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Where do Adjunctions Come From? Chimera Morphisms and Adjoint Functors in Category Theory

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Abstract: Category theory has foundational importance because it provides conceptual lenses to characterize what is important and universal in mathematics—with adjunction seeming to be the primary lens. Our topic is a theory showing "where adjoints come from". The theory is based on object-to-object "chimera morphisms", "heteromorphisms", or "hets" between the objects of *different* categories (e.g., the insertion of generators as a set-to-group map). After showing that heteromorphisms can be treated rigorously using the machinery of category theory (bifunctors), we show that all adjunctions between two categories arise (up to an isomorphism) as the representations (i.e., universal models) within each category of the heteromorphisms between the two categories. The conventional treatment of adjunctions eschews the whole concept of a heteromorphism, so our purpose is to shine a new light on this notion by showing its origin as a het *between* categories being universally represented *within* each of the two categories. This heteromorphic treatment of adjunctions shows how they can be split into two separable universal constructions. Such universals can also occur without being part of an adjunction. We conclude with the idea that it is the universal constructions (adjunctions being an important special case) that are really the foundational concepts to pick out what is important in mathematics and perhaps in other sciences, not to mention in philosophy.

Keywords: adjunctions; adjoint functors; heteromorphism; universal mapping properties

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1. Introduction: The Foundational Importance of Adjoints

Category theory is of foundational importance in mathematics but it is not "foundational" in the sense normally claimed by set theory. It does not provide some basic objects (e.g., sets) from which other mathematical objects can be constructed. Instead, the foundational role of category theory lies in providing conceptual lenses to characterize what is universal and natural in mathematics. For summary statements, see [1–3]. Two of the most important concepts are universal mapping properties and natural transformations. These two concepts are combined in the notion of adjoint functors. In recent decades, adjoint functors have emerged as the principal lens through which category theory plays out its foundational role of characterizing what is important in mathematics.

The developers of category theory, Saunders Mac Lane and Samuel Eilenberg, famously said that categories were defined in order to define functors, and functors were defined in order to define *natural* transformations. Their original paper [4] was entitled not "General Theory of Categories" but *General Theory of Natural Equivalences*. Adjoints were defined more than a decade later by Daniel Kan [5], but the realization of their foundational importance has steadily increased over time [6,7]. Now, it would perhaps not be too much of an exaggeration to see categories, functors, and natural transformations as the prelude to defining adjoint functors. As Steven Awodey stated in his text:

> The notion of adjoint functor applies everything that we have learned up to now to unify and subsume all the different universal mapping properties that we have encountered, from free groups to limits to exponentials. But more importantly, it also captures an important mathematical phenomenon that is invisible without the lens of category theory. Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics. [8] (p. 207)

Other category theorists have given similar testimonials:

> To some, including this writer, adjunction is the most important concept in category theory. [9] (p. 6)

> The isolation and explication of the notion of *adjointness* is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas. [10] (p. 438)

> Nowadays, every user of category theory agrees that [adjunction] is the concept which justifies the fundamental position of the subject in mathematics. [11] (p. 367)

Given the importance of adjoint functors in category theory and in mathematics as a whole, it would seem worthwhile to further investigate the concept of an adjunction. Whence adjoints? Where do adjoints come from? How do they arise? In this paper, we will present a theory of adjoint functors to address these questions.

Category theory groups together mathematical objects in *categories* with some common structure (e.g., sets, partially ordered sets, groups, rings, and so forth) and the appropriate morphisms between such objects. Since the morphisms are between objects of similar structure, they are ordinarily called "homomorphisms" or just "morphisms" or "homs" for short.

But there have always been other morphisms that occur in mathematical practice that are between objects with different structures (i.e., in different categories) such as an insertion-of-generator map from a set to the free group on that set. Indeed, the working mathematician might well characterize the free group $F(X)$ on a set X as the group such that for any set-to-group map $f : x \Rightarrow g$, there is a unique group homomorphism $f^* : F(x) \rightarrow g$ that factors f through the canonical insertion of generators $h_x : x \Rightarrow F(x)$, i.e., $f = f^* h_x$ ([12] (p. 69); [13] (p. 89); [14] (p. 65)). In order to contrast morphisms such as $f : x \Rightarrow g$ and $h_x : x \Rightarrow F(x)$ with the homomorphisms between objects within a category such as $f^* : F(x) \rightarrow g$, the double arrows might be called *heteromorphisms* (*hets*, for short) or, more colorfully, *chimera morphisms* (since they have a tail in one category and a head in another category). The usual machinery of category theory (bifunctors, in particular) can be adapted to give a rigorous treatment of heteromorphisms (and their compositions with homomorphisms) that is parallel to the usual bifunctorial treatment of homomorphisms.

