Chapter 11
Valuation Rings: A Better Algebraic Treatment of Boolean Algebras

Introduction

A Boolean "algebra" is not an ordinary plus-and-times algebra: it is only an algebra in the sense of "universal algebra." In order to construe a Boolean algebra as a plus-and-times algebra, it can be interpreted as a Boolean ring. But there are two ways to interpret a Boolean algebra as a Boolean ring, and the second interpretation appears as little more than an oddity. Boolean duality is expressed in ad hoc nonalgebraic terms since complementation is not a ring homomorphism in the Boolean ring. Boolean duality is thought to be dependent on the "zero-one nature" of Boolean rings.

Certain rings, called "valuation rings," have been defined by Gian-Carlo Rota for the purpose of generalizing inclusion-exclusion calculations in combinatorial theory [Rota 1971]. These rings turn out to be a natural setting to formulate and to generalize the properties of Boolean algebras using ordinary plus-and-times algebras. One key to this development is that the minimal element z of the Boolean algebra is not the zero of the valuation ring constructed from the Boolean algebra. Boolean duality is captured in two multiplications on the ring (a meet-multiplication and a join-multiplication). The complementation is a ring isomorphism that carries one multiplication into the other. This generalization of Boolean duality holds when the ring of coefficients is an arbitrary commutative ring with unity (while the classical Boolean duality takes the ring to be $\mathbf{Z}_2$). When the ring of coefficients is $\mathbf{Z}_2$, then the two Boolean rings associated with the Boolean algebra are obtained as quotients of the valuation ring. The valuation ring also linearizes the Boolean algebra in the sense that the meet, join, and complementation all become linear (or bilinear) transformations.

Propositional logic deals with the special case of free Boolean algebras. The main characterization theorem shows that the valuation rings of free Boolean algebras can be constructed as special types of polynomial rings. Thus the generalized Boolean duality can be applied to these polynomials so that each polynomial has a dual polynomial. The theorems of propositional logic such as the completeness theorem can be proven and generalized using elementary reasoning with polynomials.

Rota's Valuation Rings

Let L be a distributive lattice with maximal element u and minimal element z, and let A be a ring (always commutative with unity 1). Let F(L,A) be the free A-module on the elements of L, which consists of all the finite formal sums

$$\Sigma a_i x_i \text{ for } a_i \in A,\ x_i \in L.$$

A ring structure is established on F(L,A) by defining multiplication as $x \cdot y = x \wedge y$ for lattice elements $x,y \in L$ and then extending by linearity to all the elements of F(L,A). Let J be the submodule generated by all the elements of the form

$$x \vee y + x \wedge y - x - y \text{ for } x,y \in L.$$

LEMMA: J is an ideal.

Proof: Since J is a submodule, it is sufficient to check that J is closed under multiplication by lattice elements w. Now

$$w(x \vee y + x \wedge y - x - y) = w \wedge (x \vee y) + w \wedge (x \wedge y) - w \wedge x - w \wedge y.$$

Since L is a distributive lattice, w distributes across the join and meet to yield:

$$w(x \vee y + x \wedge y - x - y) = (w \wedge x) \vee (w \wedge y) + (w \wedge x) \wedge (w \wedge y) - w \wedge x - w \wedge y$$

which is a generator of J.

The *valuation ring* of L with values in A is defined as:

$$V(L,A) = F(L,A)/J.$$

A *valuation* on L with values in an abelian group G is a function $\nu: L \rightarrow G$ such that

$$\nu(x \vee y) + \nu(x \wedge y) = \nu(x) + \nu(y) \text{ for all x and y in L.}$$

Equation 11.1. Valuation on a Lattice

The injection L→ V(L,A) is a valuation and it is universal for valuations on L with values in an A-module.

