$ has at least one non-singleton block which represents an objectively indefinite superposition at the quantum level so its mixed state density matrix has non-zero off-diagonal coherences which lead to the non-classical interference phenomenon of QM.

To recapitulate, Table 2 gives the partition concept and the corresponding quantum concept.

Partition concept	Corresponding quantum concept
Partition with π block probs. $\Pr(B_j)$	Mixture of states with probs.
Singleton block	Eigenstate (no superposition)
Non-singleton block	Superposition of eigenstates
Discrete partition $\mathbf{1}_U$	Mixture of classical states
Indiscrete partition $\mathbf{0}_U$	Pure state
Partition density matrix $\rho(\pi)$	Quantum state as density matrix ρ
$\rho(\pi) = \sum_{j=1}^{m} \Pr(B_j) \vert b_j\rangle \langle b_j\vert$	$\rho = \sum_{j=1}^{m} \lambda_j \vert u_j\rangle \langle u_j\vert$

Table 2: Partition concepts and quantum counterparts

The notion of 'becoming' or change in classical physics is the continuous change of distinct states. QM has a different notion of becoming or change. Unitary evolution of a quantum state can take place at any level of indefiniteness while a position measurement of a particle can only take place at the classical level so the traditional notion of a particle's "trajectory" is not available in QM; it does not fit the partition notion of becoming. Instead of the classical trajectory, the "motion" of a quantum particle "is no longer that of a steady deterministic flow..., but that of states and transitions, 'flights and perchings,' in which the perchings are more stable and flights more abrupt than classical ideas would have allowed." [25, p. 198]

There is a history of conceptualizing indefinite states as opposed to definite eigenstates as the difference between "potentia" and actualities. ([26]; [37]; [27, Sec. 10.2]; [46]; [22]; [30], [29]; [10]).

> Heisenberg [26, p. 53]... used the term "potentiality" to characterize a property which is objectively indefinite, whose value when actualized is a matter of objective chance, and which is assigned a definite probability by an algorithm presupposing a definite mathematical structure of states and properties. Potentiality is a modality that is somehow intermediate between actuality and mere logical possibility. That properties can have this modality, and that states of physical systems are characterized partially by the potentialities they determine and not just by the catalogue of properties to which they assign definite values, are profound discoveries about the world, rather than about human knowledge. [46, p. 6]

At least in ordinary usage, the notion of potentialities becoming actualized as a mode of 'becoming' does not differentiate between the two modes of becoming represented in Figure 1. But the quantum theorists who use the notions of "potentia" or potentialities are referring to objectively indefinite superpositions, i.e., to the partition notion of becoming where the *real* indefinite states become more definite by the making of distinctions and then finally to a fully definite state in a (non-degenerate) measurement. The word "actuality" is often taken to refer only to the fully definite classical reality. The relevant logic is not modal logic (e.g., about modalities of possible and actual) but partition logic which deals with the mathematical concepts, partitions, that characterize indefiniteness and definiteness.[9] Shimony shows how to interpret Heisenberg's reference to Aristotle's notion of "potentia."

> The historical reference should perhaps be dismissed, since quantum mechanical potentiality is completely devoid of teleological significance, which is central to Aristotle's conception. What it has in common with Aristotle's conception is the indefinite character of certain properties of the system. [45, pp. 313-4]

Henry Margenau and R. I. G. Hughes favor the term "latency" over "potentiality" but Margenau mentions that the measurement of observables "forces them out of indiscriminacy or latency" [37, p. 10]–which once again emphasizes the indeterminacy or indefiniteness of the "latent" states. And Ruth Kastner takes indeterminacy as the primary characteristic of the potentialities [29, p. 3]. If the potentiality-actuality approach is translated into indefinite versus definite forms of reality, then that analysis is essentially the same as the partitional objective indefiniteness analysis developed here.

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9 Here "logic" means not reasoning about propositions but the older meaning of the essence of a subject matter stripped down to its bare or skeletal essentials, e.g., as partition logic is to quantum mechanics.

6. Quantum measurement: The partition version

The two main quantum concepts, the quantum notion of a state and of an observable, need to be prefigured at the skeletal level of sets. We have seen that the notions of a pure state and a mixed state are prefigured by partitions π (or the corresponding equivalence relations $\text{indit}(\pi)$) which have density matrices $\rho(\pi)$ prefiguring the representation of quantum states by density matrices (rather than state vectors, not to mention 'wave functions') with the non-zero off-diagonal entries representing the non-classical coherences involved in superpositions. The partitions π used in constructing the density matrices $\rho(\pi)$ came equipped with point probabilities.

To prefigure the other quantum notion of an observable, we also start with a partition, say $\sigma = \{C_1, ..., C_{m'}\}$ but without point probabilities. A (real-valued) *numerical attribute* on U is a function $g : U \to \mathbb{R}$. The image of the function is some set of real numbers $g(U)$. The numerical attribute defines the inverse-image partition $g^{-1} = \left\{g^{-1}(r) : r \in g(U)\right\}$ where we can take $g^{-1} = \sigma$.

The notion of measurement (always projective) in QM applies a real-valued observable (eigenvalues of a Hermitian matrix) to a quantum state represented by its density matrix ρ to yield the post-measurement density matrix $\hat{\rho}$ and that operation is called the "Lüders mixture operation" ([35]; [4, p. 279]). To define the partition version of the Lüders mixture operation, we need a "projection matrix" $P_{C_{j'}}$ for each block $C_{j'}$ in the partition σ for the 'observable' or numerical attribute. That projection matrix $P_{C_{j'}}$ is the $n \times n$ diagonal matrix with the diagonal elements equal to $\chi_{C_{j'}}(u_i)$, where $\chi_{C_{j'}} : U \to 2$ is the characteristic or indicator function for $C_{j'}$, i.e., $\chi_{C_{j'}}(u_i) = 1$ if $u_i \in C_{j'}$ and 0 otherwise. Then the partition version of the Lüders mixture operation for the 'measurement' of the state $\rho(\pi)$ by the numerical attribute $g : U \to \mathbb{R}$ where $\sigma = g^{-1}$, gives the post-measurement density matrix $\hat{\rho}(\pi)$ as:

$$\hat{\rho}(\pi) = \sum_{j'=1}^{m'} P_{C_{j'}} \rho(\pi) P_{C_{j'}}$$

Partition version of Lüders mixture operation.

Then it is a theorem [17, Theorem 1] that:

$$\hat{\rho}(\pi) = \rho(\pi \vee \sigma)$$

so the partition operation that prefigures the quantum Lüders mixture operation for a measurement is the join of partitions (or, equivalently, the intersection of equivalence relations). Joins create distinctions. Since $\text{dit}(\pi \vee \sigma) = \text{dit}(\pi) \cup \text{dit}(\sigma)$ and $\sigma = g^{-1}$, the new distinctions created by the g-measurement, i.e., $\text{dit}(\sigma) - \text{dit}(\pi)$, are the pairs $(u_i, u_k) \in \text{indit}(\pi)$ where $g(u_i) \neq g(u_k)$. Since the numerical attribute prefigures an observable operator and $(u_i, u_k) \in \text{indit}(\pi)$ prefigures u_i and u_k being in the same-block superposition indicated by $\rho(\pi)_{ik} \neq 0$, we can say that the g-measurement distinguished the (u_i, u_k) superpositions in the $\rho(\pi)$ since they had different g-(eigen)values and thus $\rho(\pi \vee \sigma)_{ik} = 0$. Thus the non-zero off-diagonal elements of $\rho(\pi)$ that got zeroed in $\rho(\pi \vee \sigma)$ are for the ik-entries where $(u_i, u_k) \in \operatorname{dit}(\sigma) - \operatorname{dit}(\pi)$.

A (proper) mixed state $\rho(\pi \vee \sigma)$ does not represent or prefigure a physical state; it represents the state of ignorance as to which (orthogonal) component (i.e., partition block with a block probability) is the actual physical state. The indiscrete partition $\mathbf{0}_U$ is the identity element for the join operation, i.e., $\mathbf{0}_U \vee \sigma = \sigma$, so in that case $\hat{\rho}(\mathbf{0}_U) = \rho(\sigma)$, so the distinctions of the 'observable' $\sigma = g^{-1}$ for $g : U \to \mathbb{R}$ have created the mixed state $\rho(\sigma)$ with those distinctions. If $\pi \vee \sigma = \mathbf{1}_U$, then the measurement is said to be non-degenerate. When $\mathbf{0}_U \vee \sigma = \sigma \neq \mathbf{1}_U$, then other partitions, e.g., $\gamma = h^{-1}$ for $h : U \to \mathbb{R}$, on the same set (or other commuting observables in the QM case) are needed to make all the distinctions. When $\mathbf{0}_U \vee \sigma \vee ... \vee \gamma = \sigma \vee ... \vee \gamma = \mathbf{1}_U$ for numerical attributes $g, ..., h : U \to \mathbb{R}$, then $g, ..., h$ are said to be a complete set of compatible attributes (CSCA) which prefigure a compete set of commuting observables (CSCO) in full QM.

For a simple example, let $\pi = \{\{a\}, \{b, c\}\}$ (without the shorthand notation) with point probabilities $\Pr(\{a\}) = \frac{1}{3}$, $\Pr(\{b\}) = \frac{1}{4}$, and $\Pr(\{c\}) = \frac{5}{12}$ in $U = \{a, b, c\}$ and $\sigma = \{\{a, b\}, \{c\}\}$ so $\sigma = g^{-1}$ for any $g : U \to \mathbb{R}$ that assigns the same g-value to a and b with a different value for c. Then the density matrix for π and the projections matrices for the blocks of σ are:

$$\rho(\pi) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{5}}{4\sqrt{3}} \\ 0 & \frac{\sqrt{5}}{4\sqrt{3}} & \frac{5}{12} \end{bmatrix}, P_{\{a,b\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{and } P_{\{c\}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the Lüders mixture operation is:

$$\begin{aligned} \hat{\rho}(\pi) &= P_{\{a,b\}}\rho(\pi) P_{\{a,b\}} + P_{\{c\}}\rho(\pi) P_{\{c\}} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{5}}{4\sqrt{3}} \\ 0 & \frac{\sqrt{5}}{4\sqrt{3}} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{5}}{4\sqrt{3}} \\ 0 & \frac{\sqrt{5}}{4\sqrt{3}} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{5}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{5}{12} \end{bmatrix}. \end{aligned}$$

Since $\pi \vee \sigma = \{a, b, c\} = \mathbf{1}_U$, we see that the post-measurement density matrix is $\hat{\rho}(\pi) = \rho(\pi \vee \sigma) = \rho(\mathbf{1}_U)$. Thus the superposition of $\{b\}$ and $\{c\}$ in π got distinguished since b and c had different g-values. The new distinctions in $\operatorname{dit}(\sigma) - \operatorname{dit}(\pi)$ are (b, c) (along with (c,b)) and those were the non-zero off-diagonal elements (coherences) of $\rho(\pi)$ that got zeroed (distinguished or decohered) in $\rho(\pi\vee\sigma)$.