With a precise notion of heteromorphisms in hand, it can then be seen that adjoint functors arise as the functors giving the representations, using homomorphisms *within* each category, of the heteromorphisms *between* two categories. That is, the basic connection, a het between two categories, is represented by a hom inside each of the two categories.

Often, one of the representations is the important one in the adjunction (with the other being a matter of conceptual bookkeeping). In the case of the free-group adjunction, the important representation is the representation of heteromorphisms $\mathrm{Het}(x, g)$ by the group of homomorphisms $\mathrm{Hom}(F(x), g)$, which is given by the natural isomorphism $\mathrm{Het}(x, g) \cong \mathrm{Hom}(F(x), g)$, which naturally pairs a het $f : x \Rightarrow g$ with the hom $f^* : F(x) \rightarrow g$. And, conversely, given a pair of adjoint functors, heteromorphisms can then be defined between (isomorphic copies of) the two categories so that the adjoints arise out of the representations of those heteromorphisms. Hence, this heteromorphic theory shows where adjoints "come from" or "how they arise". It would seem that this theory of adjoint functors was not developed in the early treatment of category theory since heteromorphisms, although present in mathematical practice, were not part of the initial machinery of category theory. The earliest heteromorphic treatment of adjunction (to the author's knowledge) was by Bodo Pareigis [15], and an extensive heteromorphic treatment of category theory was developed by Takahiro Kato [16] (using different terminology).

2. Methods: Theory of Adjoints

The cross-category object-to-object morphisms $c : x \Rightarrow a$, called *heteromorphisms* (*hets* for short) or *chimera morphisms*, will be indicated by double arrows ($\Rightarrow$) rather than single arrows ($\rightarrow$). The first question is how do heteromorphisms compose with one another? But this is not necessary. Chimeras do not need to "mate" with other chimeras to form a "species" or category; they only need to mate with the intra-category morphisms on each side to form other chimeras. The appropriate mathematical machinery to describe this is the generalization of a group acting on a set to a generalized monoid or category acting on a set (where each element of the set has a "domain" and a "codomain" to determine when composition is defined). In this case, there are two categories acting on a set, one on the left and one on the right. Given a chimera morphism $c : x \Rightarrow a$ from an object in a category $\mathbf{X}$ to an object in a category $\mathbf{A}$ and morphisms $h : x' \rightarrow x$ in $\mathbf{X}$ and $k : a \rightarrow a'$ in $\mathbf{A}$, the composition $ch : x' \rightarrow x \Rightarrow a$ is another chimera $x' \Rightarrow a$, and the composition $kc : x \Rightarrow a \rightarrow a'$ is another chimera $x \Rightarrow a'$ with the usual identity, composition, and associativity properties as illustrated in Figure 1.

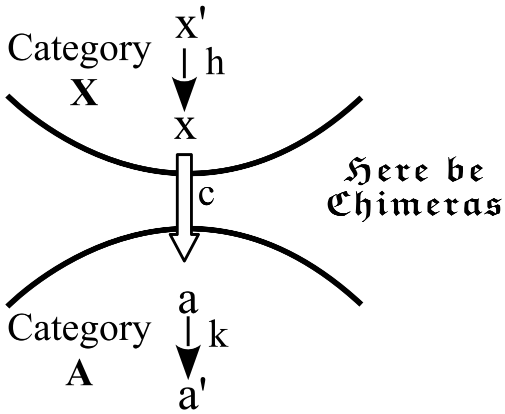

Figure 1. Composition of a het with homs on each side.

Such an action of two categories acting on a set on the left and on the right is exactly described by a bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \rightarrow \mathbf{Set}$, where $\mathrm{Het}(x, a) = \{x \Rightarrow a\}$, and where $\mathbf{Set}$ is the category of sets and set functions. Thus, the natural machinery to treat object-to-object chimera morphisms *between* categories are het-bifunctors $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \rightarrow \mathbf{Set}$ that generalize the hom-bifunctors $\mathrm{Hom} : \mathbf{X}^{op} \times \mathbf{X} \rightarrow \mathbf{Set}$ used to treat object-to-object morphisms *within* a category.

