

Chapter 12

Parallel Addition, Series-Parallel Duality, and Financial Mathematics

Introduction

In economic theory, "duality" means "convex duality," which includes duality in linear and non-linear programming as special cases. Series-parallel duality has been studied largely in electrical circuit theory and, to some extent, in combinatorial theory. Series-parallel duality also occurs in economics and finance, and it is closely related to convex duality.

When resistors with resistances a and b are placed in series, their compound resistance is the usual sum (hereafter the *series sum*) of the resistances $a+b$. If the resistances are placed in parallel, their compound resistance is the *parallel sum* of the resistances, which is denoted by the full colon:

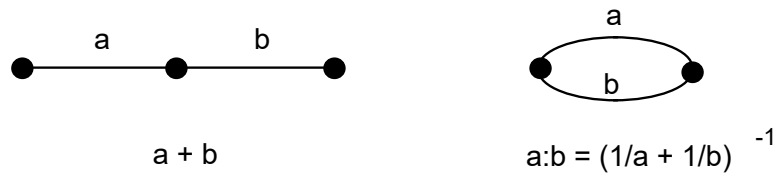
$$a:b = \left(a^{-1} + b^{-1} \right) = \frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{ab}{a+b}.$$


Figure 12.1. Series and Parallel Sums

The parallel sum is associative $x:(y:z) = (x:y):z$, commutative $x:y = y:x$, and distributive $x(y:z) = xy:xz$. On the positive reals, there is no identity element for either sum but the "closed circuit" 0 and the "open circuit" ∞ can be added to form the extended positive reals. Those elements are the identity elements for the two sums, $x+0 = x = x:\infty$.

For fractions, the series sum is the usual addition expressed by the annoyingly asymmetrical rule: "Find the common denominator and then add the numerators." The parallel

sum of fractions restores symmetry since it is defined in the dual fashion: "Find the common numerator and then (series) add the denominators."

$$\frac{a}{b} \dot{+} \frac{a}{d} = \frac{a}{b+d}$$

The parallel sum of fractions can also be obtained by finding the common denominator and taking the *parallel* sum of numerators.

$$\frac{a}{b} \dot{+} \frac{c}{b} = \frac{a+c}{b}$$

The usual series sum of fractions can be obtained by finding the common *numerator* and then taking the *parallel* sum of the denominators.

$$\frac{a}{b} + \frac{a}{d} = \frac{a}{b:d}$$

The rules for series and parallel sums of fractions can be summarized in the following four equations.

$$\frac{a}{1} + \frac{b}{1} = \frac{a+b}{1} \quad \frac{a}{1} \dot{+} \frac{b}{1} = \frac{a:b}{1}$$

$$\frac{1}{a} \dot{+} \frac{1}{b} = \frac{1}{a+b} \quad \frac{1}{a} + \frac{1}{b} = \frac{1}{a:b}$$

Series Chauvinism

From the viewpoint of pure mathematics, the parallel sum is "just as good" as the series sum. It is only for empirical and perhaps even some accidental reasons that so much mathematics is developed using the series sum instead of the equally good parallel sum. There is a whole "parallel mathematics" which can be developed with the parallel sum replacing the series sum. Since the parallel sum can be defined in terms of the series sum (or viceversa), "parallel mathematics" is essentially a new way of looking at certain known parts of mathematics.

Exclusive promotion of the series sum is "series chauvinism" or "serialism." Before venturing further into the parallel universe, we might suggest some exercises to help the reader combat the heritage of series chauvinism. Anytime the series sum seems to occur naturally in

mathematics with the parallel sum nowhere in sight, it is an illusion. The parallel sum lurks in a "parallel" role that has been unfairly neglected.

For instance, a series chauvinist might point out that the series sum appears naturally in the rule for working with exponents $x^a x^b = x^{a+b}$ while the parallel sum does not. But this is only an illusion due to our mathematically arbitrary symmetry-breaking choice to take exponents to represent powers rather than roots. Let a pre-superscript stand for a root (just as a post-superscript stands for a power) so 2x would be the square root of x . Then the rule for working with *these* exponents is ${}^a x {}^b x = {}^{a:b} x$ so the parallel sum does have a role symmetrical to the series sum in the rules for working with exponents.

Parallel Sums in High School Math

In high school algebra, parallel sums occur in the computation of completion times when activities are run in parallel. If pump A can fill a reservoir in a hours and pump B can fill the same reservoir in b hours, then running the two pumps simultaneously will fill the reservoir in $a:b$ hours.

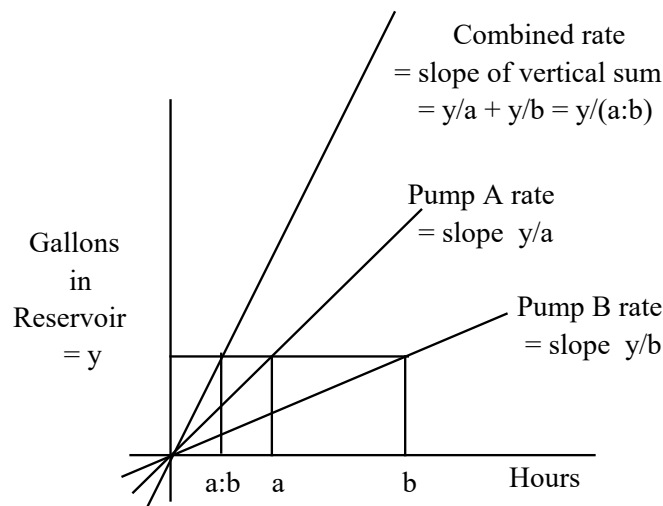


Figure 12.2. Vertical Sum of Lines

In the previous example, the slope of the vertical sum of two positively sloped straight lines was the series sum of the slopes. The "dual" of vertical sum is the horizontal sum. The

slope of the horizontal sum of two positively sloped straight lines is the parallel sum of the slopes. Inverting the previous case yields an example.

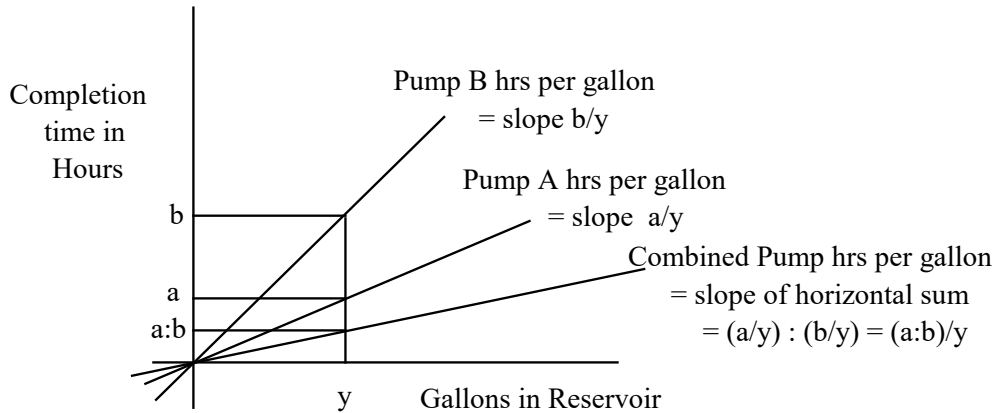


Figure 12.3. Horizontal Sum of Lines

Series-Parallel Duality and Reciprocity

The duality between the series and parallel additions on the positive reals \mathbb{R}^+ can be studied by considering the bijective reciprocity map

$$\rho: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ defined by } \rho(x) = 1/x.$$

The reciprocity map preserves the unit $\rho(1) = 1$, preserves multiplication $\rho(xy) = \rho(x)\rho(y)$, and *interchanges* the two additions:

$$\rho(x+y) = \rho(x):\rho(y) \text{ and } \rho(x:y) = \rho(x)+\rho(y).$$

The reciprocity map captures series-parallel duality on the positive reals just as an analogous anti-isomorphism (which interchanges two dual multiplications and preserves addition) on Rota's valuation rings captures Boolean duality [see previous chapter].

