Chapter 5
Are Marginal Products Created *Ex Nihilo*?

The Conventional Picture of Marginal Products Created *Ex Nihilo*

Marginal productivity (MP) theory has always played a larger importance in orthodox economics than could be justified by its purely analytical role. This is because MP theory is conventionally interpreted as showing that, in competitive equilibrium, "each factor gets what it is responsible for producing." The marginal unit of a factor is seen as being responsible for producing the marginal productivity of that factor, and each unit could be taken as the marginal unit. Consider the marginal product of labor MP_L. In competitive equilibrium, the value of the marginal product of labor $p \cdot MP_L$ (where p is the unit price of the output) is equal to w (the unit price of labor):

$$p \cdot MP_L = w$$

"Value of what a unit produces" = "Value received by a unit of the factor."

There are many problems in this conventional interpretation of MP theory [see, for example, Chapter 12 in Ellerman 1992]. Our purpose is to highlight an internal incoherence in the conventional treatment, to show how this difficulty can be overcome in a mathematically equivalent reformulation of MP theory, and to note how this reformulation leads to a rather different interpretation of the theory.

The problem (or internal incoherence) in the usual treatment is simply that a unit of a factor cannot produce its marginal product out of nothing. The factor must simultaneously use some of the other factors. If the marginal product of one man-year in a tractor factory is one tractor, how can a tractor be produced without using steel, rubber, energy, and so forth? But when that concurrent factor usage is taken into account ("priced out"), then the usual equations must be significantly reformulated. A new vectorial notion of the marginal product, the "marginal whole product," must be used in place of the conventional scalar "marginal product."

Before turning to the vectorial treatment of marginal products we must remove the seeming paradox in the scalar treatment. When we increase the labor in a tractor factory to produce more tractors, we will also have to increase the steel, rubber, energy, and other inputs necessary to produce tractors. That would spoil the attempt to take the increase in tractor output as the result of solely the increase in labor. But the so-called "marginal product of labor" is the result of a somewhat different hypothetical or conjectural change in production. It is assumed that factors are substitutable. To arrive at the "marginal product of labor" we must consider two changes: an increase in labor and a shift to a slightly more labor-intensive production technique so that the increased labor can be used together with exactly the same total amounts of the other factors. Since (following the hypothetical production shift) the other factors are used in the same total amounts, the extra output is then viewed as the "product" of the extra unit of labor, as if the extra product was produced *ex nihilo*.

Symmetry Restored: The Pluses and Minuses of Production

Nothing is produced *ex nihilo*. Labor cannot produce tractors without actually using other inputs. Production needs to be conceptualized in an algebraically symmetric manner. That is, there are both positive results (produced outputs) and negative results (used-up inputs), and they can be considered symmetrically.

For a nontechnical presentation, let Q = f(K,L) be a production function with p, r, and w as the unit prices of the outputs Q, the capital services K, and the labor services L respectively. The outputs Q are the positive product of production but there is also a negative product, namely the used-up capital and labor services K and L. Lists or vectors with three components can be used with the outputs, capital services, and labor services listed in that order. The *positive product* would be represented as (Q,0,0). The *negative product* signifying the used-up or consumed inputs could be represented as (0,–K,–L). The comprehensive and algebraically symmetric notion of the product is obtained as the (component-wise) sum of the positive and negative products. It might be called the *whole product*.

$$(Q,–K,–L) = (Q,0,0) + (0,–K,–L)$$

Equation 5.1. Whole Product = Positive Product + Negative Product

The unit prices can also be arranged in a list or vector, the *Price vector* **P** = (p,r,w) [where symbols for vectors are in bold]. The product of a price vector times a quantity vector (such as the whole product vector) is the sum of the component-wise products of prices times quantities. That sum is the value of the quantity vector.

$$\mathbf{P}\cdot(Q,0,0) = (p,r,w)\cdot(Q,0,0) = pQ$$

Equation 5.2. Value of Positive Product = Revenue

$$\mathbf{P}\cdot(0,–K,–L) = (p,r,w)\cdot(0,–K,–L) = – (rK+wL)$$

Equation 5.3. Value of Negative Product = Expenses

$$\mathbf{P}\cdot(Q,–K,–L) = (p,r,w)\cdot(Q,–K,–L) = pQ – (rK+wL)$$

Equation 5.4. Value of Whole Product = Profit

Marginal Whole Products

The alternative presentation of MP theory uses the marginal version of the whole product, which we will call the "marginal whole product." The precise mathematical development is given in the Appendix. Here we develop a heuristic discrete treatment. Given the input prices and a given level of output Q_0, there are input levels K_0 and L_0 that produce Q_0 at minimum cost $C_0 = rK_0 + L_0$.

