Chapter 6
Double-Entry Bookkeeping: Mathematical Formulation and Generalization

Introduction

Double-entry bookkeeping illustrates one of the most astonishing examples of intellectual insulation between disciplines, in this case, between accounting and mathematics. Double-entry bookkeeping (DEB) was developed during the fifteenth century and was first recorded in 1494 as a system by the Italian mathematician Luca Pacioli [1494]. Double-entry bookkeeping has been used for several centuries in the accounting systems of the market economies throughout the world. Incredibly, however, the mathematical formulation for DEB is not known, at least not in the field of accounting.

The mathematical basis behind DEB (algebraic operations on ordered pairs of numbers) was developed in the nineteenth century by Sir William Rowan Hamilton as an abstract mathematical construction to deal with complex numbers and fractions [Hamilton 1837]. The particular example of the ordered pairs construction that is relevant to DEB, called the "group of differences," is the one used in undergraduate algebra courses to construct a number system with negative numbers ("additive inverses" in technical terms) by using operations on ordered pairs of positive numbers (including zero). All that is required to grasp the connection with DEB is to make the identification:

ordered pairs of numbers in construction of positive and negative numbers
= two-sided T-accounts of DEB (debits on the left side and credits on the right side).

In view of this identification, the "group of differences" will be called the *Pacioli group*.

In spite of some attention to DEB by mathematicians [e.g., DeMorgan 1869, Cayley 1894, and Kemeny et al. 1962], this connection has been little noticed in mathematics. One (perhaps solitary) exception is the following passage in a semi-popular book by D. E. Littlewood.

> The bank associates two totals with each customer's account, the total of moneys credited and the total of moneys withdrawn. The net balance is then regarded as the same if, for example, the credit amounts of £102 and the debit £100, as if the credit were £52 and the debit £50. If the debit exceeds the credit the balance is negative.
>
> This model is adopted in the definition of signed integers. Consider pairs of cardinal numbers (a, b) in which the first number corresponds to the debit, and the second to the credit. [1960, 18]

With this exception, the author has not been able to find a single mathematics book, elementary or advanced, popular or esoteric, which notes that the ordered pairs of the group of differences construction are the T-accounts used in the business world for about five centuries. And this mathematical basis for DEB is totally unknown in the "parallel universe" of accounting.

This almost complete lack of cross-fertilization between mathematics and accounting is a topic of some interest for intellectual history and the sociology of knowledge. The story is probably rather simple from the mathematics side. Double-entry bookkeeping is apparently too mundane to hold the sustained attention of mathematicians. The real question lies on the accounting side. Over the last century, how could professional accountants and accounting professors have failed to find the mathematical basis for DEB even though it was part of undergraduate algebra?

One acid test of a mathematical formulation of a theory is the question of whether or not it facilitates the generalization of the theory. Normal bookkeeping does not deal with incommensurate physical quantities; everything is expressed in the common units of money. A long-standing question is the possible generalization of DEB to deal with incommensurates with no common measure of value.

In the literature on the "mathematics of accounting" there was a proposed "solution" to this question, a system of physical accounting that was published repeatedly [see Ijiri 1965, 1966, and 1967] and was largely accepted by the accounting community. In this system, most of the structure of DEB was lost:

- there was no balance sheet equation,
- there were no equity or proprietorship accounts,
- the temporary or nominal accounts could not be closed, and
- the "trial balance" did not balance.

It is common for certain aspects of a theory to be lost in a generalization of the theory. The accounting community had apparently accepted the failure of all these features of DEB as the necessary price to be paid to generalize DEB to incommensurate physical quantities. For instance, the systems of "Double-entry multidimensional accounting" previously published in the accounting literature [see also Charnes et al. 1972, 1976, or Haseman and Whinston 1976] had acquiesced in the absence of the balance-sheet equation.