Much of the previous work on series-parallel duality has used methods drawn from graph theory and combinatorics [e.g., MacMahon 1881, Riordan and Shannon 1942, Duffin 1965, and Brylawski 1971]. MacMahon called a series connection a "chain" and a parallel connection a "yoke" (as in ox yoke). A *series-parallel network* is constructed solely from chains and yokes (series and parallel connections). By interchanging the series and parallel connections, each series-parallel network yields a *dual* or *conjugate* series-parallel network. To obtain the dual of

an expression such as $a + ((b+c):d)$, apply the reciprocity map but, for the atomic variables, replace $1/a$ by a and so forth in the final expression. Thus the dual expression would be $a : ((b:c) + d)$ (see below).

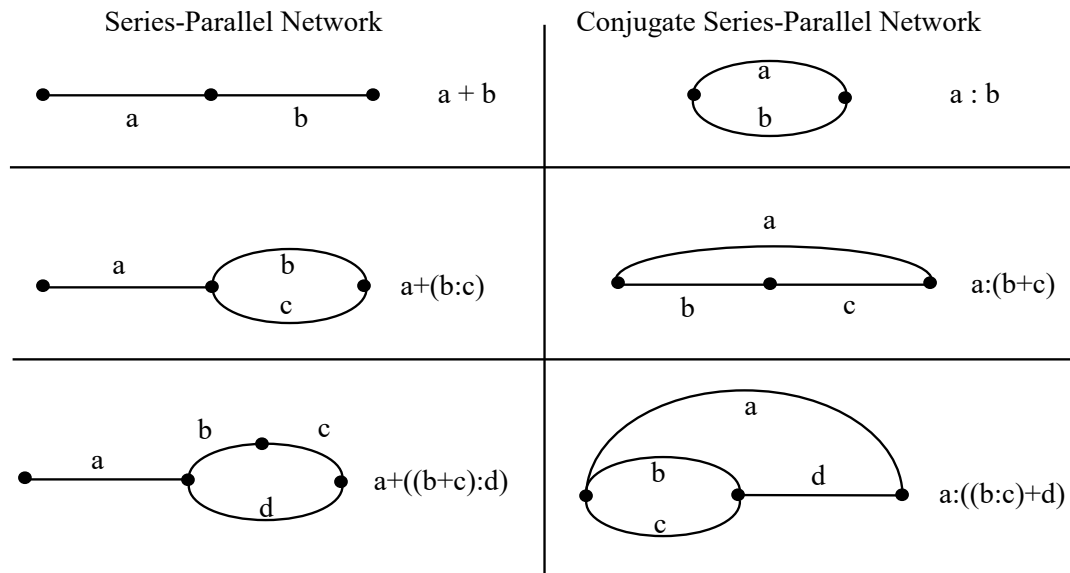


Figure 12.4. Conjugate Series-Parallel Networks

If each variable a, b, \dots equals one, then the reciprocity map carries each expression for the compound resistance into the conjugate expression. Hence if all the "atomic" resistances are one ohm, $a = b = c = d = 1$, and the compound resistance of a series-parallel network is R , then the compound resistance of the conjugate network is $1/R$ [MacMahon 1881, 1892]. With any positive reals as resistances, MacMahon's chain-yoke reciprocity theorem continues to hold if each atomic resistance is also inverted in the conjugate network.

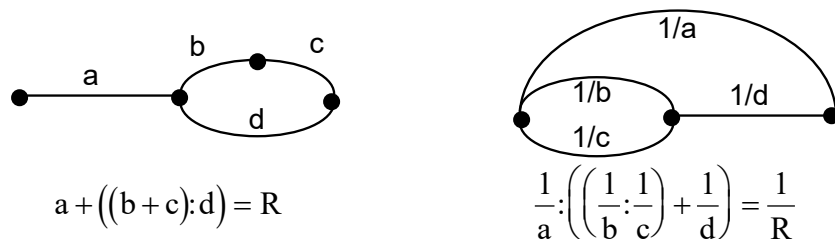


Figure 12.5. MacMahon Chain-Yoke Reciprocity Theorem

The theorem amounts to the observation that the reciprocity map interchanges the two sums while preserving multiplication and unity.

Dual Equations on the Positive Reals

Any equation on the positive reals concerning the two sums and multiplication can be dualized by applying the reciprocity map to obtain another equation. The series sum and parallel sum are interchanged. For example, the equation

$$\frac{1}{3}\left(5 + \frac{2}{5} + \frac{3}{5}\right) = 2$$

dualizes to the equation

$$3\left(\frac{1}{5} : \frac{5}{2} : \frac{5}{3}\right) = \frac{1}{2}.$$

The following equation

$$1 = (1 + x) : \left(1 + \frac{1}{x}\right)$$

holds for any positive real x . Add any x to one and add its reciprocal to one. The results are two numbers larger than one and their parallel sum is exactly one. Dualizing yields the equation

$$1 = \left(1 : \frac{1}{x}\right) + (1 : x)$$

for all positive reals x . Taking the parallel sum of any x and its reciprocal with one yields two numbers smaller than one which sum to one.

For any set of positive reals x_1, \dots, x_n , the parallel summation can be expressed using the capital P:

$$\mathbf{P}_{i=1}^n x_i = \left(\sum_{i=1}^n x_i^{-1} \right)^{-1}.$$

Equation 12.1. Parallel Summation

The binomial theorem

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

dualizes to the parallel sum binomial theorem (where "1/a" is replaced by "a" and similarly for "b"):

$$(a:b)^n = \mathbf{P}_{k=0}^n \binom{n}{k}^{-1} a^k b^{n-k}.$$

Equation 12.2. Parallel Sum Binomial Theorem

Taking $a = 1+x$ and $b = 1 + 1/x$ (and using a previous equation on the left-hand side), we have a nonobvious identity

$$1 = \mathbf{P}_{k=0}^n \binom{n}{k}^{-1} (1+x)^k \left(1 + \frac{1}{x}\right)^{n-k}$$

for any $x > 0$.