How might the categorical properties of the heteromorphisms be expressed using homomorphisms? Represent the het-bifunctors using hom-functors on the left and on the right (see any category theory text such as [17] for Alexander Grothendieck's notion of a *representable functor*). Any bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \rightarrow \mathbf{Set}$ is *represented on the left* if for each x in $\mathbf{X}$ there is a "universal" [3] object $F(x)$ in $\mathbf{A}$ and an isomorphism $\mathrm{Hom}_{\mathbf{A}}(F(x), a) \cong \mathrm{Het}(x, a)$ natural in a. This terminology "represented on the left" or "on the right" is used to agree with the terminology for left and right adjoints. In more detail, for each x in $\mathbf{X}$, there is an object $F(x)$ in $\mathbf{A}$ and a canonical het $h_x : x \Rightarrow F(x)$ such that for any het $f : x \Rightarrow a$, there is a unique $f_* : F(x) \to a$ such that the triangle commutes, i.e., $f_* h_x = f$. Intuitively, the het f between $\mathbf{X}$ and $\mathbf{A}$ is represented by f_* inside of $\mathbf{A}$. It is a standard result that the assignment $x \mapsto F(x)$ extends to a functor F and that the isomorphism is also natural in x as illustrated in Figure 2.

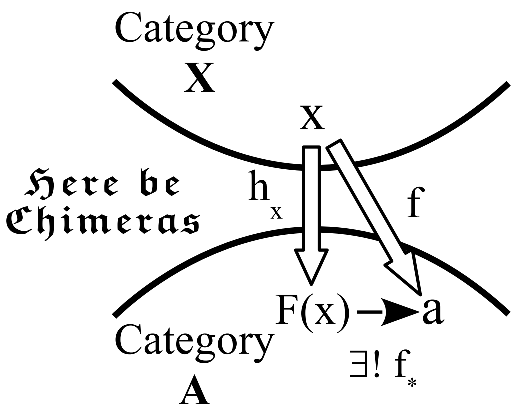

Figure 2. Het bifunctor represented on the left.

Similarly, Het is *represented on the right* if for each a there is another universal object $G(a)$ in $\mathbf{X}$ and an isomorphism $\text{Het}(x, a) \cong \text{Hom}_{\mathbf{X}}(x, G(a))$ natural in x. In more detail, for each a in $\mathbf{A}$, there is an object $G(a)$ in $\mathbf{X}$ and a canonical het $e_a : G(a) \Rightarrow a$ such that for any het $f : x \Rightarrow a$, there is a unique $f^* : x \to G(a)$ such that the triangle commutes, i.e., $e_a f^* = f$. Intuitively, the het f is represented by the hom f^* in the category $\mathbf{X}$. And similarly, the assignment $a \mapsto G(a)$ extends to a functor G and that the isomorphism is also natural in a as shown in Figure 3.

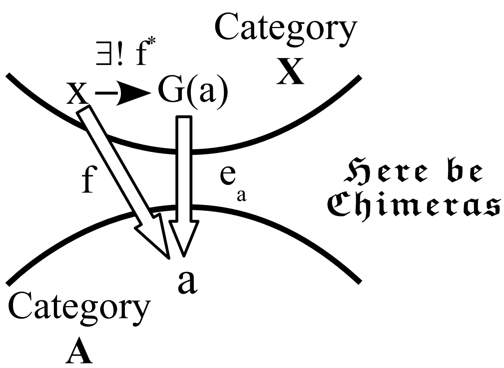

Figure 3. Het-bifunctor represented on the right.

If a het-bifunctor $\text{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ is represented on both the left and the right, then we have two functors $F : \mathbf{X} \to \mathbf{A}$, and $G : \mathbf{A} \to \mathbf{X}$ and the isomorphisms are natural in x and in a:

$$\text{Hom}_{\mathbf{A}}(Fx, a) \cong \text{Het}(x, a) \cong \text{Hom}_{\mathbf{X}}(x, Ga)$$

where $f_* \longleftrightarrow f \longleftrightarrow f^*$. It only remains to drop out the middle term $\text{Het}(x, a)$ to arrive at the "official" or usual definition of a pair of adjoint functors, which does not mention heteromorphisms [8,17].

$$\text{Hom}_{\mathbf{A}}(Fx, a) \cong \text{Hom}_{\mathbf{X}}(x, Ga).$$

At first glance, this ordinary presentation of an adjunction does not seem to have any directionality between the categories; it looks symmetrical. In the rather meager attempts at interpreting adjunctions, e.g., in illustrating a "unity of opposites" [6,7], there is still no hint of directionality. But adjoints do have a direction that comes out when adjunctions are to be composed [17] (p. 104). In the heteromorphic treatment, the directionality is obvious; it is the direction of the hets.