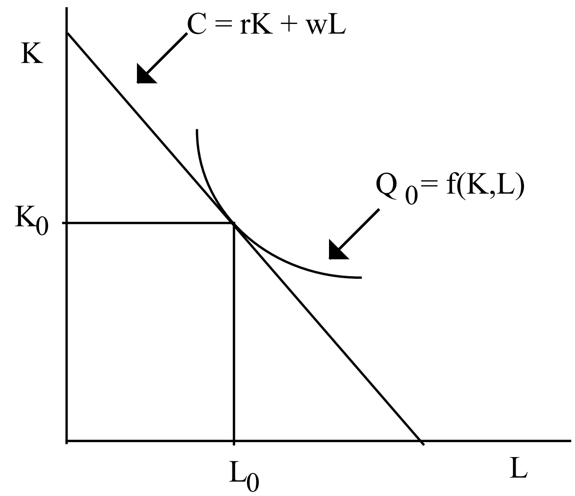

Figure 5.1. Minimum Cost to Produce Quantity Q_0

For an increase of one unit to $Q_1 = Q_0 + 1$, there will be new levels of K_1 and L_1 necessary to produce Q_1 at minimum cost.

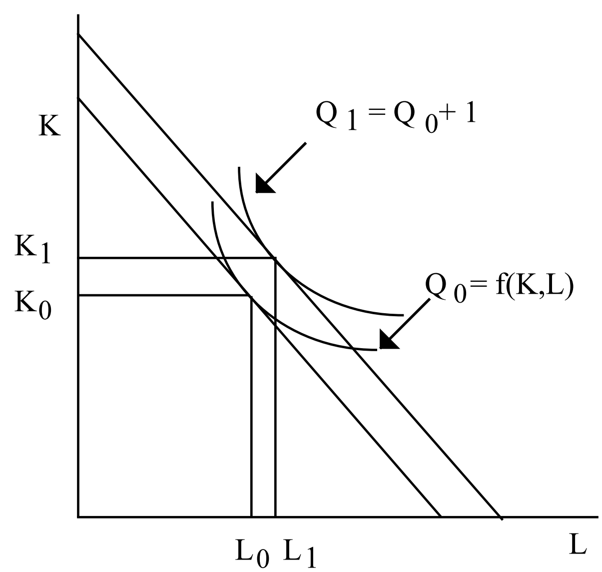

Figure 5.2. New Levels of K and L to Produce $Q_1 = Q_0+1$

Let $\Delta K = K_1–K_0$ and $\Delta L = L_1–L_0$ be the marginal increases in the amounts of capital and labor services that are necessary to produce the increase in output $\Delta Q = Q_1–Q_0 = 1$. The minimum cost of producing Q_1 is $C_1 = rK_1 + wL_1$. Since $\Delta Q = 1$, the *marginal cost* is MC = $\Delta C/\Delta Q = \Delta C = C_1 – C_0 = rK_1 + wL_1 – (rK_0 + wL_0)$.

The marginal version of the whole product is the marginal whole product which has unit output and minus the inputs necessary to produce one more unit of output at minimum cost.

$$\mathbf{MWP} = (1,-\Delta K,-\Delta L).$$

Equation 5.5. Marginal Whole Product

The value of the marginal whole product is the marginal profit, the difference between price and marginal cost.

$$\mathbf{P}\cdot(1,-\Delta K,-\Delta L) = (p,r,w)\cdot(1,-\Delta K,-\Delta L) = p - [(rK_1 + wL_1) - (rK_0 + wL_0)] = p - MC.$$

Equation 5.6. Value of Marginal Whole Product is Marginal Profit

If the marginal profit was positive at a given level of output, then profits could be increased by increasing the level of output. If the marginal profit was negative, then profits would increase by decreasing the level of output. Thus if profits are at a maximum, then the marginal profit must be zero. This is the usual result that p = MC if profits are at a maximum.

Asymmetry Between Responsible and Non-Responsible Factors

Part of the poetic charm of the conventional presentation of MP theory was that it allowed each factor to be pictured as *active*—as being *responsible* for producing its own marginal product. But we have noted the technical absurdity of, say, labor producing tractors out of nothing else. Labor must use up steel, rubber, and other inputs to produce tractors. But if that is accepted, then it is implausible to turn around and pretend that another factor is also active—that steel uses up labor, rubber, and other factors to produce tractors.

MP theory, as an analytical economic theory, does not provide any distinction between responsible or non-responsible factors. Those notions must be imported. "Responsibility" is a legal-jurisprudential notion. Poetic license and pathetic fallacy aside, only human actions can be responsible for anything. For example, the tools of the burglary trade certainly have a causal efficacy ("productivity"), but only the burglar can be charged with responsibility for the crime. The responsibility is imputed back through the tools (as "responsibility conduits") to the human user. Thus from the viewpoint of the juridical principal of imputation ("Assign legal rights and liabilities to the de facto responsible agents"), only human actions or "labor" (including managers) should legally own and be liable respectively for the positive and negative of production, i.e., for the whole product [see Ellerman 1992].