Series and Parallel Geometric Series

The following formula (and its dual) for partial sums of geometric series (starting at $i = 1$) are useful in financial mathematics (where x is any positive real).

$$\sum_{i=1}^n (1:x)^i = \sum_{i=0}^{n-1} (1:x)^{-i} : x = x \left(1 - (1:x)^n\right)$$

Equation 12.3. Partial Sums of Geometric Series

Dualizing yields a formula for partial sums of parallel-sum geometric series.

$$\mathbf{P}_{i=1}^n (1+x)^i = \mathbf{P}_{i=0}^{n-1} (1+x)^{-i} + x = \frac{x}{1 - (1+x)^{-n}}$$

Equation 12.4. Partial Sums of Dual Geometric Series

Dualization can also be applied to infinite series. Taking the limit as $n \rightarrow \infty$ in the above partial sum formulas yields for any positive reals x the dual summation formulas for series and parallel sum geometric series that begin at the index $i = 1$.

$$\sum_{i=1}^{\infty} (1:x)^i = x = \prod_{i=1}^{\infty} (1+x)^i$$

The parallel sum series in the above equation can be used to represent a repeating decimal as a fraction. An example will illustrate the procedure so let $z = .367367367\dots$ where the "367" repeats. Then since $1/a + 1/b = 1/(a:b)$, we have:

$$z = .367367\dots = \sum_{i=1}^{\infty} \frac{367}{(1000)^i} = \frac{367}{\prod_{i=1}^{\infty} (1000)^i}$$

Taking $y = x+1$ for $x > 0$ in the previous geometric series equation yields

$$\prod_{i=1}^{\infty} y^i = y - 1$$

for $y > 1$ which is applied to yield

$$z = .367367\dots = \frac{367}{\prod_{i=1}^{\infty} (1000)^i} = \frac{367}{1000 - 1} = \frac{367}{999}$$

For any positive real x , the dual summation formulas for the geometric series with indices beginning at $i = 0$ can be obtained by serial or parallel adding $1 = (1:x)^0 = (1+x)^0$ to each side.

$$\sum_{i=0}^{\infty} (1:x)^i = (1+x)$$

Equation 12.5. Geometric Series for any Positive Real x

□

$$\prod_{i=0}^{\infty} (1+x)^i = (1:x)$$

Equation 12.6. Dual Geometric Series for any Positive Real x

□

The Harmonic Mean

The *harmonic mean* of n positive reals is n times their parallel sum. Suppose an investor spends \$100 a month for two months buying shares in a certain security. The shares cost \$5 each the first month and \$10 each the second month. What is the average price per share purchased? At first glance, one might average the \$5 and \$10 prices to obtain an average price of \$7.50—but that would neglect the fact that twice as many shares were purchased at the lower price. Hence one must compute the weighted average taking into account the number of each purchased at each price, and that yields the harmonic mean of the two prices.

$$\frac{(20 \times \$5) + (10 \times \$10)}{30} = 2(\$5:\$10) = \$6\frac{2}{3}.$$

This investment rule is called "dollar cost averaging." A financial advisory letter explained a benefit of the method.

First, dollar cost averaging helps guarantee that the average *cost* per share of the shares you buy will always be lower than their average *price*. That's because when you always spend the same dollar amount each time you buy, naturally you'll buy more shares when the fund's price is lower and fewer shares when its price is higher.

[Scudder Funds 1988]

Let p_1, p_2, \dots, p_n be the price per share in each of n time periods. The average cost of the shares is the harmonic mean of the share prices, $n(p_1 : p_2 : \dots : p_n)$, and the average price is just the usual arithmetical mean of the prices, $(p_1 + p_2 + \dots + p_n)/n$.

The inequality that the average cost of the shares is less than or equal to the average of the share prices follows from

$$(A : B) + (C : D) \leq (A+C) : (B+D)$$

Equation 12.7. Lehman's Series-Parallel Inequality

for positive $A, B, C,$ and D [1962; see Duffin 1975].

This application of the series-parallel inequality can be seen by considering the prices as resistances in the following diagram (note how each of the n rows and each of the n columns contains all n resistances or prices).

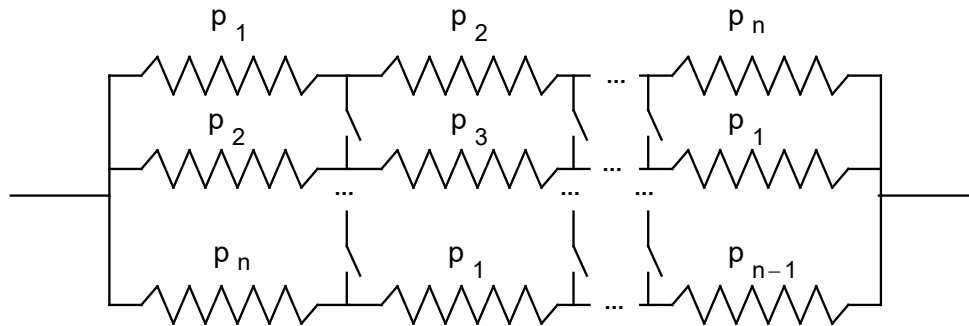


Figure 12.6. Intuitive Proof of Lehman's Inequality

When all the switches are open, the compound resistance is the parallel sum (n times) of the series sum of the prices, which is just the arithmetical mean of the prices. When the switches are closed, the compound resistance is the series sum (n times) of the parallel sum of the prices, which is the harmonic mean of the prices. Since the resistance is smaller or the same with the switches closed, we have for any positive p_1, p_2, \dots, p_n ,

$$\text{Harmonic Mean} = n \prod_{i=1}^n p_i \leq \frac{\sum_{i=1}^n p_i}{n} = \text{Arithmetical Mean}$$

Geometric Interpretation of Parallel Sum

The harmonic mean of two numbers is twice their parallel sum, just as the usual series or arithmetical mean is half their series sum. The geometric interpretation of the harmonic mean will lead us into geometric applications of the parallel sum. Draw a line FG through the point E where the diagonals cross in the trapezoid $ABDC$.

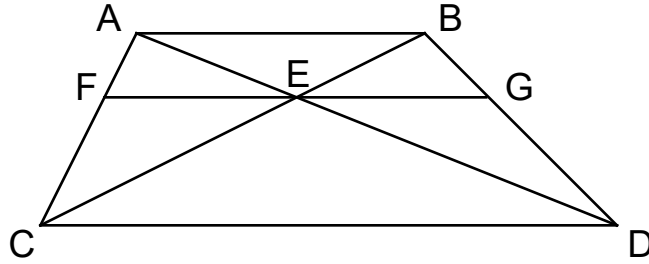


Figure 12.7. Geometry of Parallel Sum

Then FG is the harmonic mean of parallel sides AB and CD , i.e., $FG = 2(AB:CD)$. Since E bisects FG , the distance FE is the parallel sum of AB and CD , i.e., $FE = AB:CD$.

The basic geometrical fact can be restated by viewing AC as being horizontal (see following diagram). Given two parallel line segments AB and CD , draw BC and AD , which cross at E . The distance of E to the horizontal line AC in the direction parallel to AB and CD is the parallel sum $AB:CD$.