While a birepresentation of a het-bifunctor gives rise to an adjunction, do all adjunctions arise in this manner? To round out the theory, we give an "adjunction representation theorem" that shows how, given any adjunction $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$, heteromorphisms can be defined between (isomorphic copies of) the categories $\mathbf{X}$ and $\mathbf{A}$ so that (isomorphic copies of) the adjoints arise from the representations on the left and right of the het-bifunctor.

By analogy, suppose we are given any set function $f : X \to A$ from the set X to a set A. The $\text{graph}(f) = \{(x, f(x)) : x \in X\} \subseteq X \times A$ of the function is set-isomorphic to the domain of the function X. The embedding $x \longmapsto (x, f(x))$ maps X to the set-isomorphic copy of X, namely $\text{graph}(f) \subseteq X \times A$. That isomorphism generalizes to categories and to functors between categories. Given any functor $F : \mathbf{X} \to \mathbf{A}$, the domain category $\mathbf{X}$ is embedded in the product category $\mathbf{X} \times \mathbf{A}$ by the assignment $x \mapsto (x, Fx)$ (where we shorten $F(x)$ to Fx and similarly for G) to obtain the isomorphic copy $\widehat{\mathbf{X}}$ (which can be considered the graph of the functor F). Given any other functor $G : \mathbf{A} \to \mathbf{X}$, the domain category $\mathbf{A}$ is embedded in the product category by $a \mapsto (Ga, a)$ to yield the isomorphic copy $\widehat{\mathbf{A}}$ (the graph of the functor G). If the two functors are adjoints, then the properties of the adjunction can be nicely expressed by the commutativity within the one category $\mathbf{X} \times \mathbf{A}$ of "hom-pair adjunctive squares", where morphisms are pairs of homomorphisms (in contrast to a "het adjunctive square" defined later).

$$\begin{array}{ccccc} & (x, Fx) & \stackrel{(f^*, Ff^*)}{\longrightarrow} & (Ga, FGa) & \\ (\eta_x, 1_{Fx}) & \downarrow & \searrow^{(f^*, f_*)} & \downarrow & (1_{Ga}, \varepsilon_a) \\ & (GFx, Fx) & \stackrel{(Gf_*, f_*)}{\longrightarrow} & (Ga, a) & \end{array}$$

Hom-pair adjunctive square

The main diagonal (f^*, f_*) in a commutative hom-pair adjunctive square pairs together maps that are images of one another in the adjunction isomorphism $\text{Hom}_{\mathbf{A}}(Fx, a) \cong \text{Hom}_{\mathbf{X}}(x, Ga)$. If $f^* \in \text{Hom}_{\mathbf{X}}(x, Ga)$, $f_* \in \text{Hom}_{\mathbf{A}}(Fx, a)$ is the corresponding homomorphism on the other side of the isomorphism between hom-sets called its *adjoint transpose* (or later "adjoint correlate"). Since the maps on top are always in $\widehat{\mathbf{X}}$ and the maps on the bottom are in $\widehat{\mathbf{A}}$, the main diagonal pairs of maps (including the vertical maps)—which are ordinary morphisms in the product category—have all the categorical properties of heteromorphisms from objects in $\mathbf{X} \cong \widehat{\mathbf{X}}$ to objects in $\mathbf{A} \cong \widehat{\mathbf{A}}$. Hence, the heteromorphisms are abstractly defined as the *pairs of adjoint transposes*, $\text{Het}(x, a) = \{(x, Fx) \stackrel{(f^*, f_*)}{\longrightarrow} (Ga, a)\}$, and the adjunction representation theorem is that (isomorphic copies of) the original adjoints F and G arise from the representations on the left and right of this het-bifunctor.

Heteromorphisms are formally treated using bifunctors of the form $\text{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$. Such bifunctors and generalizations replacing **Set** with other categories have been studied by the Australian school under the name of *profunctors* [18], by the French school under the name of *distributors* [19], and by F. William Lawvere under the name of *bimodules* [20]. However, the guiding interpretation has been interestingly different. "Roughly speaking, a distributor is to a functor what a relation is to a mapping" [21] (p. 308) (hence the name "profunctor" in the Australian school). For instance, if **Set** was replaced with **2**, then the bifunctor would just be the characteristic function of a relation from $\mathbf{X}$ to $\mathbf{A}$. Hence, in the general context of enriched category theory, a "bimodule" $Y^{op} \otimes X \stackrel{\varphi}{\longrightarrow} \mathcal{V}$ would be interpreted as a "$\mathcal{V}$-valued relation", and an element of $\varphi(y, x)$ would be interpreted as the "truth-value of the φ-relatedness of y to x" [20] (p. 158 or p. 28 of reprint).