Since MP theory does not, by itself, provide any concept of "responsible" factors, any factor or factors could be taken as the responsible factors for analytical purposes. In the mathematical appendix, the factors $x_1,...,x_n$ will not be identified (as capital, labor, etc.), and we will arbitrarily take the first factor as being responsible. In our nontechnical presentation where the factors are identified, labor will taken as the responsible factor (but the formalism would be the same, *mutatis mutandis*, for any other choice).

As the responsible factor produces the outputs (produces the positive product), it must also use up the inputs (produce the negative product). We must calculate the positive and negative product of the marginal unit of the responsible factor, labor. We will call the vector of positive and negative marginal results of labor, the "marginal whole product of labor." The marginal whole product of labor is then compared with the opportunity cost of labor (the wage w in the model).

The marginal quantities $\Delta Q = 1$, ΔK, and ΔL that appear in the marginal whole product can be used to form the ratios $\Delta Q/\Delta K$ and $\Delta Q/\Delta L$. But these ratios are **not** the marginal products. For instance, if labor is increased by this ΔL, than an additional ΔK must be used up to product one more unit of output ($\Delta Q = 1$) in a cost-minimizing manner. The usual "marginal product" of labor is the extra product produced per extra unit of labor if the production technique is shifted so that no more extra capital is used.

In our simple model, the marginal results of labor can be calculated by dividing the marginal whole product through by ΔL to obtain $(1/\Delta L, -\Delta K/\Delta L, -1)$. Since labor also creates the marginal unit of labor (0, 0, 1), the *marginal whole product of labor* is the following vector sum.

$$\mathbf{MWP_L} = (1/\Delta L, -\Delta K/\Delta L, 0) = (1/\Delta L, -\Delta K/\Delta L, -1) + (0, 0, 1).$$

Equation 5.7. Marginal Whole Product of Labor

Multiplying through by the prices yields the corresponding value.

$$\mathbf{P}\cdot\mathbf{MWP_L} = (p,r,w)\cdot(1/\Delta L, -\Delta K/\Delta L, 0) = (p - r\Delta K)/\Delta L = w + (p - MC)/\Delta L.$$

Equation 5.8. Value of Marginal Whole Product of Labor

If $\mathbf{P}\cdot\mathbf{MWP_L}$ (the value of the fruits of the marginal unit of labor) exceeds w (the opportunity cost of the marginal unit of labor), then it is profitable to increase the use of labor to produce more output by using up more capital services. Conversely, if $\mathbf{P}\cdot\mathbf{MWP_L}$ is less then w, then the use of the marginal unit of labor does not cover its opportunity cost so it would be better to reduce the level of labor. Thus for profits to be maximized, the value of the marginal whole product of labor must equal the opportunity cost of labor.

$$\mathbf{P}\cdot\mathbf{MWP_L} = w.$$

Equation 5.9. Profit Max Implies: Value of Marginal Whole Product of Labor = Wage

Since $\mathbf{P}\cdot\mathbf{MWP_L} = w + (p - MC)/\Delta L$, the above result is equivalent to the previous p = MC.

Comparison of the Two Treatments of MP Theory

We have given an alternative treatment of MP theory. This treatment uses the juridical notion of the responsible factor (here taken as labor) to organize the presentation. The crux of the two presentations is in the two marginal conditions concerning labor:

Conventional labor equation:	$p\cdot MP_L = w$
Alternative labor equation:	$\mathbf{P}\cdot\mathbf{MWP_L} = w.$

Figure 5.3. Comparison of Two Equations for Labor

When costs are minimized, both labor conditions are equivalent to the familiar profit maximization condition p = MC.

In the conventional labor equation, p and MP_L (as well as w) are scalars. In the alternative equation, **P** and $\mathbf{MWP_L}$ are vectors (while w remains a scalar). The conventional interpretation of MP_L pictures labor as producing marginal products without using up any inputs ("virgin birth of marginal products"). The marginal whole product of labor $\mathbf{MWP_L}$ gives the picture of the marginal effect of labor as producing outputs by using up other inputs.

Since the alternative presentation gives a more realistic treatment of marginal production, one might ask why it isn't used. One "problem" in the alternative treatment is that introduces an asymmetry between labor and the nonhuman factors—or in more abstract terms, between the responsible and non-responsible factors. Since conventional production is based on all factors being treated symmetrically as being legally rentable or hirable, it is inconvenient to have a theory that suggests an alternative arrangement [as in Ellerman 1992].