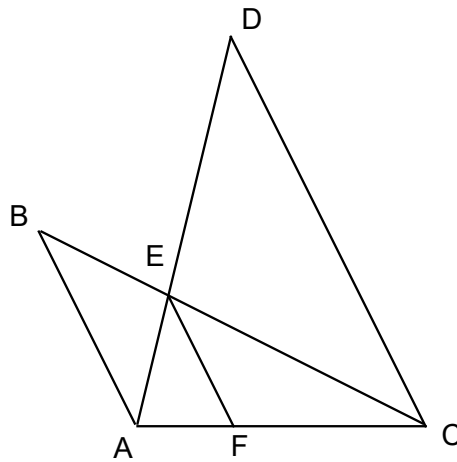


Figure 12.8. $EF = AB:CD$

It is particularly interesting to note that the distance AC is arbitrary. If CD is shifted out parallel to $C'D'$ and the new diagonals AD' and BC' are drawn (as if rubber bands connected B with C and A with D), then the distance $E'F'$ is again the parallel sum of AB and CD ($= C'D'$).

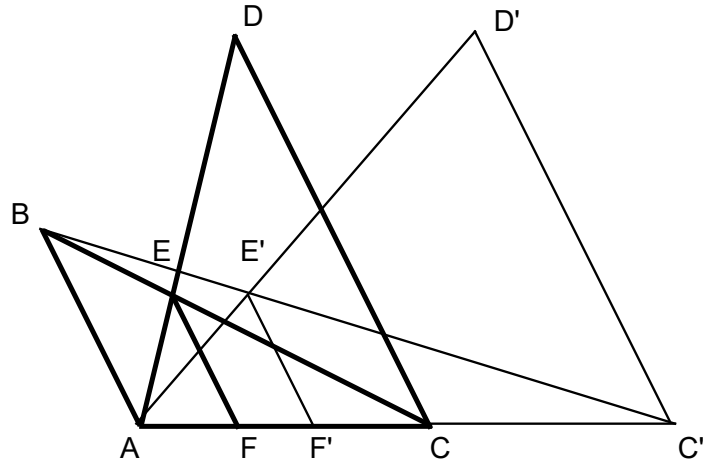


Figure 12.9. $EF = E'F' = AB:CD$ where $CD = C'D'$

The Gaussian equation for thin lenses presents the focal length $f = FE$ as the parallel sum of the distance d of the object (the arrow $A'A$) from the lens plane BEC and the distance d' of the image (the arrow $D'D$) from the lens plane.

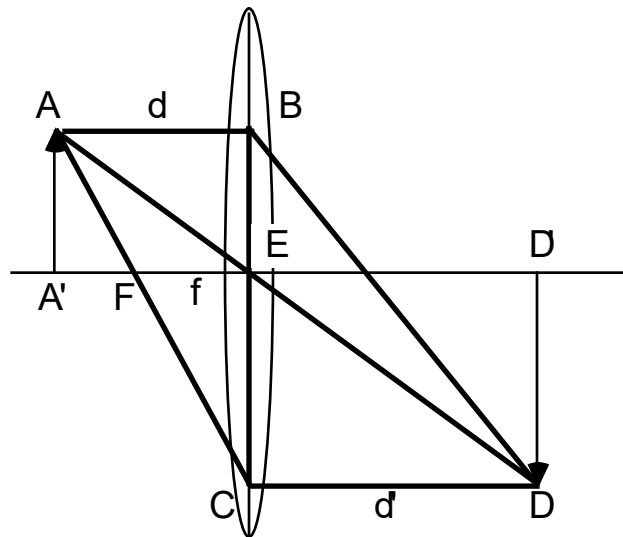


Figure 12.10. Gaussian Thin Lense Diagram

The thickened lines in the diagram show that $f = d + d'$ or, as the formula is usually written:

$$\frac{1}{d} + \frac{1}{d'} = \frac{1}{f}.$$

Equation 12.8. Gaussian Thin Lense Equation

Solutions of Linear Equations: Geometric Interpretation

Solutions to linear equations can always be presented as the parallel sum of quantities with a clear geometrical interpretation. Consider the case of two linear equations.

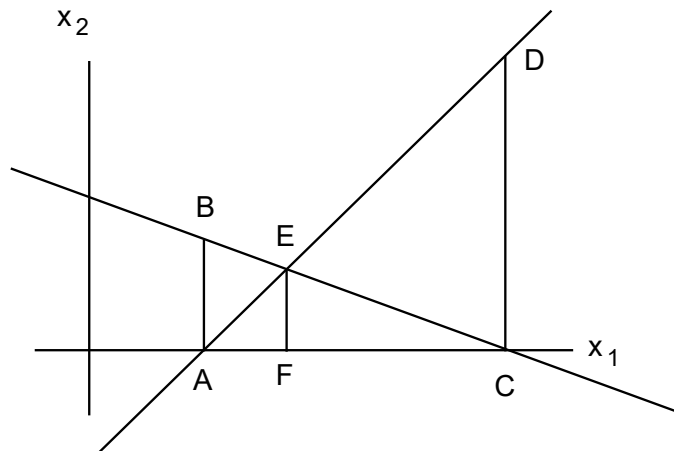


Figure 12.11. Linear Equations and Parallel Sum $EF = AB:CD$

For instance, the x_2 solution EF is the parallel sum of AB and CD , which are the "pillars" that rise up to each line from where the other line hits the x_1 floor.

This procedure generalizes to any nonsingular system of n linear equations. Suppose we wish to find the solution value of x_j . Each of the linear equations defines a hyperplane in n -space. For each $i = 1, \dots, n$, the $n-1$ hyperplanes taken by excluding the i^{th} hyperplane intersect to form a line which, in turns, intersect the $x_j = 0$ floor at some point called the "base of the i^{th} pillar." The perpendicular distance from that base point on the $x_j = 0$ floor up to the i^{th} hyperplane is the height c_i of the " i^{th} pillar" (on the $x_j = 0$ floor). If the perpendicular line through the base point does not intersect the i^{th} hyperplane, then the height c_i of the i^{th} pillar can be taken as ∞ (" ∞ " is "open circuit" element that is the identity for the parallel sum, $\infty : x = x$, and absorbs under the series sum, $\infty + x = \infty$). The parallel sum of the pillars is the solution value of x_j :

$$c_1 : c_2 : \dots : c_n = x_j \text{ solution.}$$

For $i = 1, \dots, n$, the base of the i^{th} pillar is the solution of the system of equations if the i^{th} equation is replaced by $x_j = 0$. Now replace $x_j = 0$ with the i^{th} equation but ignore the effect x_j has in the other equations, i.e., temporarily set the coefficient of x_j in the other equations equal to zero. Then the x_j solution of those modified equations is the i^{th} pillar c_i . Thus each pillar measures the effect of each equation on determining x_j if the role of x_j in the other equations is ignored. The parallel sum of these isolated effects is the x_j solution.

The proof of this result uses Cramer's Rule. Consider each column of a square matrix $A = [a_{ij}]$ as a (reversible) linear activity that uses n inputs supplied in given amounts (the b_i constants). The j^{th} activity is given by a column vector $(a_{1j}, \dots, a_{nj})^T$ (where the "T" superscript indicates transpose). Each unit of the j^{th} activity uses up a_{ij} units of the i^{th} input. With given input supplies $b = (b_1, \dots, b_n)^T$, the levels of the n activities $x = (x_1, \dots, x_n)^T$ are determined by the matrix equation $Ax = b$.