The Lüders mixture is not the end of the measurement process. The measurement returns one of the g-values, say the (degenerate) one for $\{a,b\}$. Then the Lüders Rule [27, Appendix B] gives the final density matrix which is the corresponding term in the Lüders mixture sum, e.g., $P_{\{a,b\}}\rho(\pi)P_{\{a,b\}}$, normalized so the final density matrix is the (classical) mixed state:

$$\frac{P_{\{a,b\}}\rho(\pi)P_{\{a,b\}}}{\operatorname{tr}\left[P_{\{a,b\}}\rho(\pi)P_{\{a,b\}}\right]}=\begin{bmatrix}\frac{1}{3}&0&0\\0&\frac{1}{4}&0\\0&0&0\end{bmatrix}\frac{1}{7/12}=\begin{bmatrix}\frac{4}{7}&0&0\\0&\frac{3}{7}&0\\0&0&0\end{bmatrix}.$$

7. The Yoga of Linearization

Heretofore, we have started with a partition with point probabilities and developed the corresponding *quantum state* concept (density matrix representation of states). Now we begin with a real-valued numerical attribute $g:U\to\mathbb{R}$ which yields the inverse image partition $g^{-1}=\left\{g^{-1}(r)\right\}_{r\in g(U)}$ and we develop the corresponding *quantum observable* concept. This can be done using a semi-algorithmic procedure that could be called a "yoga" [41, p. 251], in this case, the Yoga of Linearization, which is part of the mathematical folklore.

The idea is to first consider $U=\{u_1,...,u_n\}$ as just a set and then consider U as a basis set for a vector space V over any field $\mathbb{k}$ (where $\mathbb{k}=\mathbb{C}$ for the corresponding quantum observable concept). Then apply the set concept to U as a basis set and see what vector space notion is generated by that set concept applied to U. For instance, starting with the notion of a subset S of U, then that subset as a subset of a basis set will generate a subspace $[S]=\{ku_i+...+k'u_j:u_i,...,u_j\in S;k,..,k'\in\mathbb{k}\}$ of all the linear combinations of the basis vectors in S. For $S=U$, the basis set, by definition, generates the whole space $[U]=V$. The concept of cardinality of the subset corresponds to the dimension of the generated subspace, i.e., $|S|=\dim([S])$. A singleton subset $\{u_i\}$ thus generates a ray or one-dimensional subspace.

What set level concept corresponds to the quantum notion of an observable represented by a Hermitian operator with real eigenvalues? It is the set notion of a real-valued numerical attribute $g:U\to\mathbb{R}$. Taking U as a basis set U, the operator $G:V\to V$ (where $\mathbb{k}=\mathbb{C}$) with the eigenvalues being the values of g is generated by the definition $Gu_i=g(u_i)u_i$.

What quantum level concept corresponds to the set partition $g^{-1}=\left\{g^{-1}(r):r\in g(U)\right\}$. Applying g to U as a basis set, each block $g^{-1}(r)$ generates a subspace $V_r=\left[g^{-1}(r)\right]$ which is precisely the eigenspace of the operator G for the eigenvalue r. Those eigenspaces $\left\{\left[g^{-1}(r)\right]:r\in g(U)\right\}$ form a *direct-sum decomposition* (DSD) of the space V–where a DSD is defined as a set of subspaces such that every non-zero vector $v \in V$ is uniquely represented as a sum $v = \sum \{v_r : v_r \in V_r\}$.[10] The unique component vectors v_r are just the projections of v to the eigenspaces V_r. This correspondence between set partitions $\sigma = g^{-1}$ on U and DSDs $\{V_r\}_{r \in g(U)}$ of V is confirmed by noting that the DSD definition could be used to define a set partition. The projection of a subset $S \subseteq U$ to a subset $C_{j'}$ is just the intersection $S \cap C_{j'}$. Then a set of subsets $\sigma = \{C_1, ..., C_{m'}\}$ of U form a partition if and only if every non-empty subset $S \subseteq U$ is uniquely expressed as a union of non-empty subsets of the $\{C_{j'}\}_{j'=1}^{m'}$, in particular, $S = \cup_{j'=1}^{m'} (S \cap C_{j'})$–which is illustrated in Figure 4.

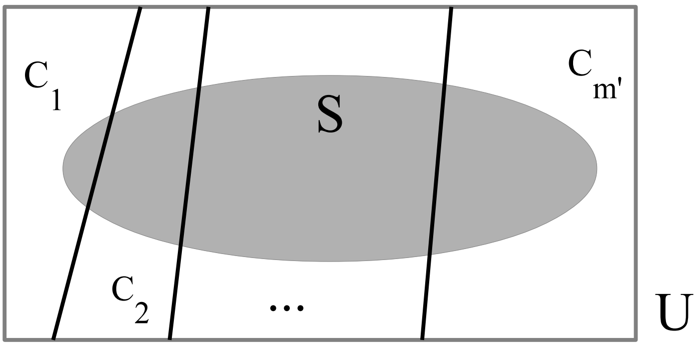

Figure 4: Partition definition of σ as the set version of a direct-sum decomposition

If the union $\cup_{j'=1}^{m'} C_{j'}$ was not all of U, then the difference set $S = U - \cup_{j'=1}^{m'} C_{j'}$ could not be represented as a union of subsets of the $C_{j'}$'s. And if some of the subsets overlapped, e.g., $S = C_1 \cap C_2 \neq \emptyset$, then that overlap set S would have two representations as a subset of C_1 and of C_2 so its representation would not be unique.

We can also run the Yoga in reverse to see the set concepts that correspond to eigenvectors, eigenvalues, and even spectral decompositions of operators. Let $g \restriction S = rS$ stand for the proposition: "The function $g : U \to \mathbb{R}$ restricted to $S \subseteq U$ has the value r on all the elements of S". Then $g \restriction S = rS$ is the set version of the eigenvector/eigenvalue equation $Gu_i = ru_i$. Thus the set version of an eigenvector is a subset $S \subseteq g^{-1}(r)$, i.e., a constant set of g. And the set version of an eigenvalue is just the constant value r of g on one of its constant sets. If $P_{V_r} : V \to V$ is the projection operator taking any vector $v \in V$ to its projection $v_r \in V_r$ to the eigenspace V_r of G for the eigenvalue r, then the spectral decomposition of G is: $G = \sum_{r \in g(U)} rP_{V_r}$. A special type of numerical attribute is the characteristic function $\chi_S : U \to \{0, 1\} \subseteq \mathbb{R}$ and the corresponding operator is defined by: $P_{[S]} u_i = \chi_S(u_i) u_i$ with the eigenvalues of 1 and 0 which is the projection operator to the subspace $[S]$ generated by $S \subseteq U$. Since characteristic functions correspond to projection

[10] Subsets dualize to partitions so the Yoga dually correlates subspaces to direct-sum decompositions and thus the Birkhoff-von-Neumann quantum logic of subspaces [6] has a correlated quantum logic of DSDs [15].

operators, the spectral decomposition $G = \sum_{r \in g(U)} r P_{V_r} : V \to V$ of G has a set version for g which is: $g = \sum_{r \in g(U)} r \chi_{g^{-1}(r)} : U \to \mathbb{R}$. Also when applied to basis sets U and U' for vector spaces V and V', the set notion of the Cartesian or direct product (bi)linearly generates by the Yoga the tensor product $V \otimes V'$ where the ordered pairs (u, u') in $U \times U'$ are denoted as the basis elements $u \otimes u'$ of $V \otimes V'$.

Table 3 summarizes the set concepts and the corresponding vector space concepts generated by the Yoga.

Set concept (skeletons)	Vector-space concept
Partition $\left\{g^{-1}(r)\right\}_{r \in g(U)}$	DSD $\{V_r\}_{r \in g(U)}$
$U = \uplus_{r \in g(U)} g^{-1}(r)$	$V = \oplus_{r \in g(U)} V_r$
$g \restriction S = rS$	$Gu_i = ru_i$
Constant set S of g	Eigenvector u_i of G
Value r on constant set S	Eigenvalue r of eigenvector u_i
Characteristic fcn. $\chi_S : U \to \{0, 1\}$	Projection operator $P_{[S]} u = \chi_S(u) u$
Spectral Decomp. $g = \sum_{r \in g(U)} r \chi_{g^{-1}(r)}$	Spectral Decomp. $G = \sum_{r \in g(U)} r P_{V_r}$
Set of r-constant sets $\wp\left(g^{-1}(r)\right)$	Eigenspace V_r of r-eigenvectors
Join $\rho(\pi \vee \sigma) = \sum_{j'=1}^{m'} P_{C_{j'}} \rho(\pi) P_{C_{j'}}$	Lüders op. $\hat{\rho} = \sum_{r \in g(U)} P_{V_r} \rho P_{V_r}$
Cartesian product $U \times U'$	Tensor product $V \otimes V'$

Table 3: Set concepts and corresponding vector space concepts

8. Reformulating the set concepts in a vector space over $\mathbb{Z}_2$

Running the Yoga in reverse, the skeletal or 'scalarless' set concepts such as the non-singleton block $\{a, b\}$ can be obtained from the corresponding quantum vector space concept such as $\alpha |a\rangle + \beta |b\rangle$ (for $\alpha, \beta \in \mathbb{C}$) by stripping away the scalars α, β and the vector space addition. But even at the set level, there is always, in effect, a 1 for the elements such as a and b in the block and a 0 for the absent terms. Those scalars 1 and 0 themselves form a field $\mathbb{k} = \mathbb{Z}_2$, the field of integers modulo 2 where $1 + 1 = 0$. Hence the skeletal set concepts for $U = \{u_1, ..., u_n\}$ can be enriched by being restated as vectors in the vector space $\mathbb{Z}_2^n \cong \wp(U)$. The non-singleton subsets in the powerset $\wp(U)$ should be thought of as the superposition of their elements, not as a subset of fully distinguished elements. This creates a 'toy' or (more respectfully) a pedagogical model of QM called "quantum mechanics over sets" or QM/sets [14] or QM/$\mathbb{Z}_2$, by taking $\mathbb{k} = \mathbb{Z}_2$ as opposed to the full quantum model by taking $\mathbb{k} = \mathbb{C}$.