One could, of course, take capital services as the active or responsible factor, define the marginal whole product of capital as $\mathbf{MWP_K} = (1/\Delta K, 0, -\Delta L/\Delta K)$, and then show that the following condition is also equivalent to profit maximization (when costs are minimized).

$$P \cdot MWP_K = r$$

Equation 5.10. Profit Max Implies: Value of Marginal Whole Product of Capital = Rental

But instead of restoring a peaceful symmetry, this only highlights the conflict since one cannot plausibly represent both capital as producing the product by using labor, and labor as producing the product by using capital. MP theory itself provides no grounds for choosing one of the conflicting pictures over the other—for choosing the picture of the burglar using the tools to commit the crime over the picture of the tools using the burglar to commit the crime. The distinction between the two pictures comes from jurisprudence, not from economics.

The conventional treatment of MP theory is clearly superior in terms of a "symmetrical" treatment of persons and things. The marginal unit of each factor can be presented as producing its marginal product (immaculately without using other inputs). The same picture can be used for each factor without any conflict.

Since the alternative treatment that acknowledges that marginal products cannot be produced *ex nihilo* seems superior on empirical grounds, orthodox economics would indeed seem to choose the conventional treatment of MP theory over the mathematically equivalent alternative treatment on "nonempirical" grounds.

Appendix

Standard MP Theory

Let $y = f(x_1,...,x_n)$ be a smooth neoclassical production function with p as the competitive unit price of the output y and $w_1,...,w_n$ as the respective competitive unit prices of the inputs $x_1,...,x_n$. The cost minimization problem involves the input prices and a given level of output y_0:

$$\text{minimize:} \quad C = \sum_{i=1}^{n} w_i x_i$$

$$\text{subject to:} \quad y_0 = f(x_1,\ldots,x_n).$$

Figure 5.4. Minimize Cost to Product Given Output

Forming the lagrangian

$$L = \sum w_i x_i + \lambda[y_0 - f(x_1,\ldots,x_n)],$$

the first-order conditions

$$\partial L/\partial x_i = w_i - \lambda\, \partial f/\partial x_i = 0 \text{ for } i = 1,...,n$$

solve to:

$$\lambda = \frac{w_1}{\partial f/\partial x_1} = \cdots = \frac{w_n}{\partial f/\partial x_n}.$$

Equation 5.11. First-Order Conditions for Cost Minimization

These equations together with the production function determine the n unknowns $x_1,...,x_n$. Varying the input prices and level of output parametrically determines the *conditional factor demand functions*:

$$\begin{aligned} x_1 &= \varphi_1(w_1,\ldots,w_n,y) \\ &\vdots \\ x_n &= \varphi_n(w_1,\ldots,w_n,y). \end{aligned}$$

Equation 5.12. Conditional Factor Demand Functions

These functions give the optimum level of the inputs to minimize the cost to produce the given level of output at the given input prices. Taking the input prices as fixed parameters, we can write the conditional factor demand functions as $x_i = \varphi_i(y)$ for i = 1,...,n. These functions define the cost-minimizing *expansion path* through input space parameterized by the level of output. Substituting into the sum for total costs yields the

$$C(y) = \sum w_i \varphi_i(y).$$

Equation 5.13. Cost Function

Differentiation by y yields the marginal cost function.

$$MC = \frac{dC}{dy} = \sum w_i \frac{\partial \varphi_i}{\partial y}.$$

Equation 5.14. Marginal Cost

The factor demand functions can also be substituted into the production function to obtain the identity:

$$y \equiv f(\varphi_1(y), \ldots, \varphi_n(y)).$$

Differentiating both sides with respect to y yields the useful equation:

$$1 = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial \varphi_i}{\partial y}.$$

Multiplying both side by the lagrange multiplier allows us to identity it as the marginal cost.

$$\lambda = \sum_{i=1}^{n} \left(\lambda \frac{\partial f}{\partial x_i} \right) \frac{\partial \varphi_i}{\partial y} = \sum w_i \frac{\partial \varphi_i}{\partial y} = MC.$$

Equation 5.15. Lagrange Multiplier of Min Cost Problem is Marginal Cost

Using the customary marginal product notation, $MP_i = \partial f/\partial x_i$ for i = 1,...,n, the first order conditions for cost minimization can be written as:

$$MC = \frac{w_1}{MP_1} = \cdots = \frac{w_n}{MP_n}.$$

Equation 5.16. Cost Minimization Conditions

The marginal products should not be confused with the reciprocals of the factor demand functions:

$$\frac{\partial f}{\partial x_i} \neq 1 \Big/ \frac{\partial \varphi_i}{\partial y}.$$

The marginal product of x_i gives the marginal increase in y when there is both a marginal increase in x_i and a shift to a more x_i-intensive production technique so that exactly the same amount of the other inputs is used. No factor prices or cost minimization is involved in the definition. The reciprocal of $\partial\varphi_i/\partial y$ gives the marginal increase in y associated with a marginal increase in x_i when there is a corresponding increase in the other inputs so as to produce the new output at minimum cost.