To isolate the effect of x_j on using the i^{th} input, replace the j^{th} activity by the column $(0, \dots, 0, a_{ij}, 0, \dots, 0)^T$ so the j^{th} activity only uses the i^{th} input. The x_j solution of the resulting equations is the i^{th} pillar c_i . Hence by Cramer's Rule, the i^{th} pillar is (if the denominator is zero, take $c_i = \infty$):

$$c_i = \frac{\begin{vmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & b_n & \dots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ \dots & \dots & a_{ij} & \dots & \dots \\ a_{n1} & \dots & 0 & \dots & a_{nn} \end{vmatrix}}.$$

The parallel sum of these pillars c_i for $i = 1, \dots, n$ will give the correct value of x_j .

The parallel sum of fractions with a common numerator is that numerator divided by the (series) sum of the denominators. The (series) sum of the denominators in the above calculation of c_i for $i = 1, \dots, n$ is the cofactor expansion of $|A|$ by the j^{th} column. Hence the result follows by Cramer's Rule:

$$x_j = \prod_{i=1}^n c_i = \frac{\begin{vmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & b_n & \dots & a_{nn} \end{vmatrix}}{|A|}.$$

Equation 12.9. Solution as Parallel Sum of "Pillars"

Duality in Financial Arithmetic

Parallel Sums in Financial Arithmetic

The parallel sum has a natural interpretation in finance so that each equation and formula in financial arithmetic can be paired with a dual equation or formula. The parallel sum "smoothes" balloon payments to yield the constant amortization payment to pay off a loan. If r is the interest rate per period, then $PV(1+r)^n$ is the one-shot balloon payment at time n that would pay off a loan with the principal value of PV . The similar balloon payments that could be paid at times $t=1, 2, \dots, n$, any one of which would pay off the loan, are

$$PV(1+r)^1, PV(1+r)^2, \dots, PV(1+r)^n.$$

But what is the equal amortization payment PMT that would pay off the same loan when paid at each of the times $t=1, 2, \dots, n$? It is simply the parallel sum of the one-shot balloon payments:

$$\begin{aligned} PMT &= PV(1+r)^1 : PV(1+r)^2 : \dots : PV(1+r)^n \\ &= \prod_{i=1}^n PV(1+r)^i. \end{aligned}$$

Equation 12.10. Amortization Payment is Parallel Sum of Balloon Payments

How does the total amount of money paid with equal loan payments compare with the one-time balloon payments? The sum of all the amortization payments $nPMT$ is the harmonic mean of the balloon payments.

This use of the parallel sum is not restricted to financial arithmetic. For example, suppose a forest of initial size PV (in harvestable boardfeet) grows at the rate r_i in the i^{th} period. Then

$$P_m = PV \prod_{i=1}^m (1 + r_i)$$

would be the *one-shot harvest* that could be obtained at the end of the m^{th} period. For instance, P_3 , P_{17} , and P_{23} are the amounts that could be harvested if the whole forest was harvested at the end of the 3rd, 17th, or the 23rd period. But what is the smooth or equal harvest PMT so that if PMT was harvested at the end of the 3rd, 17th, and the 23rd periods, then the forest would just be completely harvested at end of that last period? That smooth harvest amount is just the parallel sum of the one-time harvests:

$$\text{PMT} = P_3 : P_{17} : P_{23}.$$

The total harvest under the equal harvest method is the harmonic mean of the three one-shot harvests.

Returning to financial arithmetic, the discounted present value at time zero of n one dollar payments at the end of periods $1, 2, \dots, n$ is $a(n, r)$, the *present value of an annuity of one*.

$$a(n, r) = \frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \dots + \frac{1}{(1+r)^n}.$$

Equation 12.11. Present Value of an Annuity of One

Dualizing yields:

$$\frac{1}{a(n, r)} = (1+r)^1 : (1+r)^2 : \dots : (1+r)^n.$$

Equation 12.12. Installment to Amortize One

For the principal value of one dollar at time zero, the one-shot payments at times $1, 2, \dots, n$ that would each pay off the principal are the compounded principals $(1+r)^1, (1+r)^2, \dots, (1+r)^n$. The parallel sum $(1+r)^1 : (1+r)^2$ paid at times 1 and 2 would pay off the \$1 principal. Similarly, the parallel sum of the first three one-shot payments paid at times 1, 2, and 3 would pay off the \$1 principal, and so forth.

Suppose the constant interest rate is 20 percent per period. Then the discounted present value of two amortization payments of 1 at the end of the first and second period is principal value of the loan paid off by those payments, i.e., $55/36$:

$$a(2,.20) = \frac{55}{36} = \frac{1}{(1.2)^1} + \frac{1}{(1.2)^2}.$$

The equation dualizes to:

$$\frac{1}{a(2,.20)} = \frac{36}{55} = (1.2)^1 + (1.2)^2.$$

The amounts $(1.2)^1$ and $(1.2)^2$ are the compounded principal values of a \$1 loan so they are the one-shot or balloon payments that would pay off a loan of principal value \$1 if paid, respectively, at the end of the first or the second period. Their parallel sum, $36/55$, is the equal amortization payment that would pay off that loan of \$1 if paid at the end of both the first and second periods.

These facts can be arranged in the following dual format.

<p>Primal Fact:</p> <p>The series sum of the discounted amortization payments for a loan is the principal of the loan.</p> <p>□</p>	<p>Dual Fact:</p> <p>The parallel sum of the compounded principals of a loan is the amortization payment for the loan.</p>
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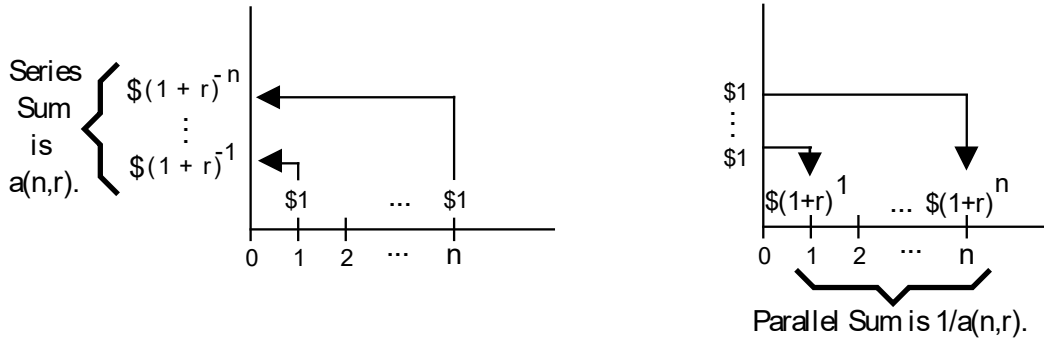


Figure 12.12. Illustrations of Primal and Dual Identities

The example illustrates some of the substitutions involved in dualizing the interpretation.

series sum	\Leftrightarrow	parallel sum
discounting	\Leftrightarrow	compounding
principals	\Leftrightarrow	payments

Future Values and Sinking Fund Deposits

Another staple of financial arithmetic is the computation of sinking fund deposits. The compounded future value at time n of n one dollar deposits at times $1, 2, \dots, n$ is $s(n,r)$, the *accumulation of one per period*.

$$s(n,r) = (1+r)^{n-1} + (1+r)^{n-2} + \dots + (1+r)^1 + 1 = a(n,r)(1+r)^n.$$

Equation 12.13. Accumulation of One per Period

The discounted values $1/(1+r)^{n-1}, \dots, 1/(1+r), 1$ of a one-dollar fund are the one-shot deposits at times $1, \dots, n-1, n$ that would each by itself yield a one-dollar future value for the sinking fund at time n . The parallel sum of these one-shot deposits is the (equal) sinking fund deposit at times $1, \dots, n-1, n$ that would yield a one-dollar fund at time n :

$$\frac{1}{s(n,r)} = \frac{1}{(1+r)^{n-1}} : \dots : \frac{1}{(1+r)^1} : 1.$$

Equation 12.14. Sinking Fund Factor

The sum of the smooth sinking fund deposits is the harmonic mean of the one-shot deposits.