THEOREM 1: Let M be an A-module and let ν:L→M be a valuation. Then there exists a unique linear transformation (i.e., an A-module homomorphism) ν^*:V(L,A)→M such that the following diagram commutes.

$$\begin{tikzcd}
L \arrow[r] \arrow[dr, "\nu"'] & V(L,A) \arrow[d, "\nu^*"] \\
 & M
\end{tikzcd}$$

Figure 11.1. L→V(L,A) as the universal valuation on L

Proof: By the universality property of the free module F(L,A), there is a unique linear transformation γ:F(L,A)→M such that the left-hand triangle in the following diagram commutes.

$$\begin{tikzcd}
L \arrow[r] \arrow[drr, "\nu"'] & F(L,A) \arrow[r] \arrow[dr, "\gamma"] & V(L,A) \arrow[d, "\nu^*"] \\
 & & M
\end{tikzcd}$$

Figure 11.2. Proof of Theorem 1

Since ν is a valuation, the kernel of γ contains J, so γ extends to a unique linear transformation ν^*:V(L,A)→M as desired.

For any x in L, $u \cdot x = u \wedge x = x$ so the maximal element u of L serves as the unity of the valuation ring V(L,A) (although the minimal element z is not the zero of the ring). In the situation of Theorem 1, if the A-module M is also an A-algebra and if the multiplicative condition $\nu(x \wedge y) = \nu(x) \cdot \nu(y)$ holds for all x and y in L, then the factor map ν^* is an A-algebra homomorphism. The map L→A that carries each lattice element to 1 is a valuation satisfying the multiplicative condition, so there is an A-algebra homomorphism ε: $V(L,A) \to A$ that takes $\Sigma a_i x_i$ to Σa_i. Hence $V(L,A)$ is an augmented algebra with the augmentation ε. Since $z \cdot \Sigma a_i x_i = z \cdot \Sigma a_i$, the minimal element z of the lattice functions as the "integral" of the augmented valuation ring that computes the augmentation $\varepsilon(\Sigma a_i x_i) = \Sigma a_i$ of any ring element.

As in any augmented algebra, there is another natural multiplication that can be put on the A-module structure of V(L,A) in order to obtain a ring:

$$f \vee g = \varepsilon(f)g + \varepsilon(g)f - fg \text{ for all } f \text{ and } g \text{ in } V(L,A).$$

Equation 11.2. Dual Join-Multiplication on V(L,A)

The join notation for this dual multiplication is appropriate since if f and g are lattice elements, then $\varepsilon(f)g + \varepsilon(g)f - fg = g + f - f \wedge g = f \vee g$ is the join of f and g. In the dual valuation ring endowed with the join-multiplication, the roles of u and z are reversed, i.e., z is the unit of the ring and u is the integral.

Generalized Boolean Duality

The dual role of the meet and join multiplications extends Boolean duality to arbitrary valuation rings. This duality is realized by the *complementation* endomorphism $\tau: V(L,A) \to V(L,A)$, which can be defined on the valuation ring by

$$\tau(f) = \varepsilon(f)(u+z) - f$$

Equation 11.3. Complementation Endomorphism on V(L,A)

for any f in V(L,A) (even if the original lattice L was not complemented). If x is a lattice element, then we have $\tau(x) = u+z-x$ so $\tau(u) = z$ and $\tau(z) = u$. Moreover, if x does have a complement $\neg x$ in L, then $\tau(\neg x) = \varepsilon(\neg x)(u+z) - \neg x = x \vee \neg x + x \wedge \neg x - \neg x = x$. Complementation is also equal to its own inverse in the sense that $\tau(\tau(f)) = f$ for any f in V(L,A).

Let $(V(L,A),\wedge)$ denote the valuation ring with the usual meet-multiplication, and let $(V(L,A),\vee)$ be the ring with the join-multiplication. Then complementation

$$\tau:(V(L,A),\wedge) \to (V(L,A),\vee)$$

is an anti-isomorphism of augmented algebras that interchanges the two multiplicative structures

$$\tau(f \cdot g) = \tau(f) \vee \tau(g)$$

and which commutes with the augmentation, i.e., $\varepsilon(\tau(f)) = \varepsilon(f)$.