The subsequent development of profunctors–distributors–bimodules has been along the lines suggested by that guiding interpretation. For instance, composition is defined between distributors as "relational" generalizations of functors to define a category of distributors in analogy with composition defined between relations as generalizations of functions, which allows the definition of a category of relations [21] (Chapter 7).

The heteromorphic interpretation of the bifunctors $\mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ is rather different. Each such bifunctor is taken as defining how the chimeras in $\mathrm{Het}(x, a)$ compose with morphisms in $\mathbf{A}$ on one side and with morphisms in $\mathbf{X}$ on the other side to form other chimeras. This provides the formal treatment of the heteromorphisms that have always existed in mathematical practice. The principal novelty here is the use of the chimera morphism interpretation of these bifunctors to carry out a whole program of interpretation for adjunctions, i.e., a *theory* of adjoint functors. In concrete examples, heteromorphisms have to be "found" as is realized in the broad classes of examples treated here. However, in general, the adjunction representation theorem shows how abstract heteromorphisms (pairs of adjoint transposes in the product category $\mathbf{X} \times \mathbf{A}$) can always be found so that any adjunction arises (up to an isomorphism) out of the representations on the left and right of the het-bifunctor of such heteromorphisms.

3. The Heteromorphic Theory of Adjoints

3.1. Het-Bifunctors

Heteromorphisms (in contrast to homomorphisms) are like mongrels or chimeras that do not fit into either of the two categories. Since inter-category heteromorphisms are not morphisms in either of the categories, what can we say about them? The one thing we can reasonably say is that heteromorphisms can be precomposed or postcomposed with morphisms within the categories (i.e., intra-category morphisms) to obtain other heteromorphisms. (The chimera genes are dominant in these mongrel matings. While mules cannot mate with mules, it is "as if" mules could mate with either horses or donkeys to produce other mules.) This is easily formalized using bifunctors similar to the hom-bifunctors $\mathrm{Hom}(x, y)$ in homomorphisms within a category. Using the sets-to-groups example to guide intuition, one might think of $\mathrm{Het}(x, a) = \{x \overset{c}{\Rightarrow} a\}$ as the set of set functions from a set x to a group a. For any $\mathbf{A}$-morphism $k : a \to a'$ and any chimera morphism $x \overset{c}{\Rightarrow} a$, intuitively, there is a composite chimera morphism $x \overset{c}{\Rightarrow} a \overset{k}{\to} a' = x \overset{kc}{\Rightarrow} a'$, i.e., k induces a map $\mathrm{Het}(x, k) : \mathrm{Het}(x, a) \to \mathrm{Het}(x, a')$. For any $\mathbf{X}$-morphism $h : x' \to x$ and chimera morphism $x \overset{c}{\Rightarrow} a$, intuitively there is the composite chimera morphism $x' \overset{h}{\to} x \overset{c}{\Rightarrow} a = x' \overset{ch}{\Rightarrow} a$, i.e., h induces a map $\mathrm{Het}(h, a) : \mathrm{Het}(x, a) \to \mathrm{Het}(x', a)$ (note the reversal of direction). The induced maps would respect identity and composite morphisms in each category. Moreover, composition is associative in the sense that $(kc)h = k(ch)$. This means that the assignments of sets of chimera morphisms $\mathrm{Het}(x, a) = \{x \overset{c}{\Rightarrow} a\}$ and the induced maps between them constitute a *bifunctor* $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ (contravariant in the first variable and covariant in the second). With this motivation, we may turn around and define *heteromorphisms* from $\mathbf{X}$-objects to $\mathbf{A}$-objects as the elements in the values of a bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$. This would be analogous to defining the homomorphisms in $\mathbf{X}$ as the elements in the values of a given hom-bifunctor $\mathrm{Hom}_{\mathbf{X}} : \mathbf{X}^{op} \times \mathbf{X} \to \mathbf{Set}$ and similarly for $\mathrm{Hom}_{\mathbf{A}} : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Set}$.

With heteromorphisms described using het-bifunctors, we can use Grothendieck's notion of a representable functor to show that an adjunction arises from a het-bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ that is "birepresentable" in the sense of being representable on both the left and right.