MP Theory with Product Vectors

For the inclusive algebraically symmetric notion of the product, we will use vectors with the outputs listed first followed by components for the inputs. The *positive product* is (y,0,...,0), the *negative product* is $(0,-x_1,...,-x_n)$, and their sum is the

$$WP = (y, -x_1, \ldots, -x_n).$$

Equation 5.17. Whole Product Vector **WP**

The whole product vector is usually called the "production vector" or "net output vector" [Varian 1984, 8] in the set-theoretic presentations using production sets rather than production functions. Assuming that costs are minimized at each output level, we can restrict attention to the whole product vectors along the expansion path:

$$WP(y) = (y, -\varphi_1(y), \ldots, -\varphi_n(y)).$$

The gradient of the whole product vector is the *marginal whole product* **MWP**.

$$MWP(y) = \nabla WP(y) = \left(1, -\frac{\partial \varphi_1}{\partial y}, \ldots, -\frac{\partial \varphi_n}{\partial y}\right)$$

Equation 5.18. Marginal Whole Product Vector **MWP**

The price vector is **P** = (p, w_1,...,w_n), the value of the whole product (the dot product of the price and whole product vectors) is the profit.

$$P \bullet WP = py - \sum w_i x_i = py - C(y),$$

Equation 5.19. Value of Whole Product = Profit

and the value of the marginal whole product is the

$$P \bullet MWP(y) = P \bullet \left(1, -\frac{\partial \varphi_1}{\partial y}, \ldots, -\frac{\partial \varphi_n}{\partial y}\right) = p - \sum w_i \frac{\partial \varphi_i}{\partial y} = p - MC.$$

Equation 5.20. Value of Marginal Whole Product = Marginal Profit

The necessary condition for profit maximization is that the marginal whole product has zero net value, which yields the familiar conditions p = MC. Substituting p for MC in the cost minimization conditions yields the central equations in the usual presentation of MP theory:

$$p\ MP_i = w_i \ \text{ for } i = 1,...,n$$

which are interpreted as showing that in competitive equilibrium, each unit of a factor is paid (w_i) the value of what it produces (p MP_i).

One Responsible Factor

We move now to the formulation of the same mathematics but with certain factors treated as responsible factors, i.e., the treatment of MP theory with responsible factors. At first we assume only one responsible factor that can be arbitrarily taken as the first factor, which provides the services x_1. In terms of totals, the responsible factor, by performing the services or actions x_1, is responsible for producing y and is responsible for using up the other inputs x_2,...x_n. Since the customary notation lists x_1 along side the other inputs, we could also picture the responsible factor as both producing and using up x_1 (which thus cancels out). Thus the *whole product of the responsible factor* is:

$$WP_1 = (y, 0, -x_2, \ldots, -x_n) = WP + (0, x_1, 0, \ldots, 0).$$

Equation 5.21. Whole Product of Responsible Factor x_1

The whole product of the responsible factor is the sum of the whole product and the services of the responsible factor.

Since we are now assuming only one responsible factor, we have the luxury of mathematically treating its actions as the independent variable. Restricting attention to the expansion path as usual and assuming $\partial\varphi_1/\partial y \neq 0$, we can invert the first factor demand function to obtain

$$y = \varphi_1^{-1}(x_1)$$

which can be substituted into the other factor demand functions to obtain the other inputs as functions of x_1:

$$x_i = \varphi_i\left(\varphi_1^{-1}(x_1)\right) \text{ for } i = 2,...,n.$$

The whole product of the responsible factor can then be expressed as a function of x_1:

$$WP_1(x_1) = \left(\varphi_1^{-1}(x_1), 0, -\varphi_2\left(\varphi_1^{-1}(x_1)\right), \ldots, -\varphi_n\left(\varphi_1^{-1}(x_1)\right)\right).$$

Equation 5.22. Whole Product of Responsible Factor x_1 as a Function of x_1

We can now present a realistic picture of the effects of a marginal increase in the responsible factor. A marginal increase in x_1 with both use up the other factors at the rate

$$\frac{\partial\varphi_i\left(\varphi_1^{-1}\right)}{\partial x_1} = \frac{\partial\varphi_i / \partial y}{\partial\varphi_1 / \partial y}$$

and will increase the output at the rate

$$\frac{\partial\varphi_1^{-1}}{\partial x_1} = \frac{1}{\partial\varphi_1 / \partial y}$$

along the expansion path. This information is given by the x_1 gradient of the whole product of the responsible factor, which is the *marginal whole product of the responsible factor*:

$$MWP_1 = \nabla WP_1(x_1) = \left(\frac{1}{\partial\varphi_1/\partial y}, 0, -\frac{\partial\varphi_2/\partial y}{\partial\varphi_1/\partial y}, \ldots, -\frac{\partial\varphi_n/\partial y}{\partial\varphi_1/\partial y} \right).$$

Equation 5.23. Marginal Whole Product of Responsible Factor

This marginal whole product vector $\mathbf{MWP_1}$ presents what the responsible factor is marginally responsible for in quantity terms. Thus it should be compared with the marginal product MP_1 in the conventional treatment of MP theory. The marginal product MP_1 is fine as a mathematically defined partial derivative. But to interpret it in terms of production, one has to consider the purely notional shift to a more x_1-intensive productive technique so that exactly the same amount of the other factors is consumed. That is not how output changes in the cost-minimizing firm. The marginal whole product $\mathbf{MWP_1}$ presents the actual marginal changes in the output and the other factors associated with a marginal increase in x_1 along the expansion path.