The vectors in $\mathbb{Z}_2^n$ can be represented by $n \times 1$ column vectors of 0's and 1's. The vector sum is addition component-wise mod 2 (so $1 + 1 = 0$). But it is more intuitive to just represent the vectors as subsets (of a set of basis elements). For the universe set $U = \{u_1, ..., u_n\}$, the standard basis vectors in $\mathbb{Z}_2^n$, the n column vectors with only one 1 in each vector, can be paired with n singletons $\{u_i\}$ for $n = 1, ..., n$ to yield an isomorphism between $\mathbb{Z}_2^n$ and the powerset $\wp(U)$. Then the vector addition of two subsets $S, T \in \wp(U)$ is the binary set operation of symmetric difference:

$$S + T = (S - T) \cup (T - S) = (S \cap T^c) \cup (T \cap S^c) = S \cup T - (S \cap T)$$

so it is the union of the two sets with the overlap taken out. The cardinality of the symmetric difference $|S + T|$ is the *Hamming distance* in coding theory which could also be expressed by $|S + T| = \sum_{u \in U} (\chi_S(u) - \chi_T(u))^2$ which prefigures the Euclidean distance in $\mathbb{C}^n$ for full QM.

For example, let's take $U = \{a, b, c\}$ so the standard or computational basis is $\{a\} \leftrightarrow [1, 0, 0]^t$, $\{b\} \leftrightarrow [0, 1, 0]^t$, and $\{c\} \leftrightarrow [0, 0, 1]^t$ (where the superscript t transposes the row vector into a column vector). But now since U is expressed as a basis set in a vector space, we see that there are many other basis sets such as $U' = \{a', b', c'\}$ where $\{a'\} = \{a, b\}$, $\{b'\} = \{b\}$, and $\{c'\} = \{b, c\}$. Those subsets also form a basis since the U-basis can be expressed in the U'-basis as follows:

$\{a'\} + \{b'\} = \{a, b\} + \{b\} = \{a\}$;
$\{b'\} = \{b\}$; and
$\{b'\} + \{c'\} = \{b\} + \{b, c\} = \{c\}$.

Too often a "vector" is just expressed as an ordered n-tuple or $n \times 1$ column vector, but that is really a vector expressed in a certain set of n basis vectors. The abstract notion of a vector, independent of any basis set, is called a *ket* (from the Dirac notation in QM [11]). Table 4 is a *ket table* that expresses the same abstract vector or ket in each row using four different basis sets.

$\mathbb{Z}_2^3$	$U = \{a, b, c\}$	$U' = \{a', b', c'\}$	$U'' = \{a'', b'', c''\}$	$U^* = \{a^*, b^*, c^*\}$
$[1, 1, 1]^t$	$\{a, b, c\}$	$\{a', b', c'\}$	$\{a'', b'', c''\}$	$\{b^*\}$
$[1, 1, 0]^t$	$\{a, b\}$	$\{a'\}$	$\{b''\}$	$\{a^*\}$
$[0, 1, 1]^t$	$\{b, c\}$	$\{c'\}$	$\{b'', c''\}$	$\{c^*\}$
$[1, 0, 1]^t$	$\{a, c\}$	$\{a', c'\}$	$\{c''\}$	$\{a^*, c^*\}$
$[1, 0, 0]^t$	$\{a\}$	$\{a', b'\}$	$\{a''\}$	$\{b^*, c^*\}$
$[0, 1, 0]^t$	$\{b\}$	$\{b'\}$	$\{a'', b''\}$	$\{a^*, b^*, c^*\}$
$[0, 0, 1]^t$	$\{c\}$	$\{b', c'\}$	$\{a'', c''\}$	$\{a^*, b^*\}$
$[0, 0, 0]^t$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 4: Ket table giving a vector space isomorphism:
$\mathbb{Z}_2^3 \cong \wp(U) \cong \wp(U') \cong \wp(U'') \cong \wp(U^*)$ where row = ket

In full QM, i.e., QM/$\mathbb{C}$, the Dirac bracket $\langle\phi|\psi\rangle$ is defined as the (basis independent) inner product of the vectors ϕ and ψ. But there are no inner products in vector spaces over finite fields like $\mathbb{Z}_2$ so we have to use the set version of the interpretation of $\langle\phi|\psi\rangle$ which is the overlap of the states ϕ and ψ. With subsets $S, T \in \wp(U)$, we have the obvious notion of overlap, namely the intersection, so in QM/sets, we define the Dirac bracket as the cardinality of the overlap:

$$\langle S|_U T\rangle = |S \cap T|,$$

where the U subscript indicates that S and T are represented in the U-basis (since the cardinality of the overlap is not basis independent). We can take the ket $|T\rangle$ as basis independent (Table 4) but when combined with the basis-dependent *bra* $\langle S|_U$ to form the bracket (bra-ket) $\langle S|_U T\rangle$, both subsets must be represented in the same basis in order to take the cardinality $|S \cap T|$ of their intersection.

The setting of $\wp(U) \cong \mathbb{Z}_2^n$ allows us to make an interesting separation between the eigenspaces of a (diagonalizable) linear operator and a direct-sum decomposition. The only linear operators on $\wp(U) \cong \mathbb{Z}_2^n$ are projection operators $S \cap () : \wp(U) \to \wp(U)$ with eigenvalues 1 and 0 so the associated DSDs of subspaces of $\wp(U)$ are always binary $\{\wp(S), \wp(S^c)\}$. But a set level numerical attribute $g : U \to \mathbb{R}$ defines the DSD $\{\wp(g^{-1}(r))\}_{r \in g(U)}$ of $\wp(U)$ which in general is not the set of eigenspaces of a linear operator on $\wp(U)$.

The norm in QM/$\mathbb{C}$ is often defined with the notation $|\psi| = \sqrt{\langle\psi|\psi\rangle}$ but that conflicts with our notation $|S|$ for the cardinality of a set S so we will denote the (basis-dependent) norm in QM/sets as: $\|S\|_U = \sqrt{\langle S|_U S\rangle} = \sqrt{|S|}$.

The restatement of the skeletal set concepts in QM/sets is summarized in Table 5.

Set concepts	Corresponding concepts in QM/sets
Universe set U	Basis $\{\{u\}\}_{u \in U}$ for $\wp(U)$
Cardinality $|U|$	Dimension of $\wp(U)$
Subset $S \subseteq U$	Subspace $[S] = \wp(S) \cong 2^S$ of $\wp(U)$
Char. fcn. $\chi_S : U \to \mathbb{Z}_2$	Projection op. $S \cap () : \wp(U) \to \wp(U)$
Partition $\{g^{-1}(r)\}_{r \in g(U)}$ of g	DSD $\{\wp(g^{-1}(r))\}_{r \in g(U)}$ of $\wp(U)$
Value r in $g(U)$	'Eigenvalue' r assoc. with $\wp(g^{-1}(r))$
Constant set $S \subseteq g^{-1}(r)$	Eigenset S in $\wp(g^{-1}(r))$
$S, T \subseteq U$, overlap $|S \cap T|$	$\sum_{u_i \in U} \chi_S(u_i)\chi_T(u_i) = \langle S|_U T\rangle$
Norm $\|S\| = \sqrt{S}$, $\|S\|^2 = |S|$	$\|S\|_U = \sqrt{\langle S|_U S\rangle} = \sqrt{|S|}$, $\|S\|_U^2 = |S|$

Table 5: Skeletal set concepts restated in QM/sets

In QM/$\mathbb{C}$, vectors can always be normalized, but in QM/sets, normalization is only performed when probabilities are calculated. In QM/$\mathbb{C}$, if $|\psi\rangle$ was not normalized, then the probability of getting an eigenvector $|u_i\rangle$ in the measurement basis in a (non-degenerate) measurement of $|\psi\rangle$ is given by the Born Rule:

$$\Pr(|u_i\rangle \mid |\psi\rangle) = \frac{\|\langle u_i|\psi\rangle\|^2}{\|\psi\|^2}.$$

Hence the corresponding calculation in QM/sets is:

$$\Pr\left(\{u_i\}|_U S\right) = \frac{\|\langle\{u_i\}|_U S\rangle\|_U^2}{\|S\|_U^2} = \frac{|\{u_i\}\cap S|}{|S|} = \begin{cases} 1/|S| \text{ if } u_i \in S \\ 0 \text{ if } u_i \notin S \end{cases}$$

(where we assume equiprobable outcomes). That is the usual conditional probability of drawing u_i given $S \subseteq U$ with equiprobable outcomes but in QM/sets, it is interpreted as the probability of the outcome $\{u_i\}$ when measuring the superposition state S represented by the pure state density matrix $\rho(S)$ with the entries $\rho(S)_{ik} = \frac{1}{|S|}$ if $u_i, u_k \in S$ and 0 otherwise.

Given a numerical attribute $g : U \to \mathbb{R}$ which defines the DSD $\left\{\wp\left(g^{-1}(r)\right)\right\}_{r\in g(U)}$, we can consider the general case of the measurement of a superposition state S (equiprobable points) by the 'observable' or attribute g. The density matrix $\rho(S)$ is the density matrix for the partition $\{S, S^c\}$ where the points of S have probability $\frac{1}{|S|}$ and the points of the complement S^c have probability 0. By the set version of the Lüders mixture operation, the density matrix $\rho(S)$ is transformed into the density matrix $\rho\left(\{S, S^c\} \vee g^{-1}\right)$ for the mixed state $\{S, S^c\} \vee g^{-1}$ where the only blocks of non-zero probability have the form $S \cap g^{-1}(r)$ for the $r \in g(U)$. Then the probability of the obtaining the 'eigenvalue' r is the block probability of $S \cap g^{-1}(r)$ which can be expressed in the form:

$$\Pr\left(r|_U S\right) = \frac{\left\|g^{-1}(r)\cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|g^{-1}(r)\cap S\right|}{|S|}.$$

This corresponds to the formula:

$$\Pr\left(r|\psi\right) = \frac{\|P_{V_r}(\psi)\|^2}{\|\psi\|^2}$$

where $P_{V_r}(\psi)$ is the projection $P_{V_r} : V \to V$ of ψ to the eigenspace V_r for r in QM/$\mathbb{C}$, just as $g^{-1}(r) \cap S$ is the projection $g^{-1}(r) \cap () : \wp(U) \to \wp(U)$ of S to the eigenspace $\wp\left(g^{-1}(r)\right)$ in QM/sets. Another way to calculate $\Pr(r|\psi)$ in QM/$\mathbb{C}$ is:

$$\Pr\left(r|\psi\right) = \operatorname{tr}\left[P_{V_r}\rho(\psi)\right].$$

In QM/sets, the projection matrix to $\wp\left(g^{-1}(r)\right)$ is the $n \times n$ diagonal matrix $P_{g^{-1}(r)}$ with the diagonal entries $\chi_{g^{-1}(r)}(u_i)$ and the corresponding formula is:

$$\Pr\left(r|_U S\right) = \operatorname{tr}\left[P_{g^{-1}(r)}\rho(S)\right] = \sum_{u_i\in g^{-1}(r)\cap S} \frac{1}{|S|} = \frac{\left|g^{-1}(r)\cap S\right|}{|S|}.$$

Table 6 summarizes concepts and formulas in QM/sets (i.e., QM/$\mathbb{Z}_2$) and the corresponding notions in QM/$\mathbb{C}$.