A function $\varphi:V(L,A)\to V(L,A)$ is *linear* if $\varphi(ax+by) = a\varphi(x)+b\varphi(y)$ for all a,b in the ring A. A function $\varphi:V(L,A)\times V(L,A)\to V(L,A)$ is *bilinear* if $\varphi(ax+by,w) = a\varphi(x,w)+b\varphi(y,w)$ and similarly for the second argument for all a,b in A. The bilinearity of meet-multiplication is just the distributivity law. But the join and complementation are not bilinear or linear as operations on a Boolean ring (considered as a $\mathbf{Z}_2$-module). The valuation ring construction linearizes all these operations. Both multiplications are bilinear and the complementation is linear on arbitrary valuation rings V(L,A).

Part of the "trick" that allows the duality represented by the complementation anti-isomorphism is the fact that z is not the zero of the ring V(L,A). A valuation ν is *normalized* if ν(z) = 0. Given a valuation ring V(L,A), the ring obtained by setting z equal to 0, i.e., V(L,A)/(z), will be called the *normalized valuation ring*. If "normalized valuation" is substituted for "valuation," then Theorem 1 will describe the universality property enjoyed by normalized valuation rings.

When L is a Boolean algebra B and A = 2 (= $\mathbf{Z}_2$), then $(V(B,2),\wedge)/(z)$ is simply B construed in the usual manner as a Boolean ring:

addition	= exclusive or (= nonequivalence)
multiplication	= meet
unity (1)	= u
zero (0)	= z.

This choice of the meet-multiplication breaks the symmetry on V(B,2) and ignores the other multiplication. But as Herbrand pointed out, a Boolean algebra can also be interpreted as a Boolean ring in another way with a join-multiplication:

addition	= equivalence

multiplication = join

unity (1) = z

zero (0) = u

[see Church 1956, 103-4 for a history of these two interpretations of a Boolean algebra as a Boolean ring]. The usual Boolean ring-theoretic treatment of Boolean algebras made no particular sense out of this alternative definition; it was just an odd footnote. This other Boolean ring can be obtained as $(V(B,2),\vee)/(u)$, the quotient of the valuation ring with the join-multiplication by setting the maximal element u equal to zero.

The (prequotient) valuation ring V(B,2) maintains the symmetry by carrying both ring structures. The two interpretations of B as a Boolean ring can be obtained as the two symmetry-breaking quotients that set either z or u equal to 0. By choosing one of the quotients, one loses not only the other multiplication but also the augmentation and the complementation anti-isomorphism.

The Boolean duality principle for Boolean algebras is usually formulated in an ad hoc nonalgebraic manner by showing that any theorem remains valid under the interchange of the meet and join, and the interchange of the minimal element z and the maximal element u. On the (prequotient) valuation ring V(B,2), this Boolean duality principle is realized algebraicly by the complementation ring anti-isomorphism, which interchanges the meet and join multiplications as well as the minimal and maximal elements.

This formulation of Boolean duality generalizes with the Boolean algebra B replaced by any distributive lattice L and with 2 (= $\mathbf{Z}_2$) replaced by any commutative ring A. Any equation in V(L,A) can be dualized by applying the complementation anti-isomorphism τ. Thus "Boolean" duality generalizes far beyond the two-valued case where the ring of coefficients is $\mathbf{Z}_2$. For instance, the valuation ring $V(B,\mathbf{Z}_n)$ might prove useful for an algebraic treatment of multivalued logic. This generalization of Boolean duality to arbitrary valuation rings was due to Ladnor Geissinger, whose papers [1973] should be consulted for further analysis.

Valuation Rings on Free Boolean Algebras

Let B(P) be the free Boolean algebra on the set P [nota bene: not the power set Boolean algebra of P]. The conventional algebraic treatment of propositional logic is based on the free Boolean algebra B(P) where P is the set of propositional variables. This treatment can be generalized to V(B(P),A) where A is any commutative ring. Instead of constructing B(P) and then V(B(P),A), we will give a direct characterization of the valuation rings V(B,A) for free Boolean algebras B as special types of polynomial rings.