The value of the marginal whole product of x_1 is the dot product:

$$P\bullet MWP_1 = \frac{\left[p - w_2 \frac{\partial\varphi_2}{\partial y} - \cdots - w_n \frac{\partial\varphi_n}{\partial y} \right]}{\partial\varphi_1/\partial y} = \frac{[p - MC]}{\partial\varphi_1/\partial y} + w_1 .$$

Equation 5.24. Value of Marginal Whole Product of Responsible Factor

Thus we have that the necessary condition for profit maximization, $p = MC$, is equivalent to

$$P\bullet MWP_1 = w_1 .$$

Equation 5.25. Profit Max Implies Value of Marginal Whole Product = Factor Price

Production is carried to the point where the value of the marginal whole product of the responsible factor is equal to its opportunity cost given by w_1. Since we are assuming cost minimization, this is also equivalent to the conventional equation:

$$p\,MP_1 = w_1.$$

Several Jointly Responsible Factors

The generalization to several jointly responsible factors is straightforward. The main mathematical difference is that we lose the luxury of parameterizing motion along the expansion path by "the responsible factor," since we now assume several such factors. Hence output will be used as the independent variable to represent motion along the expansion path.

The whole product vector **WP**(y) and the marginal whole product vector **MWP**(y) are the same as before. Suppose there are m jointly responsible factors, which we can take to be the first m factors. Intuitively, by performing the services or actions $x_1,...,x_m$, the responsible factors use up the inputs $x_{m+1},...,x_n$ and produce the outputs y. As before the *whole product of the responsible factors*, now symbolized $\mathbf{WP_r}$(y), can be presented as the sum of the whole product and the services of the responsible factors:

$$\begin{aligned} WP_r(y) &= WP + (0, \varphi_1(y), \ldots, \varphi_m(y), 0, \ldots, 0) \\ &= (y, 0, \ldots, 0, -\varphi_{m+1}(y), \ldots, -\varphi_n(y)). \end{aligned}$$

Equation 5.26. Whole Product of Responsible Factors $x_1,..., x_m$

The *marginal whole product of the responsible factors* (with variation parameterized by y) is the y gradient of $\mathbf{WP_r}$(y):

$$MWP_r(y) = \nabla WP_r(y) = \left(1, 0, \ldots 0, -\frac{\partial \varphi_{m+1}}{\partial y}, \ldots, -\frac{\partial \varphi_n}{\partial y}\right)$$

Equation 5.27. Marginal Whole Product of Responsible Factors $x_1,..., x_m$

and its value is the dot product with the price vector.

$$P \bullet MWP_r = p - w_{m+1}\frac{\partial \varphi_{m+1}}{\partial y} - \cdots - w_n \frac{\partial \varphi_n}{\partial y} = [p - MC] + \sum_{j=1}^{m} w_j \frac{\partial \varphi_j}{\partial y}$$

Equation 5.28. Value of Marginal Whole Product of Responsible Factors $x_1,..., x_m$

When producing the marginal increase in output by using up the marginal amounts of the other inputs, the responsible factors use up the marginal services

$$\left(0,\frac{\partial\varphi_1}{\partial y},\ldots,\frac{\partial\varphi_m}{\partial y},0,\ldots,0\right)$$

which have the opportunity cost of

$$\sum_{j=1}^{m} w_j \frac{\partial\varphi_j}{\partial y}.$$

Figure 5.5. Opportunity Cost of Marginal Responsible Services for Marginal Increase in y

Hence value is maximized when the responsible factors carry production to the point when the value of their marginal whole product is equal to their marginal opportunity cost which is clearly equivalent to the equation: p = MC [see Equation 5.28].

$$P \bullet MWP_r = \sum_{j=1}^{m} w_j \frac{\partial\varphi_j}{\partial y}$$

Equation 5.29. Value of Marginal Whole Product of Responsible Services = Their Opportunity Cost

Extreme Cases

It may be of some interest to take the two extreme cases when no factors or all factors are taken as being responsible.

When no factors are taken as responsible, then production is seen as a natural event rather than a human activity (on the assumption that humans are responsible factors). The product is produced and the inputs are used up—but not by anyone. No one is responsible. This is perhaps the world imagined by economists who adopt the pose of "social physicists" describing natural processes. Then we can make the identifications;

whole product of responsible factors	= whole product,
marginal whole product of responsible factors	= marginal whole product,
value of marginal whole product of responsible factors	= marginal profit,
opportunity cost of marginal responsible factors	= 0.