Concepts in QM/$\mathbb{Z}_2$	Concepts in QM/$\mathbb{C}$
Basis $\{\{u_i\}\}_{u_i \in U}$ for $\wp(U)$	Basis $\{u_i\}_{u_i \in U}$ for $[U] = V$ over $\mathbb{C}$
Dimension of $\wp(U)$	Dimension of V
Subspace $[S] = \wp(S) \cong 2^S$ of $\wp(U)$	Subspace $[S] \cong \mathbb{C}^S$ of V
Projection op. $S \cap () : \wp(U) \to \wp(U)$	Projection op. $P_{[S]} : V \to V$
DSD $\left\{\wp\left(g^{-1}(r)\right)\right\}_{r \in g(U)}$ of $\wp(U)$	DSD $\left\{\left[g^{-1}(r)\right]\right\}_{r \in g(U)}$ from $Gu_i = g(u_i)\, u_i$
'Eigenvalue' r associated with $g : U \to \mathbb{R}$	Eigenvalue r of $G : V \to V$
Eigenset S in $\wp\left(g^{-1}(r)\right)$	Eigenvector v of F in $\left[g^{-1}(r)\right] = V_r$
$\sum_{u_i \in U} \chi_S(u_i)\, \chi_T(u_i) = \langle S\|_U T\rangle$	Inner product $\langle v\|v'\rangle$
$\\|S\\|_U = \sqrt{\langle S\|_U S\rangle} = \sqrt{\|S\|}$, $\\|S\\|_U^2 = \|S\|$	Norm $\\|v\\| = \sqrt{\langle v\|v\rangle}$
$\Pr(\{u_i\}\, \|_U S) = \frac{\\|\langle \{u_i\}\|_U S\rangle\\|_U^2}{\\|S\\|_U^2}$	$\Pr(\|u_i\rangle \mid \|\psi\rangle) = \frac{\\|\langle u_i\|\psi\rangle\\|^2}{\\|\psi\\|^2}$
$\Pr(r\|_U S) = \frac{\left\\|g^{-1}(r) \cap S\right\\|_U^2}{\\|S\\|_U^2}$	$\Pr(r\|\psi) = \frac{\left\\|P_{[g^{-1}(r)]}(\psi)\right\\|^2}{\\|\psi\\|^2}$
$\Pr(r\|_U S) = \operatorname{tr}\left[P_{g^{-1}(r)}\rho(S)\right]$	$\Pr(r\|\psi) = \operatorname{tr}\left[P_{[g^{-1}(r)]}\rho(\psi)\right]$

Table 6: Corresponding concepts and formulas in QM/$\mathbb{Z}_2$ and QM/$\mathbb{C}$

9. Analysis of the double-slit experiment

Our purpose is to obtain a better understanding of the reality described by QM by showing that the mathematics of QM is the vector (Hilbert) space version of much simpler mathematics about partitions at the set level or in the reformulation of the set level treatment in vector spaces over $\mathbb{Z}_2$. Since partitions are the logic tool to describe distinctions and indistinctions (or distinguishability and indistinguishability), this shows immediately that those concepts are key to understanding quantum level reality (more below). We have seen that pure superposition states, mixed states, and classical states can be represented in a skeletal form using the lattice of partitions.

We will use the example of the double-slit experiment to introduce the skeletal version of quantum dynamics in QM/sets. The evolution of isolated quantum states is described by unitary linear transformation which are transformations that preserve the inner product and transforms an orthonormal (ON) basis into another ON basis. Since there are no inner products on vector spaces over finite fields like $\mathbb{Z}_2$, the corresponding idea would be a linear transformation that transforms a basis set into a basis set, i.e., a non-singular linear transformation. An example would be the transformation of the U-basis where $U = \{a, b, c\}$ into the U'-basis where $U' = \{a', b', c'\}$, i.e., $\{a\} \longmapsto \{a'\} = \{a, b\}$, $\{b\} \longmapsto \{b'\} = \{b\}$, and $\{c\} \longmapsto \{c'\} = \{b, c\}$. It might be noted that the basis-dependent Dirac brackets are preserved by a basis-to-basis non-singular transformation, e.g.,

$$\langle \{a, b\} \,|_U \{a, b, c\}\rangle = |\{a, b\} \cap \{a, b, c\}| = 2 = |\{a', b'\} \cap \{a', b', c'\}| =$$
$$\langle \{a', b'\} \,|_{U'} \{a', b', c'\}\rangle$$

The task now is to use QM/sets to give a schematic or skeletal treatment of the double-slit experiment that illustrates how the quantum reality differs from our ordinary classical notion of reality. The U-basis and U'-basis will suffice. The setup is a vertical screen with only three vertical coordinates or positions: $\{a\}$ = position of upper slit, $\{b\}$ = screen between the slits, and $\{c\}$ = position of lower slit. On the right is the detection wall with the same vertical positions. On the left is a source of single particles as illustrated in Figure 5.

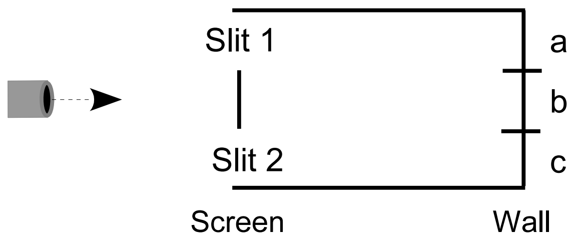

Figure 5: Skeletal setup for double-slit experiment

The particle for the emitter arrives at the screen in the pure superposition state $\{\{a, b, c\}\}$. The screen makes the distinction between the state reducing to $\{b\}$ (i.e., hitting the screen at vertical position $\{b\}$ or reducing to the smaller superposition state $\{a, c\}$. In other words, the pure superposition $\{a, b, c\}$ is transformed into the mixture $\{\{b\}, \{a, c\}\}$ (by Lüders, trivially $\{\{a, b, c\}\} \vee \{\{b\}, \{a, c\}\} = \{\{b\}, \{a, c\}\}$) and then reduces (or 'collapses') either to $\{b\}$ or $\{a, c\}$. We are interested in the case that the particle does not hit the screen at $\{b\}$ so the total state reduction is from $\{a, b, c\}$ to $\{a, c\}$ as illustrated in Figure 6 using the partition lattice.

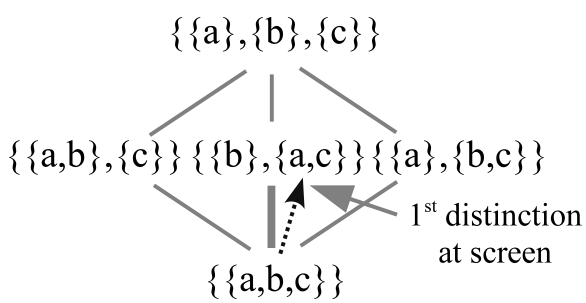

Figure 6: Screen forces the transformation $\{\{a, b, c\}\}$ to $\{\{b\}, \{a, c\}\}$

We are interested in the further evolution of the superposition $\{a, c\} = \{a\} + \{c\}$ which interprets as the superposition of going through slit 1, i.e.,$\{a\}$, and going through slit 2, i.e., $\{c\}$, where we think of each superposed state has equal amplitude. Then we need to consider two cases:

Case 1: distinctions made at the slits, i.e., detectors at one or both slits, or

Case 2: no distinctions at the slits.

In Case 1 of distinctions at the slits, the superposition $\{a,c\}$ is Lüders transformed into the mixed state $\{\{a\},\{c\}\}$ and then there is state reduction to $\{a\}$ or to $\{c\}$ at the slits each with probability $\frac{1}{2}$, e.g.,

$$\Pr\left(\{a\} \text{ at screen}|_U \{a,c\} \text{ at screen}\right) = \frac{\|\{a\}\cap\{a,c\}\|_U^2}{\|\{a,c\}\|_U^2} = \frac{|\{a\}\cap\{a,c\}|}{|\{a,c\}|} = \frac{1}{2}.$$

Since the events of going through slit 1 or slit 2 are distinguished by the detectors, we add the probabilities of the subsequent events of either the particle evolving from slit 1 or from slit 2 to the wall on the right. By the assumed dynamics, $\{a\}$ evolves to $\{a'\} = \{a,b\}$ each with probability of $\frac{1}{2}$ at the wall, e.g., $\Pr\left(\{a\} \text{ at wall}|_U \{a,b\} \text{ at wall}\right) = \frac{|\{a\}\cap\{a,b\}|}{|\{a,b\}|} = \frac{1}{2}$. Similarly, $\{c\}$ at the screen evolves to $\{c'\} = \{b,c\}$ at the wall with each position having probability $\frac{1}{2}$, e.g., $\Pr\left(\{b\} \text{ at wall}|_U \{b,c\} \text{ at wall}\right) = \frac{|\{b\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2}$. Hence we have the following chain of conditional probabilities:

$$\begin{gathered}\Pr\left(\{a\} \text{ at wall}|_U \{a,c\} \text{ at screen}\right)\\ = \Pr\left(\{a\} \text{ wall}|_U \{a,b\} \text{ wall}\right)\Pr\left(\{a,b\} \text{ wall}|_U \{a\} \text{ at scrn}\right)\Pr\left(\{a\} \text{ scrn}|_U \{a,c\} \text{ scrn}\right)\\ = \frac{1}{2}\times 1\times\frac{1}{2} = \frac{1}{4}\end{gathered}$$

where $\Pr\left(\{a,b\} \text{ at wall}|_U \{a\} \text{ at screen}\right) = 1$ by the assumed evolution. Then

$$\begin{gathered}\Pr\left(\{b\} \text{ at wall}|_U \{a,c\} \text{ at screen}\right)\\ = \Pr\left(\{b\} \text{ wall}|_U \{a,b\} \text{ wall}\right)\Pr\left(\{a,b\} \text{ wall}|_U \{a\} \text{ scrn}\right)\Pr\left(\{a\} \text{ scrn}|_U \{a,c\} \text{ scrn}\right)\\ +\Pr\left(\{b\} \text{ wall}|_U \{b,c\} \text{ wall}\right)\Pr\left(\{b,c\} \text{ wall}|_U \{c\} \text{ scrn}\right)\Pr\left(\{c\} \text{ scrn}|_U \{a,c\} \text{ scrn}\right)\\ = \left(\frac{1}{2}\times 1\times\frac{1}{2}\right)+\left(\frac{1}{2}\times 1\times\frac{1}{2}\right) = \frac{1}{2}\end{gathered}$$

where there were two distinct ways to reach $\{b\}$ at the wall so those probabilities add. And finally,

$$\begin{gathered}\Pr\left(\{c\} \text{ at the wall}|_U \{a,c\} \text{ at the screen}\right)\\ = \Pr\left(\{c\} \text{ wall}|_U \{b,c\} \text{ wall}\right)\Pr\left(\{b,c\} \text{ wall}|_U \{c\} \text{ scrn}\right)\Pr\left(\{c\} \text{ scrn}|_U \{a,c\} \text{ scrn}\right)\\ = \frac{1}{2}\times 1\times\frac{1}{2} = \frac{1}{4}.\end{gathered}$$

The probability distribution of hits at the wall is just the half-half sum of the two distributions from going through slit 1 and going through slit 2 as shown in Figure 7.