Let P be a set and let z be an element not in P. For any commutative ring A (always with unity 1), let $A[P\cup\{z\}]$ be the polynomial ring generated by the elements of P and the element z as indeterminates. Let I be the ideal generated by the elements p^2-p and $pz-z$ for all p in P and the element z^2-z. Let $A^*[P] = A[P\cup\{z\}]/I$ be the quotient so that in A*[P], the elements of P and z are all idempotent (i.e., $p^2 = p$ and $z^2 = z$) and z absorbs the elements of P (i.e., $pz = z$). Polynomial rings A*[P] generated in this way from a set P and a ring A will be called *augmented idempotent polynomial rings*.

Since z absorbs the generators of A*[P], the product fz for any f in A*[P] has the form $\varepsilon(f)z$ for some scalar $\varepsilon(f)$ in the ring A. Indeed, this defines $\varepsilon:A^*[P]\to A$, which is the *augmentation* ring homomorphism. The join-multiplication is defined on A*[P] by $f\vee g = \varepsilon(f)g + \varepsilon(g)f - fg$, and the complementation homomorphism is defined by $\tau(f) = \varepsilon(f)(1+z) - f$ for any f and g in A*[P].

Our main characterization theorem is that the augmented idempotent polynomial rings are precisely the valuation rings on free Boolean algebras [see appendix for the proof].

CHARACTERIZATION THEOREM: $A^*[P] \cong V(B(P),A)$ for any set P and ring A.

The *idempotent polynomial rings* $A[P]/(p^2-p)$ are constructed without the extra element z and by dividing by the ideal generated by the element p^2-p for p in P.

COROLLARY 1: The idempotent polynomial rings are the normalized valuation rings of free Boolean algebras, i.e.,

$$A[P]/(p^2–p) \cong V(B(P),A)/(z)$$

[for a direct proof, see Ellerman and Rota 1978].

COROLLARY 2: The free Boolean algebras construed as Boolean rings (meet-multiplication) are the idempotent polynomial rings over $\mathbf{Z}_2$, i.e.,

$$\mathbf{Z}_2[P]/(p^2–p) \cong V(B(P),\mathbf{Z}_2)/(z) \cong B(P).$$

These normalized valuation rings on free Boolean algebras (i.e., the "unaugmented" idempotent polynomial rings) do not have the augmentation homomorphism, the complementation isomorphism, or the join-multiplication. Hence we will continue the development using the augmented idempotent polynomial rings.

Duality in Augmented Polynomial Rings

Each element of A*[P] is an equivalence class of polynomials. Since all the indeterminates are idempotent and z absorbs the other indeterminates, each class can be represented by a polynomial f in the *standard form,* which is a sum of first degree monomials $ax_1 \cdots x_n$ involving only x's in P, a constant term $a_u 1$ and a *co-constant* term $a_z z$ for a, a_u and a_z in A:

$$f = \Sigma\{ax_1 \cdots x_n\} + a_z z + a_u 1.$$

Equation 11.4. Standard Form Polynomial in Augmented Idempotent Polynomial Rings

The "Boolean duality principle" on augmented idempotent polynomial rings expresses the fact that complementation is an anti-isomorphism that interchanges the two multiplications, and interchanges 1 and z. Any polynomial transforms into a dual polynomial, and any polynomial equation transforms into a dual polynomial equation. The dual of a polynomial in standard form can be computed.

$$
\begin{aligned}
\tau(f) &= \tau(\Sigma\{ax_1\cdots x_n\} + a_z z + a_u 1) \\
&= \Sigma\{a\tau(x_1\cdots x_n)\} + a_u z + a_z 1 \\
&= \Sigma\{a[1+z - x_1\cdots x_n]\} + a_u z + a_z 1 \\
&= -\Sigma\{ax_1\cdots x_n\} + (\Sigma a + a_u)z + (\Sigma a + a_z)1 \\
&= \Sigma\{-ax_1\cdots x_n\} + (\varepsilon(f) - a_z)z + (\varepsilon(f) - a_u)1
\end{aligned}
$$

Equation 11.5. Dual of a Polynomial in Standard Form

If the augmentation of a polynomial in $A^*[P]$ is zero, then the dual of the polynomial is simply the negative of the polynomial. The polynomial $1+z$ is self-dual. The dual of the dual polynomial is the original polynomial.