Figure 5.6. Extreme Case of No Responsible Factors

Thus $P{\bullet}MWP_r = p - MC$ so when the value of the marginal whole product of the responsible factors is set equal to their marginal opportunity costs, then we simply have p – MC = 0.

If all the factors are taken as responsible then we are in the magical world of poets where "all the factors co-operate together to produce the product." In this world we can make the identifications:

whole product of responsible factors	= positive product,
marginal whole product of responsible factors	= (1,0,...,0)
value of marginal w.p. of responsible factors	= p,
opportunity cost of marginal responsible factors	= MC.

Figure 5.7. Extreme Case of All Responsible Factors

Thus $\mathbf{P{\cdot}MWP_r} = p$ so when the value of the marginal whole product of the responsible factors is set equal to their marginal opportunity costs MC, then we again have the equation p = MC

Several Products

To illustrate the generalization to several products, we consider an example with two products y_1 and y_2. The production possibilities can be given in the form:

$$F(y_1,y_2,x_1,...,x_n) = 0.$$

Given the output levels y_1 and y_2, the cost minimization problem is:

$$\text{minimize:} \quad C = \sum_{i=1}^{n} w_i x_i$$

$$\text{subject to:} \quad F(y_1,y_2,x_1,...,x_n) = 0.$$

Figure 5.8. Cost Minimization Problem with Two Outputs

The lagrangian is

$$L = \sum_{i=1}^{n} w_i x_i - \lambda F(y_1, y_2, x_1, \ldots, x_n)$$

and the first-order conditions are $w_i - \lambda\partial F/\partial x_i$ for i = 1,...,n. Determining the cost-minimizing input levels in terms of the given output levels yields the conditional factor demand functions

$$x_i = \varphi_i(y_1, y_2) \text{ for } i = 1,...,n.$$

Equation 5.30. Conditional Factor Demand with Two Outputs

Substituting into the cost sum yield the cost function $C(y_1,y_2)$ and the marginal costs

$$MC_j = \frac{\partial C}{\partial y_j} = \sum_{i=1}^{n} w_i \frac{\partial \varphi_i}{\partial y_j} \text{ for } j = 1,2.$$

Equation 5.31. Marginal Costs of the Two Outputs

The whole product vector (parameterized by y_1 and y_2) is

$$WP(y_1, y_2) = \left(y_1, y_2, -\varphi_1(y_1, y_2), \ldots, -\varphi_n(y_1, y_2)\right)$$

and the two marginal whole products with respect to y_1 and y_2 are the two gradients with respect to those variables:

$$\nabla_1 WP = \left(1, 0, -\frac{\partial \varphi_1}{\partial y_1}, \ldots, -\frac{\partial \varphi_n}{\partial y_1}\right)$$

$$\nabla_2 WP = \left(0, 1, -\frac{\partial \varphi_1}{\partial y_2}, \ldots, -\frac{\partial \varphi_n}{\partial y_2}\right)$$

Equation 5.32. Marginal Whole Products with Respect to the Two Outputs

With output unit prices p_1 and p_2, the value of the marginal whole products must be zero for profits to be maximized:

$$P \bullet \nabla_j WP = (p_1, p_2, w_1, \ldots, w_n) \bullet \nabla_j WP = p_j - MC_j = 0 \text{ for } j = 1,2.$$

Equation 5.33. Profit Maximization Conditions for Multiple Outputs

Let the first m factors be the responsible factors as before. The whole product of the responsible factors is the sum of the whole product and the services of the responsible factors:

$$WP_r(y_1, y_2) = \left(y_1, y_2, 0, \ldots, 0, -\varphi_{m+1}(y_1, y_2), \ldots, -\varphi_n(y_1, y_2)\right)$$

Equation 5.34. Whole Product of Responsible Factors and the marginal whole products of the responsible factors would be the two gradients with respect to y_1 and y_2. The values of those marginal whole products are:

$$P\bullet\nabla_j WP_r = p_j - MC_j + \sum_{i=1}^{n} w_i \frac{\partial \varphi_i}{\partial y_j} \text{ for } j = 1,2.$$

Equation 5.35. Value of Marginal Whole Products of Responsible Factors for each of the Two Outputs

To produce a marginal increase in y_1, the responsible factors must used their actions, which have the marginal opportunity cost:

$$\sum_{i=1}^{n} w_i \frac{\partial \varphi_i}{\partial y_1}$$

Figure 5.9. Marginal Opportunity Cost of Responsible Factors for Marginal Increase in y_1

and similarly for y_2. Value is maximized when the responsible factors carry production of each output to the point when the value of their respective marginal whole product is equal to their respective marginal opportunity costs:

$$P\bullet\nabla_j WP_r = \sum_{i=1}^{n} w_i \frac{\partial \varphi_i}{\partial y_j}$$

Equation 5.36. Profit Max Implies Value of Marginal Whole Products of Responsible Factors Is Their Opportunity Cost

which is clearly equivalent to pj = MCj for j = 1,2 [see Equation 5.35]. This example with several products helps to motivate the next multi-product model where there is no substitution.