The dual interpretations might be stated as follows.

<p>The series sum of the n compounded one-dollar deposits is the sinking fund that is accumulated by the one-dollar deposits.</p>	<p>The parallel sum of the n discounted one-dollar funds is the deposit that accumulates to a one-dollar sinking fund.</p>
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Infinite Streams of Payments

The formulas for amortization payments can be extended to an infinite time horizon.

This involves a financial interpretation for the dual geometric series with indices beginning at $i = 1$:

$$\sum_{i=1}^{\infty} (1:x)^i = x = \mathbf{P} \sum_{i=1}^{\infty} (1+x)^i.$$

Taking $x = 1/r$ so that $1:x = 1:1/r = 1/(1+r)$ in the series summation yields the fact that the discounted present value of the constant stream of one-dollar payments at times 1, 2, ... is reciprocal of the interest rate $x = 1/r$.

$$\sum_{i=1}^{\infty} (1+r)^{-i} = \frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \dots = \frac{1}{r}.$$

Equation 12.15. Perpetuity Capitalization Formula

Taking $x = r$ in the parallel summation yields the fact that the parallel sum of compounded values of one dollar is the interest rate r , the constant payment at $t = 1, 2, \dots$ that pays off a principal value of one dollar. Thus the dual to the annuity capitalization formula is the fact that the constant income stream of r is the equivalent of the capital of \$1.

<p style="text-align: center;">The series sum of the stream of discounted \$1 amortization payments (which is the principal amortized by a \$1 amortization payment) is the reciprocal of the interest rate, $(1+r)^{-1} + (1+r)^{-2} + \dots = r^{-1}$.</p>	<p style="text-align: center;">The parallel sum of the stream of compounded \$1 principals (which is the payment that amortizes a \$1 principal) is the interest rate, $(1+r)^1 : (1+r)^2 : \dots = r$.</p>
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Adding Groups of Payments

The first use of the parallel sum was based on the fact that given a series of balloon payments at different times that each would pay off the same principal, then their parallel sum was the smooth payment that would pay off the principal if paid at all of those times. This fact can be generalized. For each balloon payment substitute a stream of equal payments. For example, suppose that PMT_1 paid at each of a set of times will pay off the principal of PV, and suppose that PMT_2 paid at each of another set of times will pay off the same principal PV. Then the parallel sum $PMT_1 : PMT_2$ paid at the combined set of times will pay off the principal PV.

This more general use of the parallel sum can be demonstrated by dualizing the identities in financial arithmetic obtained by considering two groups of payments, n payments at the times $t = 1, 2, \dots, n$ and then m payments at $n+1, n+2, \dots, n+m$. One-dollar payments at times $1, \dots, n$ have the present value $a(n, r)$. One-dollar payments at times $n+1, \dots, n+m$ have the value $a(m, r)$ at time n and the value $(1+r)^{-n} a(m, r)$ at time zero. The present value of the combined stream of payments is $a(n+m, r)$ so:

$$a(n+m, r) = a(n, r) + \frac{a(m, r)}{(1+r)^n}.$$

Dualizing, $1/a(n, r)$ is the amortization payment at times $1, \dots, n$ that pays off a \$1 principal while $(1+r)^n/a(m, r)$ is the payment at time $n+1, \dots, n+m$ that pays off the same principal of \$1 at time zero. Thus the parallel sum of the payments is the payment at the combined times $1, \dots, n,$

$n+1, \dots, n+m$ that pays off the \$1 principal. But $1/a(n+m,r)$ is the amortization payment at those times for a \$1 principal so we have the dual identity:

$$\frac{1}{a(n+m,r)} = \frac{1}{a(n,r)} \cdot \frac{(1+r)^n}{a(m,r)}.$$

Multiplying the series-sum identity through by $(1+r)^{n+m}$ and using $s(k,r) = (1+r)^k a(k,r)$ yields the corresponding identity for sinking fund deposits:

$$s(n+m,r) = (1+r)^m s(n,r) + s(m,r)$$

which dualizes to:

$$\frac{1}{s(n+m,r)} = \frac{(1+r)^{-m}}{s(n,r)} \cdot \frac{1}{s(m,r)}.$$

The deposit $(1+r)^{-m}/s(n,r)$ at times $1, \dots, n$ accumulates to a future fund value of \$1 at time $n+m$ as does the deposit $1/s(m,r)$ at times $n+1, \dots, n+m$. Hence their parallel sum deposited at the combined times accumulates to \$1 at time $n+m$, but $1/s(n+m,r)$ deposited at the same times accumulates to the same fund value so it is the same deposit.

The present value of an annuity of one can be expressed by the formula $a(n,r) = [1-(1+r)^{-n}]/r$ which can be rearranged as $1/r = (1+r)^{-n}/r + a(n,r)$. This formula can be obtained by considering two groups of payments, the payments at times $1, \dots, n$ and the infinite stream of payments thereafter.

<p>The time zero present value of an infinite stream of one-dollar payments, i.e. the reciprocal of the interest rate, is the series sum of the value of the infinite stream at time n discounted to time zero and the present value of the \$1 payments at $t = 1, \dots, n$:</p> $\frac{1}{r} = \frac{(1+r)^{-n}}{r} + a(n, r).$	<p>The payment that in an infinite stream amortizes a \$1-principal at time zero, i.e. the interest rate, is the parallel sum of the payment that amortizes the time n compounded \$1-principal and the payment at $t = 1, \dots, n$ that amortizes the \$1 principal:</p> $r = r(1+r)^n \cdot \frac{1}{a(n, r)}.$
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The payment $a(n,r)^{-1}$ at $t = 1, \dots, n$ amortizes a \$1 principal as does the payment $r(1+r)^n$ at $t = n+1, \dots$ so the parallel sum of the two payments will amortize the \$1 principal if paid at the combined times $t = 1, \dots, n, n+1, \dots$. However, the amortization payment of r at the same times will amortize the same \$1 principal, so the two payments are equal.