Each equation in the polynomial algebra has a dual equation. For instance, the fact that z is absorbing on P under the usual multiplication is expressed by the equation $z(1-p) = 0$ for any p in P. Applying complementation to both sides yields $\tau(z(1-p)) = \tau(0)$, which simplifies to:

$$1\vee(p-1) = \tau(z) \vee\tau(1-p) = \tau(z(1-p)) = \tau(0) = 0, \text{ or}$$

$$1\vee(p-1) = 0.$$

That dual equation expresses the fact that 1 is absorbing on P for the join-multiplication. The dual of $x\vee\tau(x) = 1$ is $x\tau(x) = z$.

Equations involving both multiplications can also be dualized. For instance, the simple case of the inclusion-exclusion principle

$$x\vee y = x + y - xy$$

dualizes to

$$(1+z-x)(1+z-y) = (1+z-x) + (1+z-y) - (1+z-x)\vee(1+z-y)$$

for x and y in P. To use the overcount-undercount interpretation, consider two subsets X and Y in a finite universe U. Interpret x as the cardinality #(X) of X, $1+z$ as the cardinality #(U) of the universe U, $1+z-x$ as the cardinality $\#(\neg X)$ of the complement of X, $x\vee y$ as the cardinality $\#(X\cup Y)$ of the union of X and Y, and so forth. The overcount-undercount principle says that the number of elements in either X or Y can be calculated as the number elements of of X plus the number of elements of Y minus the number of elements that are both X and Y:

$$\#(X \cup Y) = \#(X) + \#(Y) - \#(X \cap Y).$$

Equation 11.6. Overcount-Undercount Principle

The dual principle says that the number of elements that are both non-X and non-Y can be calculated by adding the elements of non-X and non-Y and then subtracting the number of elements that are either non-X or non-Y:

$$\#(\neg X \cap \neg Y) = \#(\neg X) + \#(\neg Y) - \#(\neg X \cup \neg Y).$$

Equation 11.7. Dual to Overcount-Undercount Principle

The dual applies the overcount-undercount principle to the complements. Because of our focus on propostional logic, we will use idempotent variables, but polynomial duality could be investigated on general augmented polynomial rings.

Generalized Propositional Logic

Valuation rings of free Boolean algebras allow propositional logic to be generalized in the context of an arbitrary ring of coefficients. The characterization of the valuation rings of free Boolean algebras as the augmented idempotent polynomial rings allows generalized propositional logic to be treated using a simple "plus-and-times" algebra of polynomials.

A polynomial f in A*[P] is said to be a "provable" polynomial if it equals its augmentation (times unity) in A*[P], and f is a "refutable" polynomial if it equals its augmentation times z in A*[P][see Halmos 1963, pp.. 45-46 to motivate these definitions in the case $A = \mathbf{Z}_2$]:

f is *provable* if $f = \varepsilon(f)1$ in A*[P], and

f is *refutable* if $f = \varepsilon(f)z$ in A*[P].

Equation 11.8. Definition of Provable and Refutable Polynomials

For instance, $x + \tau(x) - x\tau(x)$ is provable and $x\tau(x)$ is refutable for any x in P. The complementation interchanges provable and refutable polynomials.

Every map $w:P\cup\{z\}\rightarrow A$ such that w(p) and w(z) are idempotent and w(z) absorbs w(p) for all p in P will extend to an A-algebra homomorphism $w^*:A^*[P]\rightarrow A$ (and all such homomorphisms induce such a map on the generators). A polynomial is said to be a "tautology" if it is always mapped to its augmentation (times unity, which is the image of the unity in the polynomial ring), and f is said to be a "contradiction" if it is always mapped to its augmentation times the image of z, i.e, for all A-algebra homomorphisms $w^*:A^*[P]\rightarrow A$,

if $w^*(f) = \varepsilon(f)1$ [$= \varepsilon(f)w^*(1)$], f is a *tautology*, and

if $w^*(f) = \varepsilon(f)w^*(z)$, f is a *contradiction*.