> For instance, the convenient idea of an accounting identity is lost since the dimensional and metric comparability it assumes is no longer present except under special circumstances. [Ijiri 1967, 333]

When DEB is mathematically formulated using the group of differences, then the generalization to vectors of incommensurate physical quantities is immediate and trivial. All of the normal features of DEB (such as the balance-sheet equation, the equity account, the temporary accounts, and the trial balance) are preserved in the generalization [see Ellerman 1982, 1986]. Thus the "accepted" generalized model of DEB was simply a failed attempt at generalization which had been received as a successful generalization that unfortunately had to "sacrifice" certain features of DEB.

In spite of the results that can be obtained from a simple border crossing between mathematics and accounting, the successful mathematical treatment and generalization of double-entry bookkeeping (first published over a decade ago) will take years if not decades to become known and understood in accounting.

The Pacioli Group

We will develop the Pacioli group using vectors (ordered lists of numbers) for multidimensional accounting. The usual case of scalar accounting can be identified with the special case using one dimensional vectors. A vector $x = (x_1,...,x_n)$ is *non-negative* if and only if all its components x_i are non-negative (positive or zero). The ordered pairs of non-negative vectors will be called *T-accounts* and will be denoted as follows for vectors d and c.

[d // c] = [debit vector // credit vector]

Equation 6.1. Definition of T-Accounts

The left-hand side (LHS) vector d is the debit entry and the right-hand side (RHS) vector c is the credit entry. A fraction is also an ordered pair of whole numbers or integers where the two integers are the numerator and denominator of the fraction.

The algebraic operations on T-accounts are much like the operations on fractions except that addition is substituted for multiplication. In order to illustrate the additive-multiplicative analogy between T-accounts and fractions, the basic definitions will be developed in parallel columns.

T-accounts add together by adding debits to debits and credits to credits
[w // x] + [y // z] = [w+y // x+z].
The identity element for addition is the zero T-account [0 // 0]. Given two T-accounts [w // x] and [y // z], the *cross-sums* are the two vectors obtained by adding the credit entry in one T-account to the debit entry in the other T-account. The equivalence relation between T-accounts is defined by setting two T-accounts *equal* if their cross-sums are equal:
[w // x] = [y // z] if w+z = x+y.
The negative or additive inverse of a T-account is obtained by reversing the debit and credit entries:
– [w // x] = [x // w].
Given two vectors $x = (x_1,...,x_n)$ and $y = (y_1,...,y_n)$, let max(x,y) be the vector with the maximum of x_i and y_i as its i^{th} component, and let min(x,y) be the vector with the minimum of x_i and y_i as its i^{th} component.

Fractions multiply together by multiplying numerator with numerator and denominator with denominator
(w/x)(y/z) = (wy/xz).
The identity element for multiplication is the unit fraction 1/1. Given two fractions w/x and y/z, the *cross-multiples* are the two integers obtained by multiplying the numerator of one faction with the denominator of the other fraction. The equivalence relation between fractions is defined by setting two fractions *equal* if their cross-multiples are equal:
w/x = y/z if wz = xy.
The multiplicative inverse of a fraction is obtained by reversing the numerator and denominator:
$(w/x)^{-1} = x/w$.
Given two integers w and x, let lcm(w,x) be the least common multiple of w and x, and let gcd(w,x) be the greatest common divisor of w and x (the largest integer dividing both).

Two non-negative vectors x and y are said to be *disjoint* if min(x,y) = 0. A T-account [x // y] is in *reduced form* if x and y are disjoint. Every T-account [x // y] has a unique reduced representation [x–min(x,y) // y–min(x,y)].	Two integers w and x are said to be *relatively prime* if gcd(w,x) = 1. A fraction w/x is in *lowest terms* if w and x are relatively prime. Each fraction w/x has a unique representation in lowest terms (w/gcd(w,x))/(x/gcd(w,x)).
Consider the T-account [(12, 3, 8) // (10, 5, 4)]. The minimum of the debit and credit vectors is (10, 3, 4) so the reduced form representation is [(2, 0, 4) //(0, 2, 0)].	Consider the fraction 28/35. The greatest common divisor of the numerator and denominator is 7 so the fraction in lowest terms is 4/5.