Duality can be applied as well to identities involving a continuously compounded interest rate r . For instance, the present value of a continuous payment stream of one dollar per period from time 0 to time T is:

$$\int_0^T e^{-rt} dt = \frac{1 - e^{-rT}}{r}$$

while the value of the same stream from time T onward is:

$$\int_T^\infty e^{-rt} dt = \frac{e^{-rT}}{r}.$$

The series sum of the two present values is $1/r$, the present value of the same stream from time 0 onwards:

$$\frac{1 - e^{-rT}}{r} + \frac{e^{-rT}}{r} = \frac{1}{r}.$$

Dualizing, the continuous payment stream from time 0 to time T that pays off a principal of one dollar is $r/[1 - e^{-rT}]$ and the continuous stream from time T onward that pays off the same principal is r/e^{-rT} . Hence the parallel sum of these streams is r , the stream from time 0 onward that pays off the one-dollar principal:

$$\frac{r}{1 - e^{-rT}} + \frac{r}{e^{-rT}} = r.$$

Principal Values as Parallel Sums

In previous formulas using the parallel sum in financial arithmetic, the parallel sum of two or more amortization payments, each of which when paid at some series of times pays off the same principal, yields the amortization payment that will pay off the same principal when paid at the combined series of times. In the dual formulas, principal values were obtained as series sums. It is also possible to obtain amortization payments as series sums so that the dual formulas will then yield principal values as parallel sums.

The amortization payment at times $t = 1, \dots, n$ that will pay off the principal of 1 at time 0 is expressed as a series sum by the following interesting formula:

$$\frac{1}{a(n, r)} = \frac{1}{s(n, r)} + r.$$

Equation 12.16. Amortization Payment = Sinking Fund Factor plus Interest

The formula can be easily interpreted. Suppose that at the times $1, \dots, n$ the interest r on a principal of 1 paid. In addition, the sinking fund deposit of $1/s(n, r)$ was made at the times $1, \dots, n$. Those sinking fund deposits accumulate to the fund of 1 at time n so the full principal of 1 could be paid all at once at time n along with the last interest payment. Thus the constant stream of payments $r + 1/s(n, r)$ at the times $1, \dots, n$ will pay off the principal of 1 at time 0. The constant payment $1/a(n, r)$ at those times will pay off the same principal so the two payments must be equal.

Since the above formula expresses an amortization payment as a series sum, the dual equation will express a principal value as a parallel sum:

$$a(n, r) = s(n, r) \cdot \frac{1}{r}.$$

The present value of the series of payments of 1 at the times 1, ..., n is equal to the parallel sum of the sinking fund $s(n, r)$ accumulated at time n from the same payments and the principal $1/r$ at time 0 would generate the series of 1's as just the interest on that principal. This duality can be expressed in the familiar way.

$\frac{1}{a(n, r)} = \frac{1}{s(n, r)} + r.$ <p>The amortization payment at $t = 1, \dots, n$ for the principal value of 1 at time 0 is the series sum of the sinking fund deposit at $t = 1, \dots, n$ to accumulate the sinking fund amount of 1 at time n and the interest on the principal of 1.</p>	$a(n, r) = s(n, r) \cdot \frac{1}{r}.$ <p>The principal value at time 0 of the amortization payments of 1 at $t = 1, \dots, n$ is the parallel sum of the sinking fund at time n accumulated by the sinking fund deposits of 1 at $t = 1, \dots, n$ and the principal with the interest of 1.</p>
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Summary of Duality in Financial Arithmetic

Each equation involving the series sum, the parallel sum, and multiplication can be dualized by interchanging the series and parallel sum, taking the reciprocal of each constant or variable term, and leaving multiplication unchanged. Equations in financial arithmetic expressing principal values as a series sum dualize to equations expressing amortization payments as parallel sums. An equation expressing the amount of a sinking fund as a series sum dualizes to an equation expressing a sinking fund deposit as a parallel sum. In the last section,

we saw an equation expressing an amortization payment as a series sum, which dualized to an equation expressing a principal value as a parallel sum.

In addition to precise mathematical procedure for dualizing equations, it is also possible to establish the duality at a conceptual level so that verbal statements of an equation can be immediately dualized into the verbal statement for the dual equation. This conceptual duality can be summarized in the following table.

Concept in Financial Arithmetic	Example	Dual Example	Dual Concept
Series Sum	$x+y$	$1/x : 1/y$	Parallel Sum
Discounting	$(1+r)^n$	$1/(1+r)^n$	Compounding
Principal Value	$a(n,r)$	$1/a(n,r)$	Amortization Payment
Sinking Fund Amount	$s(n,r)$	$1/s(n,r)$	Sinking Fund Payment
Interest on Principal of 1	r	$1/r$	Principal with Interest of 1

Appendix: Series-Parallel Algebras

Commutative Series-Parallel Algebras

A *commutative series-parallel (SP) algebra* is a set containing a unit element 1, having a multiplication (indicated by juxtaposition) and having two additions (indicated by "+" and ":") that satisfy the following axioms:

- a. Multiplicative Identity: $x1 = 1x = x$
- b. Associativity: $(x(yz)) = ((xy)z)$
 $(x+(y+z)) = ((x+y)+z)$
 $(x:(y:z)) = ((x:y):z)$
- c. Commutativity: $xy = yx$

$$x+y = y+x$$

$$x:y = y:x$$

d. Distributivity: $x(y+z) = (xy)+(xz)$

$$x(y:z) = (xy):(xz).$$

e. Modular Law: $(x+y)(x:y) = xy$.

The standard example of a commutative series-parallel algebra is the positive reals \mathbb{R}^+ with parallel addition defined as above. A commutative SP algebra is a commutative bi-semiring where the additions are related by the modular law (5). The series and parallel sum operations are *dual* in the sense that the axioms (1)-(5) remain the same under the interchange of the two sums. This *series-parallel duality principle* is analogous to the Boolean duality principle, which states that any theorem of Boolean algebra remains valid under the interchange of conjunction and disjunction (including the interchange of the null element and unit). Any proof and theorem for commutative series-parallel algebras remains valid under the interchange of series and parallel sums.

The series-parallel duality exhibited in the axioms for commutative series-parallel algebras does not presuppose a reciprocity map ρ . A *series-parallel division algebra* is an SP algebra that is also a multiplicative group (i.e., each element has a multiplicative inverse). In a SP division algebra, the reciprocity map $\rho(x) = x^{-1}$ is an anti-isomorphism between the two additive structures, i.e., it interchanges the two sums and preserves multiplication and the unit. The positive reals \mathbb{R}^+ and the positive rationals \mathbb{Q}^+ are examples of commutative SP division algebras.

Every commutative group has an extension that is a series-parallel division algebra. Let G be a commutative multiplicative group with the identity 1. A *free bi-semiring* $\text{BSR}(G)$ is constructed from G by taking all finite formal products and sums (no additive inverses) using two sum operations: a series sum $x+y$ and a parallel sum $x:y$. For any elements $x, y,$ and z thus formed, the universal relations expressed in the first four axioms (above) are imposed in

BSG(G). The *series-parallel algebra of G*, denoted SP(G), is formed by taking the quotient of the bi-semiring BSR(G) by the modular law.

The *reciprocity map* $\rho: \text{SP}(G) \rightarrow \text{SP}(G)$ is defined first on the elements of G by $\rho(g) = g^{-1}$ and then it is extended to all the elements of SP(G) by preserving multiplication and interchanging series and parallel sums:

$$\rho(xy) = \rho(x)\rho(y)$$

$$\rho(x+y) = \rho(x) : \rho(y)$$

$$\rho(x : y) = \rho(x) + \rho(y).$$

By the duality principle, ρ respects the universal relations (1)-(5) defining series-parallel algebras so it is a well-defined self-map on SP(G) which preserves multiplication and interchanges series and parallel sums.