Equation 11.9. Definition of Tautologous and Contradictory Polynomials

This generalizes the usual "truth table" definitions of tautologies and contradictions. The complementation interchanges tautologies and contradictions. Clearly a theorem is a tautology and a refutable polynomial is a contradiction. The converse is the generalization of the completeness theorem.

GENERALIZED COMPLETENESS THEOREM: If a polynomial f in A*[P] is a tautology, then f is provable.

Proof: Since by assumption $w^*(f - \varepsilon(f)) = 0$ for all homomorphisms w*, it suffices to prove the theorem in the form: if $w^*(f) = 0$ in A for all w*, then $f = 0$ in A*[P]. This can be shown using elementary reasoning about polynomials. Let f be represented in standard form:

$$f = \Sigma\{ax_1\cdots x_n\} + a_z z + a_u 1.$$

By taking different choices of a $w:P\cup\{z\}\rightarrow A$ mapping the x_i's in P and the z to 1's and 0's, the coefficients in A can all be shown to be 0 (but if $w(z) = 1$ then $w(x) = 1$ for all x in P).

If $w(x) = w(z) = 1$ for all x in P, then $w^*(f) = \Sigma a + a_z + a_u = \varepsilon(f) = 0$.

If $w(x) = w(z) = 0$ for all x in P, then $w^*(f) = a_u = 0$ so $a_u = 0$.

If $w(x) = 1$ for all x in P and $w(z) = 0$, then $w^*(f) = \Sigma a = 0$ so by the above, $a_z = 0$.

For each single x_i in P, set $w(x_i) = 1$ and the other $w(x) = 0$ and $w(z) = 0$, so the coefficient of each x_i must be 0.

For each pair x_i and x_j in P, set $w(x_i) = w(x_j) = 1$ and the other $w(x) = 0$ and $w(z) = 0$, so the coefficient of $x_i x_j$ must be 0.

By continuing the reasoning for each triple of x's and so forth, all the coefficients of f are shown to be 0.

The classical completeness theorem for propositional logic is the special case $A = \mathbf{Z}_2$.

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Appendix: Proof of Characterization Theorem

CHARACTERIZATION THEOREM: The valuation rings over free Boolean algebras are the augmented idempotent polynomial rings, i.e., V(B(P),A) ≅ A*[P] for any set P and commutative ring (with unity) A.

Proof: Let $B_0 = \{f \in A^*[P]: f^2 = f \text{ and } e(f) = 1\}$ be the set of idempotents of A*[P] with unitary augmentation. B_0 becomes a Boolean algebra with the operations defined by $f \vee g = f + g - fg$, $f \wedge g = fg$, and $\neg f = 1+z-f$. Since the generators P are included in B_0, there exists, by the universality property of the free Boolean algebra B(P), a unique Boolean algebra homomorphism $\nu: B(P) \rightarrow B_0 \subseteq A^*[P]$ that commutes with the insertion of P. Since ν is a Boolean algebra homomorphism, we have

$\nu(z) = z$, $\nu(x \wedge y) = \nu(x) \wedge \nu(y) = \nu(x) \cdot \nu(y)$, and $\nu(x \vee y) = \nu(x) \vee \nu(y) = \nu(x) + \nu(y) - \nu(x \wedge y)$

for any x and y in B(P). Hence ν is a multiplicative valuation on B(P) with values in the A-algebra A*[P], so by Theorem 1 (for multiplicative valuations), there exists a unique A-algebra homomorphism ν* such that the following diagram commutes.

$$\begin{tikzcd} B(P) \arrow[r, "\text{canonical}"] \arrow[dr, "\nu"'] & V(B(P),A) \arrow[d, "\nu^*"] \\ & A^*[P] \end{tikzcd}$$