That completes the construction of the *Pacioli group* P^n where each element is an ordered pair [x // y] of non-negative n-dimensional vectors. The Pacioli group P^n is isomorphic with all of $\mathbf{R}^n$ (the set of all n-vectors with positive and negative components) under two isomorphisms: the debit isomorphism, which maps [w // x] to w–x, and the credit isomorphism, which maps [w // x] to x–w. In order to translate from T-accounts [x // y] to general vectors z, one only need specify whether to use the debit or credit isomorphism. This will be done by labeling the T-account as *debit balance* or *credit balance*. Thus if a T-account [x // y] is debit balance, the corresponding vector is x–y, and if it is credit balance, then the corresponding vector is y–x.

The Double-entry Method

The double-entry method of accounting is a method of using the Pacioli group to perform additive algebraic operations on equations such as the conventional balance sheet equation:

Assets = Liabilities + Net Worth.

Equation 6.2. Balance-Sheet Equation

Vector equations are first *encoded* in the Pacioli group. A T-account equal to the zero T-account [0 // 0] is called a *zero-account*. Equations encode as zero-accounts. Since the vectors in a T-account must be non-negative, we must first develop a way to separate out the positive and negative components of a vector. The *positive part* of a vector x is $x^+ = \max(x,0)$, the maximum of x and the zero vector [note that "0" is used depending on the context to refer to the zero scalar or the zero vector]. The *negative part* of x is $x^- = -\min(x,0)$, the negative of the minimum of x and the zero vector. Both the positive and negative parts of a vector x are non-negative vectors. Every vector x has a "Jordan decomposition" $x = x^{+} - x^{-}$.

Given any equation in $\mathbf{R}^n$, $w + ... + x = y + ... + z$, each left-hand side (LHS) vector w is encoded as a debit-balance T-account $[w^{+} // w^{-}]$ and each right-hand side (RHS) vector y is encoded as a credit-balance T-account $[y^{-} // y^{+}]$. Then the original equation holds if and only the sum of the encoded T-accounts is a zero-account:

$$w + ... + x = y + ... + z$$

if and only if

$$[w^{+} // w^{-}] + ... + [x^{+} // x^{-}] + [y^{-} // y^{+}] + ... + [z^{-} // z^{+}] \text{ is a zero-account.}$$

Equation 6.3. Encoding an Equation as a Zero T-Account

Given the equation, the sum of the encoded T-accounts is called the *equational zero-account* of the equation. Since only plus signs can appear between the T-accounts in an equational zero-account, the plus signs can be left implicit. The listing of the T-accounts in an equational zero-account (without the plus signs) is the *ledger*.

Changes in the various terms or "accounts" in the beginning equation are recorded as *transactions*. Transactions must be recorded as valid algebraic operations which transform equations into equations. Since equations encode as zero-accounts, a valid algebraic operation would transform zero-accounts into zero-accounts. There is only one such operation in the Pacioli group: add on a zero-account. Zero plus zero equals zero. The zero-accounts representing transactions are called *transactional zero-accounts*. The listing of the transactional zero-accounts is the *journal*.

A series of valid additive operations on an equation can then be presented in the following standard scheme:

Beginning Equational Zero-Account
\+ Transactional Zero-Accounts
= Ending Equational Zero-Account

or, in more conventional terminology,

Beginning Ledger

+ <u>Journal</u>

= Ending Ledger.

The process of adding the transactional zero-accounts to the initial ledger to obtain the ledger at the end of the accounting period is called *posting the journal to the ledger*. The fact that a transactional zero-account is equal to [0 // 0] is traditionally expressed as the *double-entry principle* that transactions are recorded with equal debits and credits. The summing of the debit and credit sides of a computed equational zero-account to check that it is indeed a zero-account is traditionally called the *trial balance*.