It can then be verified that for any x in SP(G), $\rho(x)x = 1$ so it carries each element to its multiplicative inverse. In particular, this shows that SP(G) is a group itself, i.e., a SP division algebra, which could be called the *series-parallel completion of G*. For example, the series-parallel extension of the trivial group is the positive rationals, i.e., $\text{SP}(\{1\}) = \mathbb{Q}^+$. Intuitively, a series-parallel circuit with resistance equal to any positive rational can be constructed solely from one ohm resistances.

Adding Ideal Elements

The ideal elements 0 and ∞ can be added to the positive reals to form the extended positive reals. In the electrical interpretation, 0 is the "short circuit" of zero resistance and ∞ is the "open circuit" with infinite resistance. The operations in the extended reals are series sum, parallel sum, and multiplication.

Adding a short circuit in parallel to any resistance gives a short circuit so for any x , $x : 0 = 0 : x = 0$. Adding a short circuit in series with any resistance does not change that resistance so for any x : $x + 0 = 0 + x = x$. Adding an open circuit in parallel to any resistance does not change

that resistance so for any x , $x : \infty = \infty : x = x$. Adding an open circuit in series with any resistance creates an open circuit, so for any x : $x + \infty = \infty + x = \infty$.

Multiplication is only a partially defined operation on the extended positive reals since the product of 0 and ∞ is undefined. Elsewhere, 0 and ∞ are absorbing:

$$0x = x0 = 0 \text{ for } x \neq \infty \text{ and } \infty x = x\infty = \infty \text{ for } x \neq 0.$$

The definition $0\infty = 1$ is tempting but it leads to disaster, e.g.,

$$1 = \infty(0) = \infty(0 + 0) = \infty 0 + \infty 0 = 1 + 1 = 2,$$

so the product of 0 and ∞ is left undefined. However, the reciprocity map $\rho(x) = 1/x$ can be directly extended by $\rho(0) = \infty$ and $\rho(\infty) = 0$ (instead of trying to make sense of $\rho(0) = 1/0 = \infty$ or $\rho(\infty) = 1/\infty = 0$). Then $x + \infty = \infty + x = \infty$ dualizes to $x : 0 = 0 : x = 0$ (replacing $1/x$ by x after applying the reciprocity map) so the duality results apply as well to the extended positive reals.

An *extended commutative series-parallel algebra* has elements 0 (the "closed circuit") and ∞ (the "open circuit") such that 0∞ is undefined and that satisfy the following axioms:

- f. $x : 0 = 0 : x = 0$ and $x + 0 = 0 + x = x$ for any x .
- g. $x : \infty = \infty : x = x$ and $x + \infty = \infty + x = \infty$ for any x .
- h. $0x = x0 = 0$ for $x \neq \infty$ and $\infty x = x\infty = \infty$ for $x \neq 0$.

The standard example of an extended commutative S-P algebra is the extended non-negative reals, i.e., the positive reals \mathbb{R}^+ together with 0 and ∞ . The above axioms transform into one another under the interchange of the sums and of the ideal elements 0 and ∞ . Thus series-parallel duality extends to extended commutative SP algebras in the sense that any axiom and thus any proof and theorem remains valid under the interchange of the sums and of the ideal elements 0 and ∞ .

Noncommutative Series-Parallel Algebras

A general not-necessarily-commutative *series-parallel algebra* has two additions, denoted + and :, and multiplication defined on a set S satisfying the following axioms:

- a. **Multiplicative Identity.** $x1 = 1x = x$

- b. Associativity: $(x(yz)) = ((xy)z)$
 $(x+(y+z)) = ((x+y)+z)$
 $(x:(y:z)) = ((x:y):z)$
- c. Commutativity. $x+y = y+x$
 $x:y = y:x$
- d. Right-Distributivity over + $(x + y)z = (xz + yz)$
Left-Distributivity over : $z(x : y) = (zx : zy)$
- e. Modular Law. $(1 : x)(1 + x) = x$
- f. Two-Sided Inverses. if $xy = 1$ then $yx = 1$.

A general *duality principle* is obtained in any series-parallel algebra by interchanging series addition (+) with parallel addition (:) and reversing the order of multiplications. Swapping the sums and permuting the products transforms any of the above axioms into another axiom so all theorems would be preserved under that transformation.

Any group, not necessarily commutative, generates a series-parallel algebra that is a group, i.e., a series-parallel division algebra, that is its series-parallel completion. On any SP division algebra, duality is realized by the reciprocity map that interchanges the two additions, permutes the order of multiplication, and preserves the unit.

The principal model for a noncommutative SP algebra is the algebra of monotonic increasing real functions on a single real variable. Series and parallel sums are respectively the vertical and horizontal sums of functions. Multiplication is the composition of functions so the reciprocal or inverse of an element is its functional inverse.

Series-Parallel Duality as the "Derivative" of Convex Duality

Series-parallel duality can be related to the form of duality most familiar in economics, namely the duality of convex functions that includes duality in linear and nonlinear programming [see Rockafellar 1970].

General S-P duality is to the convex duality of functions as the derivative of a function is to the function. The derivatives of differentiable strictly convex functions of a single variable are monotonic increasing functions. In addition to the usual sum of convex functions, there is a "dual" [Rockafellar 1970, 145] addition for convex functions, namely the "infimal convolution" [Rockafellar 1970, 34]:

$$(f \oplus g)(x) = \inf_y \{f(x - y) + g(y)\}.$$

For instance, if a firm can produce the same type of output in a number of plants, the firm's cost function is the infimal convolution of the plant cost functions. The derivatives of the ordinary and dual (infimal convolution) sums of differentiable strictly convex functions are, respectively, the vertical (series) and horizontal (parallel) sums of the derivatives. The "inverse" or dual of a convex function $f(x)$ is its "convex conjugate":

$$f^*(x) = \sup_y \{xy - f(y)\}.$$

The derivative of the convex conjugate is the inverse of the derivative in the SP algebra of monotonic increasing functions.

The relationship between convex duality and series-parallel duality is summarized in the following table.

Convex Duality		Derivative →	Series-Parallel Duality	
Primal	Dual		Primal	Dual
Convex Function	Convex Conjugate		Increasing Function	Inverse Function
Sum of Convex Functions	Infimal Convolution		Vertical Sum	Horizontal Sum

For a simple example of SP duality as the derivative of convex duality, consider the strictly convex functions of the form $y = ax^2/2$ for positive constants a that correspond by differentiation to the positively sloped straight lines through the origin. Given two such

functions $ax^2/2$ and $bx^2/2$, their series sum is, of course, $(a+b)x^2/2$. Furthermore, their infimal convolution is $(a : b)x^2/2$. The derivatives of these dual sums of convex functions are the vertical (series) and horizontal (parallel) sums of the derivatives in the SP algebra of monotonic increasing functions. The convex conjugate of $ax^2/2$ is $(1/a)x^2/2$ [note that composition is not defined for convex functions—only for their derivatives]. The derivatives are multiplicative inverses in the SP algebra of monotonic increasing functions.

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