Given any bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$, it is *representable on the left* if for each $\mathbf{X}$-object x, there is an $\mathbf{A}$-object Fx that represents the functor $\mathrm{Het}(x, -)$, i.e., there is an isomorphism $\psi_{x,a} : \mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Het}(x, a)$ natural in a. For each x, let h_x be the image of the identity on Fx, i.e., $\psi_{x,Fx}(1_{Fx}) = h_x \in \mathrm{Het}(x, Fx)$. We first show that h_x is a universal element for the functor $\mathrm{Het}(x, -)$ and then use that to complete the construction of F as a functor. For any $f \in \mathrm{Het}(x, a)$, let $f_* = \psi_{x,a}^{-1}(f) : Fx \to a$. Then, naturality in a means that the following diagram commutes.

$$\begin{array}{ccccc} & \mathrm{Hom}_{\mathbf{A}}(Fx, Fx) & \cong & \mathrm{Het}(x, Fx) & \\ {\scriptstyle \mathrm{Hom}(Fx, f_*)} & \downarrow & & \downarrow & {\scriptstyle \mathrm{Het}(x, f_*)} \\ & \mathrm{Hom}_{\mathbf{A}}(Fx, a) & \cong & \mathrm{Het}(x, a) & \end{array}$$

$\mathrm{Het}(x, a)$ representable on the left

Chasing 1_{Fx} around the diagram yields that $f = \mathrm{Het}(x, f_x)(h_x)$, which can be written as $f = f_* h_x$. Since the horizontal maps are isomorphisms, f_* is the unique map $f_* : Fx \to a$ such that $f = f_* h_x$. Then, (Fx, h_x) is a *universal element* (in Mac Lane's sense [17] (p. 57)) for the functor $\mathrm{Het}(x, -)$ or equivalently $1 \xrightarrow{h_x} \mathrm{Het}(x, Fx)$ is a *universal arrow* [17] (p. 58) from 1 (the one point set) to $\mathrm{Het}(x, -)$. Then, for any $\mathbf{X}$-morphism $j : x \to x'$, $Fj : Fx \to Fx'$ is the unique $\mathbf{A}$-morphism such that $\mathrm{Het}(x, Fj)$ fills in the right vertical arrow in the following diagram.

$$\begin{array}{ccccc} & 1 & \xrightarrow{h_x} & \mathrm{Het}(x, Fx) & \\ {\scriptstyle h_{x'}} & \downarrow & & \downarrow & {\scriptstyle \mathrm{Het}(x, Fj)} \\ & \mathrm{Het}(x', Fx') & \xrightarrow{\mathrm{Het}(j, Fx')} & \mathrm{Het}(x, Fx') & \end{array}$$

It is easily checked that such a definition of $Fj : Fx \to Fx'$ preserves identities and composition using the functoriality of $\mathrm{Het}(x, -)$ so we have a functor $F : \mathbf{X} \to \mathbf{A}$. It is a further standard result that the isomorphism is also natural in x (e.g., [17] (p. 81) or the "parameter theorem" [22] (p. 525)).

Given a bifunctor $\mathrm{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$, it is *representable on the right* if for each $\mathbf{A}$-object a, there is an $\mathbf{X}$-object Ga that represents the functor $\mathrm{Het}(-, a)$, i.e., there is an isomorphism $\varphi_{x,a} : \mathrm{Het}(x, a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga)$ natural in x. For each a, let e_a be the inverse image of the identity on Ga, i.e., $\varphi_{Ga,a}^{-1}(1_{Ga}) = e_a \in \mathrm{Het}(Ga, a)$. For any $f \in \mathrm{Het}(x, a)$, let $f^* = \varphi_{x,a}(f) : x \to Ga$. Then, naturality in x means that the following diagram commutes.

$$\begin{array}{ccccc} & \mathrm{Het}(Ga, a) & \cong & \mathrm{Hom}_{\mathbf{X}}(Ga, Ga) & \\ {\scriptstyle \mathrm{Het}(f^*, a)} & \downarrow & & \downarrow & {\scriptstyle \mathrm{Hom}(f^*, Ga)} \\ & \mathrm{Het}(x, a) & \cong & \mathrm{Hom}_{\mathbf{X}}(x, Ga) & \end{array}$$