It remains to decode the ending equational zero-account to obtain the equation that results from the algebraic operations represented in the transactions. The T-accounts in an equational zero-account can be arbitrarily partitioned into two sets DB (debit balance) and CB (credit balance). T-accounts [w // x] in DB are decoded as w–x on the left side of the equation, and T-accounts [w // x] in CB are decoded as x–w on the right side of the equation. Given a zero-account, this procedure yields an equation. In an accounting application, the T-accounts in the final equational zero-account would be partitioned into sets DB and CB according to the side of the initial equation from which they were encoded.

To illustrate encoding and decoding equations, consider the vector equation

$$(6, -3, 10) + (-2, 5, 2) = (4, 2, 8).$$

Equation 6.4. Sample Vector Equation to be Encoded

It encodes as the equational zero-account

$$[(6, 0, 10) // (0, 3, 0)] + [(0, 5, 2) // (2, 0, 0)] + [(0, 0, 0) // (4, 2, 8)].$$

Figure 6.1. Equation Encoded as a Zero T-Account

To illustrate decoding, consider another equational zero-account,

$$[(8, 1, 4) // (2, 3, 6)] + [(1, 13, 3) // (5, 4, 2)] + [(2, 1, 3) // (4, 8, 2)].$$

Figure 6.2 Sample Zero T-Account to be Decoded

Let the first two T-accounts be debit balance and the third one credit balance. Then the equational zero-account decodes as the vector equation

$$(6, -2, -2) + (-4, 9, 1) = (2, 7, -1).$$

Equation 6.5. Decoded Equation

Given the additive-multiplicative analogy between the double-entry T-accounts and the double-entry fractions, one could develop a whole system of multiplicative double-entry bookkeeping [see Ellerman 1982, 58-66 for the theory with an example].

A Simple Example of Value Accounting

Consider an example of a company with the simplified initial balance sheet equation:

Assets	=	Liabilities	+	Equity
15,000	=	10,000	+	5,000.

Equation 6.6. Beginning Scalar Balance Sheet

The balance sheet equation encodes as an equational zero-account which, by leaving out the plus signs, becomes the following initial ledger of T-accounts.

Assets	Liabilities	Equity
[15,000 // 0]	[0 // 10,000]	[0 // 5,000]

Figure 6.3 Beginning Ledger of T-Accounts

A transaction will change two or more of the accounts. The fact that a transaction changes two or more accounts has nothing to do with the "doubleness" of double-entry bookkeeping. DEB is a system of *recording* transactions that uses the *double*-sided T-accounts of positive numbers (or the double-sided fractions in the multiplicative case). Any other way of recording the transaction (e.g., using positive and negative numbers) would also have to change two or more accounts in an equation. If one item in a equation changes, then clearly one or more other items in the equation must also change in order for the equation to still be true.

Consider three transactions in a productive firm.

1. $1,200 of input inventories are used up in production.

2. $1,500 of product is produced and sold.

3. $800 principal payment is made on a loan.

Each transaction is then encoded as a transactional zero-account and added to the equational zero-account.

	Assets	Liabilities	Equity
	[15,000 // 0]	[0 // 10,000]	[0 // 5,000]
1.	[0 // 1,200]		[1,200 // 0]
2.	[1,500 // 0]		[0 // 1,500]
3.	[0 // 800]	[800 // 0]	
	=========	=========	=========
Totals	[16,500 // 2,000]	[800 // 10,000]	[1,200 // 6,500]
= (in reduced form)	[14,500 // 0]	[0 // 9,200]	[0 // 5,300].

Figure 6.4. Initial Ledger + Journal = Ending Ledger

Each T-account is decoded according to how whether it was encoded as debit balance or credit balance to obtain the ending balance sheet equation.