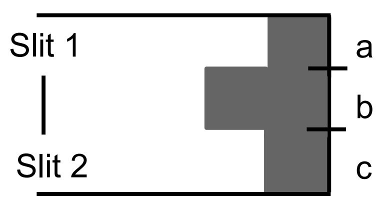

Figure 7: Probabilities at the wall with detectors at slits

In Case 1, the detectors at the slits turn the superposition $\{a,c\}$ into the mixed $\{\{a\},\{c\}\}$ at the classical level, i.e., part of classical reality represented by the discrete partition $\{\{a\},\{b\},\{c\}\}$. And then each state evolves $\{a\} \rightsquigarrow \{a'\} = \{a,b\}$ and $\{c\} \rightsquigarrow \{c'\} = \{b,c\}$ as represented in Figure 8. This evolution is from the classical level so it is classically understandable with no quantum interference effects.

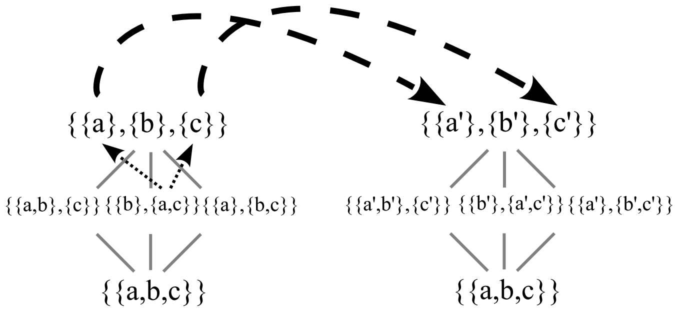

Figure 8: Detectors at slits force superposition $\{a,c\}$ to $\{a\}$ or to $\{c\}$ at the *classical* level and then separate evolution to $\{a'\} = \{a,b\}$ or $\{c'\} = \{b,c\}$

Case 2 is with no detectors at the slits so the superposed state $\{a,c\}$ is not distinguished at the screen so they evolve by the assumed dynamics, i.e., $\{a,c\} = \{a\} + \{c\} \rightsquigarrow \{a,b\} + \{b,c\} = \{a,c\}$. The probabilities are then:

$$\Pr(\{a\} \text{ at wall}|_U \{a,c\} \text{ at screen})$$
$$= \Pr(\{a,c\} \text{ at wall}|_U \{a,c\} \text{ at screen}) \Pr(\{a\} \text{ at wall}|_U \{a.c\} \text{ at wall}) = 1 \times \tfrac{1}{2} = \tfrac{1}{2}.$$

Due to the cancellation of the superposed $\{b\}$'s at the wall,

$$\Pr(\{b\} \text{ at wall}|_U \{a,c\} \text{ at screen})$$
$$= \Pr(\{a,c\} \text{ at wall}|_U \{a,c\} \text{ at screen}) \Pr(\{b\} \text{ at wall}|_U \{a,c\} \text{ at wall}) =$$
$$1 \times \tfrac{|\{b\} \cap \{a,c\}|}{|\{a,c\}|} = 0.$$

And also:

$$\Pr(\{c\} \text{ at wall}|_U \{a,c\} \text{ at screen})$$
$$= \Pr(\{a,c\} \text{ at wall}|_U \{a,c\} \text{ at screen}) \Pr(\{c\} \text{ at wall}|_U \{a.c\} \text{ at wall}) = 1 \times \tfrac{1}{2} = \tfrac{1}{2}.$$

Hence the probability distribution of hits of the particle at the wall shows the characteristic probability strips due to the interference and cancellation, i.e., $\{a,b\} + \{b,c\} = \{a,c\}$, as shown in Figure 9.

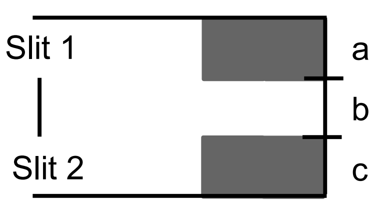

Figure 9: Probability strips due to interference with evolution of superposition $\{a, c\}$

In Case 1 of detectors at the slits, the detectors rendered the particle into a classical state $\{a\}$ or $\{c\}$ at the slits which then evolved at the classical level to $\{a'\}$ or $\{c'\}$ respectively. But in Case 2, the lack of detectors allowed the particle to stay in the superposition state $\{a, c\}$ so it did not rise to the classical level.

The partition lattice introduces the all-important idea of levels of indefiniteness in the particle's state 'below' the classical fully definite level. In classical reality (represented by the discrete partition), everything is distinguished so it represents the fully definite level of reality of our classical intuitions. But QM shows that there are other levels of reality, i.e., different levels of indefiniteness represented at the skeletal level by a partition lattice 'beneath' the discrete partition. Our classical intuition demands to know in Case 2: "Which slit did the particle go through?". But that assumes the particle had reached the classical level of definiteness; instead it stayed in Case 2 in the superposition state $\{a, c\}$ at the slits and then evolved by the non-singular dynamics *at that level of indefiniteness*

$$\{a, c\} = \{a\} + \{c\} \rightsquigarrow \{a'\} + \{c'\} = \{a, b\} + \{b, c\} = \{a, c\}.$$

This evolution at the non-classical level of indefiniteness is illustrated in Figure 10.[11]

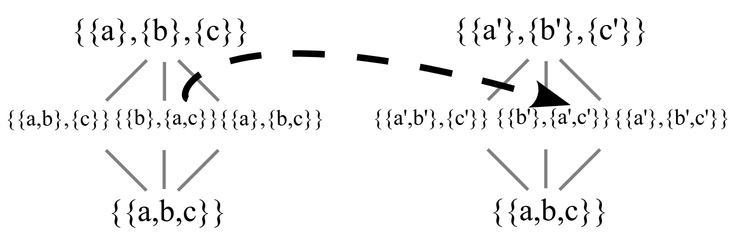

Figure 10: Evolution at a non-classical level of indefiniteness

And then $\{a, c\}$ at the wall, which is a U-basis detector, reduces with half-half probabilities to the classical states $\{a\}$ or $\{c\}$ that we can register as shown in Figure 9.

The major 'takeaway' from this analysis (illustrated using the two-slit experiment) is that there are levels of indefiniteness at the quantum level of reality that are below the classical level of full definiteness–and that evolution can take place at those lower levels of indefiniteness which is the sort of evolution that exhibits the non-classical interference

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[11] One crude metaphor for the quantum process of becoming as moving up to higher levels of definiteness might be 3D printing where the structure or object being constructed slowly becomes more definite. At any stage before it is finished, there are different ways the 3D printer could complete the construction. To push this metaphorical crutch even further, the evolution in the 2-slit experiment (without detection at the slits) is taking place before the printer has reached the classical level of the particle going through slit 1 or slit 2. A 2D metaphor for quantum becoming would be the sketch pad of the police artist that starts out completely indefinite (blank) and ends up with a rather definite portrait.

effects. There is a natural measure of the levels of indefiniteness in the lattice of partitions, namely logical entropy, which also extends to measure the indefiniteness in quantum states ([16]; [36]).[12]

10. What characterizes a measurement?

10.1. Feynman's analysis of measurement

There is considerable controversy in the philosophy of QM about just what constitutes a measurement. One theory, with some following, is that a "measurement" has to involve an interaction between the quantum level state being measured and a macroscopic system that leads to a type of "decoherence" [55]. The notion of measurement in standard von Neumann/Dirac QM involves a state reduction or collapse which is a different type of process than the unitary transformations described by the solutions to the (time dependent) Schrödinger equation. How could there be any fundamental distinction to separate unitary evolution from the non-unitary measurement?

> It indeed seems necessary to admit that "measurements" are ubiquitous, and occur even in places and times where there are no human experimenters. But it also seems hopeless to think that we will be able to give an appropriately sharp answer to the question of what, exactly, differentiates the 'ordinary' processes (where the usual dynamical rules apply) from the 'measurement-like' processes (where the rules momentarily change). [39, p. 64]
>
> [I]t seems unbelievable that there is a fundamental distinction between "measurement" and "non-measurement" processes. Somehow, the true fundamental theory should treat all processes in a consistent, uniform fashion. [39, p. 245]

What could be that "fundamental distinction" between measurements (in the sense of state reductions) and non-measurements?

Richard Feynman was notoriously sceptical about the importance of the philosophy of physics.[13] It seemed beneath him to enter into the philosophical controversies about QM. Yet, he gave examples that counteracted certain claims and stated positions that

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[12] The partition lattices herein are drawn starting with Figure 1 so that different levels are classified by different logical entropies.

[13] Feynman notoriously claimed that "philosophy of science is as useful to scientists as ornithology is to birds."

answered certain 'questions' without bothering to point out the significance to philosophers of QM. For instance, on the claim that measurement requires the interaction between microscopic and macroscopic systems, Feynman gave examples of measurements, i.e., non-unitary evolution, at the quantum level with no human involvement or interaction with a macroscopic system. But more importantly, he gave the criterion that separated those measurements from ordinary unitary evolution.

We have already seen that the set version of a projective measurement was described by the Lüders mixture operation wherein a state specified by a set level density matrix $\rho(\pi)$ was being measured by a set level numerical attribute $g : U \to \mathbb{R}$ represented by the partition $\sigma = g^{-1}$. The result of the Lüders mixture operation was to change the state $\rho(\pi)$ into the state $\rho(\pi \vee \sigma)$. For the partition join, $\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$, so it creates the new distinctions $\operatorname{dit}(\sigma) - \operatorname{dit}(\pi) = \operatorname{dit}(\sigma) \cap \operatorname{indit}(\pi)$. Thus, in the interaction represented by the join of π with σ, a distinction of σ that was not a distinction π is a new distinction. If the interaction where π is measured by σ did not create any new distinctions, then $\operatorname{dit}(\sigma) - \operatorname{dit}(\pi) = \emptyset$, i.e., $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$, and then the join operation representing the measurement has no effect. That is, $\pi \vee \sigma = \pi$, so no measurement took place in the interaction. No distinctions (or distinguishings), no measurement. Thus we see that the difference between a measurement and a non-measurement is making distinctions or not.

Feynman pointed out that making distinctions was the key part of measurement as early as 1951 [18] and restated the point clearly in his *Lectures*.