$\mathrm{Het}(x, a)$ representable on the right

Chasing 1_{Ga} around the diagram yields that $c = \mathrm{Het}(f^*, a)(e_a) = e_a f^*$, so (Ga, e_a) is a universal element for the functor $\mathrm{Het}(-, a)$ and that $1 \xrightarrow{e_a} \mathrm{Het}(Ga, a)$ is a universal arrow from 1 to $\mathrm{Het}(-, a)$. Then, for any $\mathbf{A}$-morphism $k : a' \to a$, $Gk : Ga' \to Ga$ is the unique $\mathbf{X}$-morphism such that $\mathrm{Het}(Gk, a)$ fills in the right vertical arrow in the following diagram.

$$\begin{array}{ccccc} & 1 & \xrightarrow{e_a} & \mathrm{Het}(Ga, a) & \\ {\scriptstyle e_{a'}} & \downarrow & & \downarrow & {\scriptstyle \mathrm{Het}(Gk, a)} \\ & \mathrm{Het}(Ga', a') & \xrightarrow{\mathrm{Het}(Ga', k)} & \mathrm{Het}(Ga', a) & \end{array}$$

In a similar manner, it is easily checked that the functoriality of G follows from the functoriality of $\mathrm{Het}(-, a)$. Thus, we have a functor $G : \mathbf{A} \to \mathbf{X}$ such that Ga represents the functor $\text{Het}(-,a)$, i.e., there is a natural isomorphism $\varphi_{x,a} : \text{Het}(x,a) \cong \text{Hom}_{\mathbf{X}}(x, Ga)$ natural in x. And in a similar manner, it can be shown that the isomorphism is natural in both variables.

Thus, given a bifunctor $\text{Het} : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Set}$ representable on *both* sides, we have the following adjunction natural isomorphisms:

$$\text{Hom}_{\mathbf{A}}(Fx, a) \cong \text{Het}(x,a) \cong \text{Hom}_{\mathbf{X}}(x, Ga).$$

Starting with $f \in \text{Het}(x,a)$, the corresponding $f^* \in \text{Hom}_{\mathbf{X}}(x, Ga)$ and $f_* \in \text{Hom}_{\mathbf{A}}(Fx, a)$ are called *adjoint correlates* or *transposes* of one another. Starting with $1_{Fx} \in \text{Hom}_{\mathbf{A}}(Fx, Fx)$, its adjoint correlates are the *het unit* $h_x \in \text{Het}(x, Fx)$ and the ordinary unit $\eta_x \in \text{Hom}_{\mathbf{X}}(x, GFx)$, where this usual unit η_x might also be called the "hom unit" to distinguish it from its het correlate. Starting with $1_{Ga} \in \text{Hom}_{\mathbf{X}}(Ga, Ga)$, its adjoint correlates are the *het counit* $e_a \in \text{Het}(Ga, a)$ and the usual (hom) counit $\varepsilon_a \in \text{Hom}_{\mathbf{A}}(FGa, a)$. Starting with any $f \in \text{Het}(x,a)$, the two factorizations $f_* h_x = f = e_a f^*$ combine to give what we will later call the "het adjunctive square" with f as the main diagonal [as opposed to the hom-pair adjunctive square previously constructed, which had (f^*, f_*) as the main diagonal].

The conventional (heterophobic) presentation of an adjunction as a natural isomorphism between two hom-sets makes it seem like an atom that cannot be split. But the heteromorphic treatment involves two natural isomorphisms so it shows that an adjunction splits into two separate elementary parts that just represent universal mapping properties and thus could be called "half-adjunctions" (or "semi-adjunction"), or even better just "universal constructions" [16].

When the het-bifunctor is representable on the left, $\text{Hom}_{\mathbf{A}}(Fx, a) \cong \text{Het}(x,a)$, that is, *left universal* or *left half-adjunction*. Or if the het-bifunctor is representable on the right, $\text{Het}(x,a) \cong \text{Hom}_{\mathbf{X}}(x, Ga)$, that is, a *right universal* or *right half-adjunction*. Then, a left half-adjunction plus a right half-adjunction equals an adjunction. For many adjunctions, only one of the half-adjunctions is the important one. The other half-adjunction is trivial and is *only* needed to state the universal mapping property without using heteromorphisms. This raises the question, to be discussed later, whether or not the concept of category theory that is of foundational importance is the universal construction, where adjunctions occur as the particularly nice special cases of birepresentations where left and right universals (or half-adjunctions) combine to make an adjunction.

3.2. Adjunction Representation Theorem

Adjunctions may be and usually are presented without any thought to any underlying heteromorphisms. However, given any adjunction, there is always an "abstract" associated het-bifunctor given by the main diagonal maps in the commutative hom-pair adjunctive squares:

$$\text{Het}(\widehat{x}, \widehat{a}) = \{\widehat{x} = (x, Fx) \overset{(f^*, f_*)}{\longrightarrow} (Ga, a) = \widehat{a}\}$$

Het-bifunctor for any adjunction from hom-pair adjunctive squares.