Assets	=	Liabilities	+	Equity
14,500	=	9,200	+	5,300.

Equation 6.7. Ending Balance-Sheet Equation

A Simple Example of Multidimensional Property Accounting

When the scalars (single non-negative numbers) of value accounting are replaced by non-negative vectors, then the vectors can be interpreted as representing the physical amounts of different types of property. We will consider a simple model where there are only three types of property: cash, outputs, and inputs. These goods will be listed in that order in each three-dimensional vector.

Let the initial asset vector be (9000, 40, 50) so the firm has $9000 cash, 40 units of the output in inventory, and 50 units of the input in inventory. The firm also has a $10000 liability represented by the vector (10000, 0, 0) so the equity vector (Assets – Liabilities) is given by the vector (–1000, 40, 50). Thus the initial balance sheet (vector) equation is:

Assets	=	Liabilities	+	Equity
(9000, 40, 50)	=	(10000, 0, 0)	+	(–1000, 40, 50).

Equation 6.8. Initial Vector Balance-Sheet Equation

This encoded as the following equational zero-account or ledger:

Assets	Liabilities	Equity
[(9000, 40, 50)//(0, 0, 0)]	[(0, 0, 0)//(10000, 0, 0)]	[(1000, 0, 0)//(0, 40, 50)].

Figure 6.5. Initial Vector T-Accounts in Ledger

The underlying production process is very simple. Two units of the inputs are combined to make one unit of the output. Hence the following physical transactions underlie the previous value transactions (where we split the production and sale of the outputs are the transactions 2a and 2b).

1. 30 units of the inputs are used up in production.
2a. 15 units of the product are produced.
2b. 15 units of the product are sold for $100 each.
3. $800 principal payment is made on a loan.

These transactions are then encoded as transactional zero-accounts and added to the ledger T-accounts.

	Assets	Liabilities	Equity
	[(9000, 40, 50)//(0, 0, 0)]	[(0, 0, 0)//(10000, 0, 0)]	[(1000, 0, 0)//(0, 40, 50)].
1.	[(0, 0, 0)//(0, 0, 30)]		[(0, 0, 30)//(0, 0, 0)]
2a.	[(0, 15, 0)//(0, 0, 0)]		[(0, 0, 0)//(0, 15, 0)]
2b.	[(1500, 0, 0)//(0, 15, 0)]		[(0, 15, 0)//(1500, 0, 0)]
3.	[(0, 0, 0)//(800, 0, 0)]	[(800, 0, 0)//(0, 0, 0)]	
	=================	=================	=================
	[(10500, 55, 50)//(800, 15, 30)]	[(800, 0, 0)//(10000, 0, 0)]	[(1000, 15, 30)//(1500, 55, 50)]
=	[(9700, 40, 20)//(0, 0, 0)]	[(0, 0, 0)//(9200, 0, 0)]	[(0, 0, 0)//(500, 40, 20)]

Figure 6.6. Initial Vector Ledger + Journal = Ending Vector Ledger

where the last line of ledger accounts is in reduced form. The reduced accounts are then decoded to obtain the ending balance-sheet equation:

Assets	=	Liabilities	+	Equity
(9700, 40, 20)	=	(9200, 0, 0)	+	(500, 40, 20).

Equation 6.9. Ending Vector Balance-Sheet Equation

Given a set of prices or valuation coefficients, the vectors can be evaluated so that the vector accounts of property accounting collapse to the scalar accounts of value accounting. For instance, suppose that the prices per unit are (cash, output, input) = (1, 100, 40). Multiplying the physical quantities times their price and adding up yields the balance-sheet equation of the previous example of value accounting.

Assets	=	Liabilities	+	Equity
14,500	=	9,200	+	5,300.

Equation 6.10. Scalar Equation = Price Vector times Vector Equation

Thus we see how property accounting can use double-entry accounting with vectors to trace out the property transactions that underlie the value transactions recorded in conventional accounting.

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