> If you could, *in principle*, distinguish the alternative *final* states (even though you do not bother to do so), the total, final probability is obtained by calculating the *probability* for each state (not the amplitude) and then adding them together. If you *cannot* distinguish the final states *even in principle*, then the probability amplitudes must be summed before taking the absolute square to find the actual probability.[20, p. 3-9]

Feynman gave several quantum level examples to explain the point. Suppose a particle was scattered off of atoms in a crystal and that interaction did not make distinctions, i.e., nature made no distinction between scattering off one atom or another. Then no measurement or state reduction takes place so unitary evolution continues which involves adding the *amplitudes* of scattering off the alternative atoms to reach the amplitude of the particle reaching a certain final state. However, if say all the atoms had spin-down and if a particle scattering off an atom flipped the spin, then the interaction does make the distinction between those alternatives and the *probabilities* of those state reductions are added to get the probability of the particle reaching the final state. ([20, Section 3.3], [21, pp. 17-8]) We previously saw in the analysis of the two-slit experiment how no measurement (i.e., no distinctions) at the slits meant adding amplitudes (Case 2) and how measurement (i.e., distinctions) at the slits meant adding probabilities (Case 1).

The difference between when a particle in a superposition state undergoes an interaction that constitutes a measurement (i.e., a state reduction to a more definite state) or not is whether or not the interaction would distinguish between the superposed eigenstates. Hence the implicit principle behind the Feynman analysis of measurement or state reduction to a more definite state is:

If the interaction distinguishes between superposed eigenstates,
then a distinction (state reduction) is made.
The State Reduction Principle

The set version of the Lüders mixture operation for a projective measurement was $\rho(\pi \vee \sigma) = \hat{\rho}(\pi) = \sum_{j'=1}^{m'} P_{C_{j'}} \rho(\pi) P_{C_{j'}}$ and $\text{indit}(\pi \vee \sigma) = \text{indit}(\pi) \cap \text{indit}(\sigma)$. Hence a superposition $(u_i, u_k) \in \text{indit}(\pi)$ survives as a superposition in the post-measurement state $\pi \vee \sigma$ if and only if $(u_i, u_k) \in \text{indit}(\sigma)$ where the projection matrices $P_{C_{j'}}$ are for the blocks $C_{j'} \in \sigma = g^{-1}$ which were the constant sets of the numerical attribute $g : U \to \mathbb{R}$. Contrapositing, a superposition $(u_i, u_k) \in \text{indit}(\pi)$ is turned into a distinction in $\pi \vee \sigma$ if and only if the interaction represented by the σ-measurement of $\rho(\pi)$ distinguished u_i and u_k, i.e., $(u_i, u_k) \in \text{dit}(\sigma)$. This is the proof of the set version of the State Reduction Principle.

Theorem (State Reduction Principle–set case). Given a non-zero off-diagonal element $\rho(\pi)_{ik}$ (representing a superposition in π), if σ distinguishes u_i and u_k, i.e., $(u_i, u_k) \in \text{dit}(\sigma)$, or in terms of the attribute values, $g(u_i) \neq g(u_k)$, then (and only then) $\rho(\pi)_{ik}$ is zeroed in the Lüders mixture operation $\hat{\rho}(\pi) = \sum_{j'=1}^{m'} P_{C_{j'}} \rho(\pi) P_{C_{j'}}$ representing the σ-measurement of the state $\rho(\pi)$.

The Hilbert space version is just the *mutatis mutandis* version of the set case, where the projections P_{C_j} to the constant sets of the numerical attribute $g : U \to \mathbb{R}$ are replaced by the projections P_{V_λ} to the eigenspaces of the Hermitian operator G.

Theorem (State Reduction Principle–Hilbert space case). Given a non-zero off-diagonal element ρ_{ik} (representing a superposition in the quantum state ρ), if the observable G distinguishes between the u_i and u_k, i.e., the eigenvectors u_i and u_k of G have different G-eigenvalues $\lambda_i \neq \lambda_k$, then (and only then) ρ_{ik} is zeroed in the Lüders mixture operation $\hat{\rho} = \sum_\lambda P_{V_\lambda} \rho P_{V_\lambda}$ representing (the interaction that is) the G-measurement of the state ρ.

And if no distinctions are made, then no measurement or state reduction takes place. A macroscopic human apparatus for measurement amplifies the quantum level state reductions to the macroscopic level, but such human level considerations play no role in the *theory* of QM.

10.2. Von Neumann's Type I and Type II processes

In von Neumann's classic treatment of QM [50], he famously divided quantum processes into two types:

Type I: A state reduction (or "collapse of the wave packet") that is part of a measurement; and

Type II: The evolution of an isolated quantum system according to Schrödinger's (time dependent) equation.

As previously noted, there is controversy about "the question of what, exactly, differentiates the 'ordinary' processes (where the usual dynamical rules apply) from the 'measurement-like' processes (where the rules momentarily change)." [39, p. 64]

We have argued that measurements or state reductions, i.e., Type I processes, can be characterized by the making of distinctions. Hence the natural characterization of a Type II process would be one that does not make distinctions between quantum states. But how can that be formulated? The extent to which two quantum states are indistinct (i.e., that they overlap) is given by their inner product or Dirac bracket (and ditto for sets in QM/sets). For instance, if there is zero indistinctness (no overlap) between two states, then they are fully distinct or orthogonal. Hence the natural characterization of a process that makes no distinctions is a process that preserves the measure of indistinctness or distinctness between states, i.e., a linear transformation that preserves inner products (a unitary transformation). This characterization of the Type II non-measurement process as a unitary transformation is also a process that evolves according to the solutions of the Schrödinger equation by Stone's Theorem ([48]; [27, p. 114]). This analysis of Type I and II processes meshes completely with Feynman's account of when to add probabilities (when alternative outcomes are distinguishable) and when to add amplitudes (when alternative outcomes are indistinguishable).

10.3. Weyl's informal description of measurement

Arthur Eddington associated quantum mechanics with a sieve.

> In Einstein's theory of relativity the observer is a man who sets out in quest of truth armed with a measuring-rod. In quantum theory he sets out armed with a sieve. [12, p. 267]

Hermann Weyl approvingly quotes Eddington about the idea of a sieve which Weyl calls a "grating." [53, p. 255] Then Weyl, in effect, uses the Yoga of Linearization to develop the idea of a grating as a set partition and as a vector space direct-sum decomposition (DSD) [53, pp. 255-257]. He starts with a numerical attribute, e.g., $g : U \rightarrow \mathbb{R}$, which defines a partition on a set or "aggregate [which] is used in the sense of 'set of elements with equivalence relation'" [53, p. 239] with blocks of equal attribute value, e.g., $\left\{g^{-1}(r)\right\}_{r \in g(U)}$. Then he moves to the quantum case where the "aggregate of n states has to be replaced by an n-dimensional Euclidean vector space" [53, p. 256. "Euclidean" is older terminology for an inner product space.]. Then he describes the vector space notion of a grating as the "splitting of the total vector space into mutually orthogonal subspaces" so that "each vector $\vec{x}$ splits into r component vectors lying in the several subspaces" [53, p. 256], i.e., a direct-sum decomposition of the space. Finally Weyl notes that "Measurement means application of a sieve or grating" [53, p. 259].

This idea of measurement as applying a grating can be illustrated as in Figure 11.

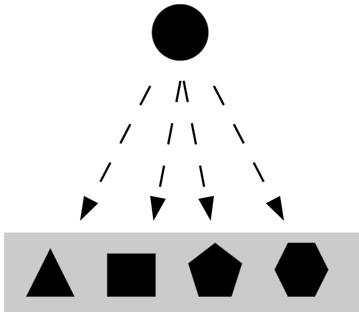

Figure 11: Indefinite shape passing through a sieve or grating to get a definite shape

The round doughball shape is considered as a visual 'superposition' of the four definite polygonal shapes. In a measurement, the doughball, with various probabilities, passes through one of the holes in the grating and takes on a definite eigen-shape. The interaction between the superposed blob and the grating or sieve forces a distinction; the indefinite blob cannot pass through the grating without "collapsing to an eigenstate." A state reduction ('measurement') from an indefinite superposition to a more definite state takes place when the particle in the superposition state undergoes an interaction that makes a distinction, i.e., that acts like a grating as in Figure 11.

11. Commuting, non-commuting, and conjugate observable operators

11.1. The partition analysis

At first, observable operators may seem to have little to do with partitions. But we have seen that the vector space version of a partition, e.g., $\sigma = g^{-1}$ for $g : U \to \mathbb{R}$, is the DSD of eigenspaces of the operators. Consider two observables $F, G : V \to V$ with the respective DSDs of eigenspaces $\{W_j\}_{j=1}^{m}$ and $\left\{V_{j'}\right\}_{j'=1}^{m'}$. For set partitions, the important operation was the partition join so let's try a join-like operation on these two DSDs by considering the set of non-zero vector spaces $\left\{W_j \cap V_{j'}\right\}$. Unlike the case of set partitions, the result of this operation on DSDs may not span the whole space V. Since the non-zero vectors in each intersection $W_j \cap V_{j'}$ are eigenvectors for both operators, let's call the space spanned by them, the *simultaneous eigenvector space* $\mathcal{SE}$.

To consider the commutativity of the operators F and G, one considers the *commutator* $[F, G] = FG - GF : V \to V$. The commutator is a linear operator so it has a kernel

$\ker[F,G]$ which is the subspace of V consisting of all the vectors v such that $[F,G](v) = 0$, the zero vector. The usual definition of commutativity is that F and G *commute* if their commutator is the zero operator, i.e., maps all vectors in V to 0, so in that case $\ker[F,G] = V$. It is a theorem [17] that:

$$\mathcal{SE} = \ker[F,G].$$

This means that the commutativity or non-commutativity of observable operators can be defined in terms of their vector-space partitions or DSDs of eigenspaces: F, G *commute* if $\mathcal{SE} = V$ and are *non-commuting* if $\mathcal{SE} \neq V$. Then we can define: F, G are *conjugate* if $\mathcal{SE} = \mathbf{0}$ (the zero space) which makes sense because then F and G have no eigenvectors in common so if a particle is in an definite state or eigenstate of one observable it cannot be in a definite state of a conjugate observable.