An Example Without Substitution

We have criticized the usual interpretation of MP_i as the "product of the marginal unit of x_i" on a number of grounds. For instance, a marginal increase in x_i cannot produce an increase in the output out of thin air. Other inputs will be needed. The definition of the partial derivative MP_i however assumes substitutability in the sense that there is a shift to a slightly more x_i intensive productive technique so that more output can be produced using exactly the same amount of the other factors. Yet we have shown that such an imaginary shift is not necessary to interpret marginal productivity theory. By using vectorial notions of the product, the theory can be expressed quite plausibly using marginal whole products computed along the cost-minimizing expansion path.

The luxury of the alternative treatment of MP theory becomes a necessity when there is no substitutability as in a Leontief input-output model. Hence we will give the alternative treatment of MP theory in such a model.

We will consider an example where there are n commodities $x_1,...,x_n$ and labor L where the latter is taken as the services of the responsible factor. The technology is specified by the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ where a_{ij} gives the number of units of the i^{th} good needed per unit of the j^{th} good as output. Thus for the output column vector $\mathbf{x} = (x_1,...,x_n)^T$ (the superscript "T" denotes the transpose), the vector of required commodity inputs is $\mathbf{Ax}$. The labor requirements per unit are given by the vector $\mathbf{a_0} = (a_{01},...,a_{0n})$, so the total labor requirement is the scalar $L = \mathbf{a_0x}$.

Let $\mathbf{p} = (p_1,...,p_n)$ be the price vector and let w be the wage rate. We assume that the outputs and inputs are separated by one time period (a "year") and that r is the annual interest rate. The competitive equilibrium condition is usually stated as the zero-profits condition with no mention of marginal productivity or the like. With labor taking its income at the end of the year, the zero-profit condition for any output vector is:

$$\mathbf{px} = (1+r)\mathbf{pAx} + w\mathbf{a_0x}.$$

Since this must hold for any $\mathbf{x}$, we can extract the following vector equation.

$$\mathbf{p} = (1+r)\mathbf{pA} + w\mathbf{a_0}$$

Equation 5.37. Competitive Equilibrium Condition

We now show how this condition can be derived using MP-style reasoning with products represented as vectors. The whole product will be a 2n+1 component column vector since the output vector $\mathbf{x}$ is produced a year after the input vector $\mathbf{Ax}$. The following notation for the whole product is self-explanatory:

$$WP = \begin{bmatrix} x \\ -Ax \\ -a_0 x \end{bmatrix}.$$

Equation 5.38. Whole Product Vector **WP**

The whole product of the responsible factor is, as always, the sum of the whole product and the services of the responsible factor (since the factor is represented as both producing and using up its own services):

$$WP_L = WP + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_0 x \end{bmatrix} = \begin{bmatrix} x \\ -Ax \\ 0 \end{bmatrix}$$

Equation 5.39. Whole Product of Labor Vector $\mathbf{WP_L}$

To consider output variations, we use the output unit vectors $\boldsymbol{\delta}_j = (0,...,0,1,0,...,0)^T$ where the "1" is in the j^{th} place. The marginal whole product of the responsible factor with respect to the j^{th} output is will be symbolized as:

$$\nabla_j WP_L = \begin{bmatrix} \delta_j \\ -A\delta_j \\ 0 \end{bmatrix}.$$

Equation 5.40. Marginal Whole Product of Labor with Respect to the j^{th} Output

and the required labor is $\mathbf{a_0 \delta_j} = a_{0j}$ with the opportunity cost of wa_{0j}. The price vector stated in year-end values is $\mathbf{P} = (\mathbf{p}, (1+r)\mathbf{p}, w)$ so the value of the marginal whole product of labor is:

$$P \bullet \nabla_j WP_L = p_j - (1+r)pA\delta_j.$$

Equation 5.41. Value of Marginal Whole Product of Labor with respect to the j^{th} Output

When the value of that marginal whole product of the responsible factor with respect to the j^{th} output is set equal to opportunity cost of the necessary labor wa_{0j} for $j = 1,...,n$, then we again have the same equilibrium conditions:

$$\mathbf{p} - (1+r)\mathbf{pA} = w\mathbf{a_0}.$$

Equation 5.42. Competitive Equilibrium Condition Expressed as: Value of Marginal Whole Product of Labor with Respect to Each Output = Its Opportunity Cost.

Thus the alternative presentation of MP theory with product vectors and responsible factors can be used in models without substitution where the conventional marginal products are undefined.

References

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