Figure 11.3. Existence of A-Algebra Homomorphism $\nu^*: V(B(P),A) \rightarrow A^*[P]$

Let $w: P \cup \{z\} \rightarrow V(B(P),A)$ be the insertion of the elements of $P \cup \{z\}$ into the A-algebra V(B(P),A). Then by the universality property of the polynomial ring $A[P \cup \{z\}]$, there exists a unique A-algebra homomorphism w' such that the left-hand triangle in the following diagram commutes.

$$
\begin{tikzcd}
P\cup\{z\} \arrow[r] \arrow[drr, "w"'] & A[P\cup\{z\}] \arrow[r, "\text{proj.}"] \arrow[dr, "w'"'] & A^*[P] \arrow[d, "w^*"] \\
 & & V(B(P),A)
\end{tikzcd}
$$

Figure 11.4. Existence of A-Algebra Homomorphism $w^*:A^*[P]\rightarrow V(B(P),A)$

Then for any p in P, we have $w'(p^2–p) = w'(p)^2–w'(p) = p–p = 0$, $w'(pz–z) = w'(p)z–z = z–z = 0$, and $w'(z^2–z) = 0$ so I is contained in the kernal of w'. Hence there exists a unique A-algebra homomorphism w* such that $w' = w^*\cdot\text{proj.}$, i.e., the right-hand triangle commutes.

Now consider the following diagram.

$$
\begin{tikzcd}
 & A^*[P] \arrow[d, "w^*"] \\
A[P\cup\{z\}] \arrow[ur, "\text{proj.}"] \arrow[r, "w'"'] \arrow[dr, "\text{proj.}"'] & V(B(P),A) \arrow[d, "\nu^*"] \\
 & A^*[P]
\end{tikzcd}
$$

Figure 11.5. ν^*w^* is the Identity on A*[P]

We have just seen that the upper triangle commutes. To see that the lower triangle commutes, note that ν^*w' and proj. are both A-algebra homomorphisms $A[P\cup\{z\}]\rightarrow A[P\cup\{z\}]/I = A^*[P]$ which commute with the insertion of $P\cup\{z\}$. By the universality property of $A[P\cup\{z\}]$, there is only one such map so $\nu^*w' = \text{proj.}$ Since $w' = w^*\cdot\text{proj.}$, we have that $\text{proj.} = \nu^*w^*\cdot\text{proj.}$, i.e., the outer triangle commutes. By the universality property of quotient rings, the identity is the unique A-algebra homomorphism $A^*[P]\rightarrow A^*[P]$ that commutes with the projections, so ν^*w^* is the identity on A*[P].

It remains to consider the following diagram.

$$
\begin{tikzcd}
 & V(B(P),A) \arrow[d, "\nu^*"] \\
B(P) \arrow[ur, "\text{canonical}"] \arrow[r, "\nu"'] \arrow[dr, "\text{canonical}"'] & A^*[P] \arrow[d, "w^*"] \\
 & V(B(P),A)
\end{tikzcd}
$$

Figure 11.6. w*ν* is the Identity on V(B(P),A)

We have seen that the upper triangle commutes. To see that the lower triangle commutes, note that when the A-algebra homomorphism w* is restricted to B_0, then it is a Boolean algebra homomorphism into the Boolean algebra of idempotents of V(B(P),A). Hence w*ν and the canonical map are both Boolean algebra homomorphisms B(P)→V(B(P),A), which agree on the insertion of P. By the universality property of the free Boolean algebra B(P), there is only one such map so w*ν = canonical, i.e., the lower triangle commutes and thus the outer triangle commutes as well. By the universality property of V(B(P),A) (Theorem 1), the identity is the unique A-algebra homomorphism V(B(P),A)→V(B(P),A), which commutes with the canonical map as a multiplicative valuation, so w*ν* is the identity on V(B(P),A). Hence we have the isomorphism: A*[P] ≅ V(B(P),A).