Since the join-like operation on the DSDs did not necessarily give a set of subspaces that span the whole space, that operation could only be considered a proper *join* of DSDs when the DSDs commute. Or as Weyl put it: "Thus combination of two gratings presupposes commutability...". [53, p. 257]

11.2. Conjugacy in QM/sets

We have seen that conjugacy can be defined in terms of DSDs and in QM/sets the only linear operators are projection operators. One can define conjugate DSDs in $\mathbb{Z}_2^3$ using the computational U-basis and the U^*-basis of Table 4 where $\{a^*\} = \{a,b\}$, $\{b^*\} = \{a,b,c\}$, and $\{c^*\} = \{b,c\}$. Consider any two one-to-one numerical attributes on U and U^*, e.g., $f : U \to \mathbb{R}$ where $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$, and similarly for $h : U^* \to \mathbb{R}$. Then the DSD determined by f is:

$$\left\{\wp\left(f^{-1}(1)\right), \wp\left(f^{-1}(2)\right), \wp\left(f^{-1}(3)\right)\right\} = \{\{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{c\}\}\}$$

and the DSD determined by $h : U^* \to \mathbb{R}$ is:

$$\left\{\wp\left(h^{-1}(1)\right), \wp\left(h^{-1}(2)\right), \wp\left(h^{-1}(3)\right)\right\} = \{\{\emptyset, \{a^*\}\}, \{\emptyset, \{b^*\}\}, \{\emptyset, \{c^*\}\}\}$$

which expressed in the computational basis is:

$$\{\{\emptyset, \{b,c\}\}, \{\emptyset, \{a,b,c\}\}, \{\emptyset, \{b,c\}\}\}.$$

Then it is easily seen that all the intersections of the subspaces of the two DSDs have only the zero vector (i.e., the null set in the set rendition of $\mathbb{Z}_2^3$) in common. Hence those two DSDs are conjugate.

It is also possible to have conjugate (projection) operators in $\mathbb{Z}_2^n$ for even numbers $n \geq 4$. For $n = 4$, consider the usual U-basis $= \{\{a\}, \{b\}, \{c\}, \{d\}\}$ and $\hat{U}$-basis $= \left\{\{\hat{a}\}, \left\{\hat{b}\right\}, \{\hat{c}\}, \left\{\hat{d}\right\}\right\} = \{\{b,c,d\}, \{a,c,d\}, \{a,b,d\}, \{a,b,c\}\}$ of $\mathbb{Z}_2^4$ where $\{\hat{a}\} =$

$\{b,c,d\}$,..., $\left\{\hat{d}\right\} = \{a,b,c\}$. Let $f = \chi_{\{a,b\}} : U \rightarrow \mathbb{Z}_2$ so $f(a) = f(b) = 1$ and $f(c) = f(d) = 0$ with the eigenspaces $\wp\left(f^{-1}(1)\right) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$ and $\wp\left(f^{-1}(0)\right) = \{\emptyset, \{c\}, \{d\}, \{c,d\}\}$. Let $g = \chi_{\{\hat{b},\hat{c}\}} : \hat{U} \rightarrow \mathbb{Z}_2$ so $g\left(\hat{b}\right) = g(\hat{c}) = 1$ and $g(\hat{a}) = g\left(\hat{d}\right) = 0$ with the eigenspaces $\wp\left(g^{-1}(1)\right) = \left\{\emptyset, \left\{\hat{b}\right\}, \{\hat{c}\}, \left\{\hat{b},\hat{c}\right\}\right\}$. To consider the intersections of the eigenspaces, we need to restate the $\wp\left(\hat{U}\right)$ eigenspaces in the computational basis. They are: $\wp\left(g^{-1}(1)\right) = \{\emptyset, \{a,c,d\}, \{a,b,d\}, \{b,c\}\}$ and $\wp\left(g^{-1}(0)\right) = \left\{\emptyset, \{\hat{a}\}, \left\{\hat{d}\right\}, \left\{\hat{a},\hat{d}\right\}\right\} = \{\emptyset, \{b,c,d\}, \{a,b,c\}, \{a,d\}\}$. The two direct-sum decompositions are:

$$\begin{gathered}\left\{\wp\left(f^{-1}(1)\right), \wp\left(f^{-1}(0)\right)\right\} \\ = \{\{\emptyset, \{a\}, \{b\}, \{a,b\}\}, \{\emptyset, \{c\}, \{d\}, \{c,d\}\}\} \\ \text{and} \\ \left\{\wp\left(g^{-1}(1)\right), \wp\left(g^{-1}(0)\right)\right\} \\ = \{\{\emptyset, \{a,c,d\}, \{a,b,d\}, \{b,c\}\}, \{\emptyset, \{b,c,d\}, \{a,b,c\}, \{a,d\}\}\}.\end{gathered}$$

Then all the four intersections of eigenspaces have only $\emptyset$ in common so $\mathcal{SE} = \{\emptyset\} = \mathbf{0}$ and f and g are conjugate.

11.3. Complete Sets of Commuting Observables or CSCOs: partition joins at work

The set case: The join of two or more partitions is the least upper bound of the partitions in the refinement partial ordering so the join is more (or equally) refined than the partitions in the join. The limit in joins is when the join has the maximum refinement of the discrete partition $\mathbf{1}_U$ where the blocks have cardinality of one. When the join of a set of partitions, *compatible* in the sense of being defined on the same set U, is $\mathbf{1}_U$, and the partitions are all inverse-images of numerical attributes, then that set is a *Complete Set of Compatible Attributes* or *CSCA*, and then the elements of the set U can be uniquely characterized by an ordered set of the attribute values.

For example, take $U = \{a,b,c,d\}$ and let $\pi = \{\{a,b\}, \{c,d\}\}$ and $\sigma = \{\{a,c\}, \{b,d\}\}$ be the inverse-image partitions of the attributes $f, g :\rightarrow \mathbb{R}$ where $f(a) = f(b) = 0$, $f(c) = f(d) = 1$ and $g(a) = g(c) = 2$ and $g(b) = g(d) = 3$. Then f and g form a CSCA since $\pi \vee \sigma = \mathbf{1}_U$, and the elements of U can be characterized by the ordered pairs of (f,g)-values. The pairs for a, b, c, and d are respectively: $(0,2)$, $(0,3)$, $(1,2)$, and $(1,3)$.

The quantum case: By the Yoga of Linearization, all this carries over, *mutatis mutandis*, to the quantum case. Dirac's *Complete Set of Commuting Operators* or *CSCO* [11] is a set of commuting operators so that the join of their DSDs is a DSD with subspaces of dimension one (instead of subsets of cardinality one as in a CSCA). Then the simultaneous eigenvectors spanning the space can be characterized by ordered tuples of the eigenvalues. For example in $\mathbb{C}^4$, let $|a\rangle$,..., $|d\rangle$ be an orthonormal basis and consider the two operators F and G defined by the above f and g. That is: $F|a\rangle = f(a)|a\rangle$,..., $F|d\rangle = f(d)|d\rangle$ and similarly for G and g. Then $|a\rangle$,..., $|d\rangle$ is a set of simultaneous eigenvectors that span the whole space so F and G commute. The four subspaces in the join of the two DSDs of eigenspaces are the subspaces of dimension one generated by the four basis vectors. Hence F and G form a CSCO and the four simultaneous eigenvectors $|a\rangle$,..., $|d\rangle$ can be characterized by the respective ordered pairs of F, G-eigenvalues: $(0, 2)$, $(0, 3)$, $(1, 2)$, and $(1, 3)$.

12. Indistinguishability of like particles: a partition analysis

The indistinguishability of like particles (same intrinsic and extrinsic properties) is *prima facie* evidence for there being levels of objective indefiniteness where there is no 'digging deeper' to discover or create differences. A CSCO is a complete description of the quantum state of a particle so particles sharing that description are objectively indistinguishable.

> In quantum mechanics, however, identical particles are truly indistinguishable. This is because we cannot specify more than a complete set of commuting observables for each of the particles; in particular, we cannot label the particle by coloring it blue. [43, p. 446]

The partition analysis distills out the essence of the matter in a skeletal form. The simplest example is for two particles each with two quantum states which we can signify as h (heads) and t (tails). In QM/$\mathbb{C}$, the analysis would use the tensor product but at the skeletal level of sets as in QM/sets, the tensor product is the same as the direct or Cartesian product of sets. The Cartesian product of the possible states for the two particles is:

$$\{h, t\} \times \{h, t\} = \{(h, h), (h, t), (t, h), (t, t)\}.$$

Hence we can use the lattice of partitions on a four element set $\{a, b, c, d\}$ as in Figure 3 by making the identifications: $a = (h, h)$, $b = (h, t)$, $c = (t, h)$, and $d = (t, t)$. Then we obtain the lattice of partitions in Figure 12 where we have used the shorthand where the superposition $\{(h, t), (t, h)\}$ is abbreviated $(h, t)(t, h)$ and different blocks are separated with the semi-colon so a partition $\{\{(h, h)\}, \{(h, t), (t, h)\}, \{(t, t)\}\}$ is abbreviated as $\{(h, h); (h, t)(t, h); (t, t)\}$.

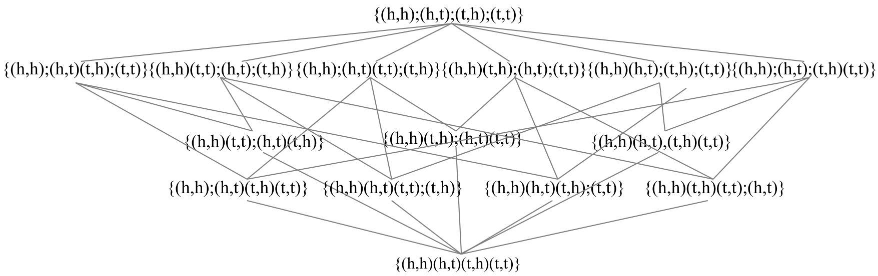

Figure 12: Lattice of partitions on four element set $U = \{(h,h),(h,t),(t,h),(t,t)\}$

Then to consider the lattice of possible states for two bosons, we cross out all the states that are not symmetric under the permutation of the two particles (i.e., interchange the left and right elements in the ordered pairs). For instance, the state $\{(h,h)(t,t);(h,t);(t,h)\}$ is not symmetric since the permutation transforms the state $\{(h,t)\}$ into the state $\{(t,h)\}$ but there is no fact-of-the-matter difference between the permuted states so the partition that treats them as distinct blocks is disallowed as a possible physical state. The crossing out of all the non-symmetric states is illustrated in Figure 13.

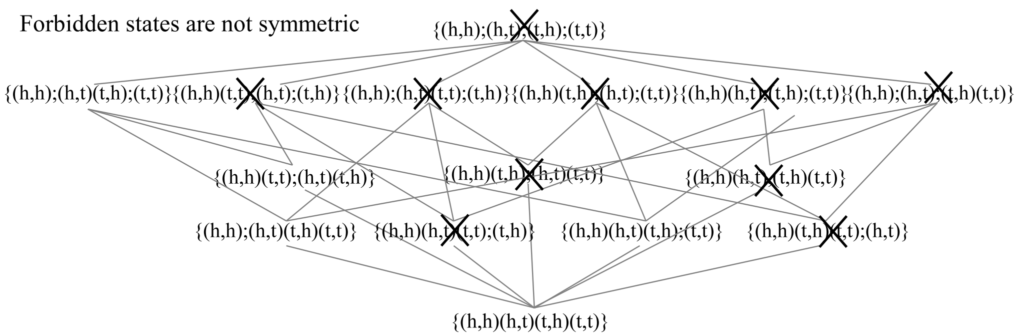

Figure 13: Lattice with non-symmetric states crossed out

Deleting the forbidden states gives the sublattice of possible boson mixed and pure states in Figure 14.

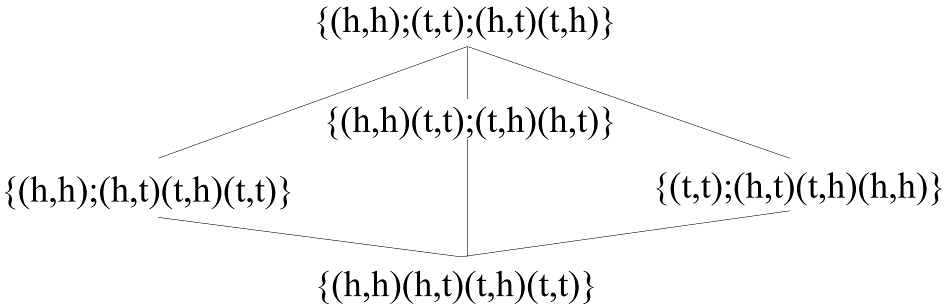

Figure 14: Lattice of possible mixed and pure boson states (two particles and two particle states)

Note that the classical state is disallowed since there is no factual difference between $\{(h,t)\}$ and $\{(t,h)\}$ so they cannot be treated as distinct states. Nature cannot become more definite. Those three states in the mixture $\{(h,h)\,;(t,t)\}\,;(h,t)\,(t,h)\}$ are the most definite boson states; the remaining indefiniteness in the superposition $\{(h,t)\,(t,h)\}$ is objective.

For fermions, repetitions are not allowed which is the Pauli exclusion principle. When we cross out all the mixtures that involved the repetitions $\{(h,h)\}$ or $\{(t,t)\}$, then the only possible fermion state is the superposition $\{(h,t)\,(t,h)\}$.

In QM/$\mathbb{C}$, when permutation symmetry was not required, there were four eigenstates corresponding to $\{(h,h)\}$, $\{(h,t)\}$, $\{(t,h)\}$, $\{(t,t)\}$ which form a basis for the four-dimensional space $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the three-dimensional subspace generated by the possible boson states $\{(h,h)\}$, $\{(t,t)\}$, $\{(h,t)\,(t,h)\}$, and the one-dimensional subspace generated by the only possible fermion state $\{(h,t)\,(t,h)\}$.

To illustrate the difference between the classical Maxwell-Boltzmann (MB), Bose-Einstein (BE), and Fermi-Dirac (FD) statistics, compute the probability of flipping two coins of the same type and getting the state of different outcomes. Hence there are two particles of the same type and two states h and t like two coins with heads and tails as the states. What is the probability that one "coin" will be "heads" and the other "tails"–with equal probability assigned to each distinct state?

- Classical coins: $\Pr_{MB}\left(\{(h,t)\,,(t,h)\}\right) = \frac{1}{2}$.
- Boson coins: $\Pr_{BE}\left(\{(h,t)\,(t,h)\}\right) = \frac{1}{3}$.
- Fermion coins: $\Pr_{FD}\left(\{(h,t)\,(t,h)\}\right) = 1$.

The two fermion coins *have* to be in different states (i.e., with probability 1) which illustrates the Pauli exclusion principle. In the bosonic case, the two classical outcomes $\{(h,t)\}$ and $\{(t,h)\}$ differ only by a permutation of particles of the same type so that counts only as one state out of three equiprobable states. The skeletal partition-version of the 3-dimensional bosonic subspace was illustrated in Figure 14. The probability of getting the same outcomes $\{(h,h)\}$ or $\{(t,t)\}$ is $\frac{2}{3}$ in the bosonic case in comparison with the classical MB probability of $\frac{1}{2}$ which illustrates the "social" tendency of bosons to "want" to be in the same state.

13. Conclusions

Our method has been to show that the mathematics of QM comes from the vector (Hilbert) space version of the mathematics of partitions. Since partitions are the logical level tool to describe distinctions and indistinctions, this shows that the concepts of distinctions and indistinctions or distinguishability and indistinguishability are the key concepts to analyze QM, e.g., following Feynman to separate von Neumann's type I and type II processes or measurement from non-measurements.

> Feynman's approach is based on the contrast between processes that are *distinguishable* within a given physical context and those that are *indistinguishable* within that context. A process is distinguishable if some record of whether or not it has been realized results from the process in question; if no record results, the process is indistinguishable from alternative processes leading to the same end result. [47, p. 314]

By correlating the set level mathematics of partitions with concepts in full quantum mechanics, we have shown that partitions provide a skeletal set level model of quantum notions. The key notion of becoming is the change or jump from an indefinite state to a more definite state by being in-formed with distinctions.[14] In particular, the lattice of partitions provides a skeletal model of the different levels of definiteness and indefiniteness corresponding to classical states, mixed states (consisting of orthogonal pure states), and pure states. The discrete partition has fully distinct states so it represents the classical state of affairs and thus it uniquely satisfies the partition logic version of the Principle of the Identity of Indistinguishables.

The set level concepts can also be viewed as resulting from stripping away the scalars from the quantum vectors. If one also strips away the vector space notion of addition then one has the skeletal set model. But if one retains the vector space addition, then the set level concepts can be represented in the vector space where the vectors are naturally interpreted as sets, namely $\mathbb{Z}_2^n \cong \wp(U)$. Then the set level material yields the pedagogical model of quantum mechanics over sets (QM/sets or QM/$\mathbb{Z}_2$).

This skeletal model was used to analyze the two-slit experiment. Our classical intuitions assume a fully definite world without the objective indefiniteness of superposition states–as evidenced in the question: "Which slit does the particle go through when there are no detectors at the slits?". But one crucial takeaway was that quantum states do not have to rise to the classical level in order to evolve. At the lower level of definiteness than the classical level, the superposition state of the particle, $|\text{going through slit } 1\rangle + |\text{going through slit } 2\rangle$, evolves showing the pattern of interference. In that interference phenomenon, the 'mystery' results from the invalid classical assumption that evolution can only occur at the classical level of fully distinct states. With no distinctions at the slits, the particle's definiteness does not rise to the classical level so it unitarily evolves from that indefinite state to show interference before registering on the wall.

The two basic quantum notions that needed to be accounted for are the notions of

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[14] Perhaps this is how to interpret John A. Wheeler's "it from bit" speculations [54].

quantum states and quantum observables. We saw that both notions can be modeled at the set level by partitions (or equivalence relations) with added structure. There are two equivalent ways to model quantum states, state vectors or density matrices. If there are point probabilities for the points in the universe set, then an equivalence relation on the set is naturally modeled by a density matrix that prefigures the orthogonal decomposition of a density matrix in the math of QM.

$$\rho(\pi) = \sum_{j=1}^{m} \Pr(B_j) |b_j\rangle \langle b_j| \text{ prefigures } \rho = \sum_{i=1}^{n} \lambda_i |u_i\rangle \langle u_i| .$$

Quantum States

If instead we consider a partition as the inverse-image partition of a numerical attribute, then that suffices to define an observable operator in full QM where the blocks in the partition correspond to the eigenspaces of the observable and the partition itself corresponds to the direct-sum decomposition of the vector space into eigenspaces.

$$\sigma = \biguplus_{r \in g(U)} g^{-1}(r) \text{ prefigures } V = \oplus_\lambda V_\lambda$$

Quantum Observables

If the density matrix $\rho(\pi)$ is for the equivalence relation $\operatorname{indit}(\pi)$ and the numerical attribute's inverse-image partition is $\sigma = g^{-1}$, then the set level measurement of π by σ is described by the set level version of the Lüders mixture operation that describes (projective) measurement in full QM. The result is the density matrix $\rho(\pi \vee \sigma)$ for the partition join $\pi \vee \sigma$. Thus the Lüders mixture operation of measurement in QM also has a set level version in the partition join operation.

$$\rho(\pi \vee \sigma) = \sum_{r \in g(U)} P_{g^{-1}(r)} \rho(\pi) P_{g^{-1}(r)} \text{ prefigures } \hat{\rho} = \sum_{\lambda} P_{V_\lambda} \rho P_{V_\lambda}$$

Quantum Measurements

Hence quantum states, quantum observables, and quantum measurement all have skeletal level models using partitions. That is natural since the reality described by QM is non-classical in having objective indefiniteness which is modeled at set level by non-singleton blocks in a partition and in full QM by superposition states. These results together show that the mathematics[15] of QM is the vector (Hilbert) space version of the

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[15] Our target is the mathematics of QM, not the physics of QM which is obtained by a quantization of classical physics.

mathematics, or indeed, the logic of partitions.[16] All this is not an accident or coincidence. The tools of the logic of partitions such as the lattice of partitions and the partition join operation provide a skeletal model of quantum mechanics.

Our ultimate goal is to better understand the non-classical quantum level world described by quantum mechanics. The simplified models in terms of set partitions or in QM/sets show that the world described by QM is a world characterized by the non-classical notion of objective indefiniteness in superposition states. This partition approach bears out the views of Abner Shimony.

> From these two basic ideas alone – indefiniteness and the superposition principle – it should be clear already that quantum mechanics conflicts sharply with common sense. If the quantum state of a system is a complete description of the system, then a quantity that has an indefinite value in that quantum state is objectively indefinite; its value is not merely unknown by the scientist who seeks to describe the system. Furthermore, since the outcome of a measurement of an objectively indefinite quantity is not determined by the quantum state, and yet the quantum state is the complete bearer of information about the system, the outcome is strictly a matter of objective chance – not just a matter of chance in the sense of unpredictability by the scientist. Finally, the probability of each possible outcome of the measurement is an objective probability. Classical physics did not conflict with common sense in these fundamental ways. [44, p. 47]

Instead of adding new hidden variables (Bohmian mechanics), new stochastic equations (spontaneous collapse), or new other worldly interpretations of measurement (many-worlds), Shimony suggested that we should interpret the formalism or mathematics of QM *literally* to try to better understand quantum reality.

> These statements ... may collectively be called "the Literal Interpretation" of quantum mechanics. This is the interpretation resulting from taking the formalism of quantum mechanics literally, as giving a representation of physical properties themselves, rather than of human knowledge of them, and by taking this representation to be complete. [46, pp. 6-7]

Our analysis suggests that the mathematics and logic of partitions is the skeleton key to unlock a better *literal* understanding of quantum reality. The "formalism of quantum mechanics" is the Hilbert space version of the set level partition mathematics to describe indefiniteness and definiteness. That is the Objective Indefiniteness (or Literal)

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16 Further analysis, e.g., logical entropy, group representation theory, and proofs are developed in [17].

Interpretation of quantum mechanics, the use of the mathematics of distinctions and indistinctions to better understand the reality described so well by quantum mechanics.

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