The diagonal maps are closed under precomposition with maps from $\widehat{\mathbf{X}}$ and postcomposition with maps from $\widehat{\mathbf{A}}$. Associativity follows from the associativity in the ambient category $\mathbf{X} \times \mathbf{A}$.

The representation is accomplished essentially by putting a $\widehat{hat}$ on objects and morphisms embedded in $\mathbf{X} \times \mathbf{A}$. The categories $\mathbf{X}$ and $\mathbf{A}$ are represented, respectively, by the subcategory $\widehat{\mathbf{X}}$ with objects $\widehat{x} = (x, Fx)$ and morphisms $\widehat{f} = (f^*, Ff^*)$ and by the subcategory $\widehat{\mathbf{A}}$ with objects $\widehat{a} = (Ga, a)$ and morphisms $\widehat{g} = (Gf_*, f_*)$. The *twist functor* $(F, G) : \mathbf{X} \times \mathbf{A} \to \mathbf{X} \times \mathbf{A}$ defined by $(F, G)(x, a) = (Ga, Fx)$ (and similarly for morphisms) restricted to $\widehat{\mathbf{X}} \cong \mathbf{X}$ is $\widehat{F}$ has the action of F, i.e., $\widehat{F}\widehat{x} = (F, G)(x, Fx) = (GFx, Fx) = \widehat{Fx} \in \widehat{\mathbf{A}}$

and similarly for morphisms. The twist functor restricted to $\widehat{\mathbf{A}} \cong \mathbf{A}$ yields $\widehat{G}$, which has the action of G, i.e., $\widehat{G}\widehat{a} = (F,G)(Ga,a) = (Ga,FGa) = \widehat{Ga} \in \widehat{\mathbf{X}}$ and similarly for morphisms. These functors provide representations on the left and right of the abstract het-bifunctor $\mathrm{Het}(\widehat{x},\widehat{a}) = \{\widehat{x} \overset{(f^*,f_*)}{\longrightarrow} \widehat{a}\}$, i.e., the natural isomorphism

$$\mathrm{Hom}_{\widehat{\mathbf{A}}}(\widehat{F}\widehat{x},\widehat{a}) \cong \mathrm{Het}(\widehat{x},\widehat{a}) \cong \mathrm{Hom}_{\widehat{\mathbf{X}}}(\widehat{x},\widehat{G}\widehat{a}).$$

This birepresentation of the abstract het-bifunctor gives an isomorphic copy of the original adjunction between the isomorphic copies $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{A}}$ of the original categories. This hom-pair representation is summarized in the following:

> **Adjunction Representation Theorem:** Every adjunction $F : \mathbf{X} \rightleftarrows \mathbf{A} : G$ can be represented (up to isomorphism) as arising from the left and right representing universals of a het-bifunctor $\mathrm{Het} : \widehat{\mathbf{X}}^{op} \times \widehat{\mathbf{A}} \to \mathbf{Set}$, giving the heteromorphisms from the objects in a category $\widehat{\mathbf{X}} \cong \mathbf{X}$ to the objects in a category $\widehat{\mathbf{A}} \cong \mathbf{A}$.

As a historical note [17] (p. 103), Mac Lane noted that Bourbaki "missed" the notion of an adjunction because Bourbaki focused on the left representations of bifunctors $W : \mathbf{X}^{op} \times \mathbf{A} \to \mathbf{Sets}$. Mac Lane remarks that given $G : \mathbf{A} \to \mathbf{X}$, they should have taken $W(x,a) = Hom_{\mathbf{X}}(x,Ga)$ and then focused on "the symmetry of the adjunction problem" to find Fx so that $Hom_{\mathbf{A}}(Fx,a) \cong Hom_{\mathbf{X}}(x,Ga)$. But Mac Lane thus missed the completely symmetrical adjunction problem, which is the following: given $W(x,a)$, find both Ga and Fx such that $Hom_{\mathbf{A}}(Fx,a) \cong W(x,a) \cong Hom_{\mathbf{X}}(x,Ga)$. For more on the history of adjunctions and heteromorphisms, see [23].

3.3. Het Adjunctive Squares

We previously used the representations of $\mathrm{Het}(x,a)$ to pick out universal elements, the het unit $h_x \in \mathrm{Het}(x,Fx)$, and the het counit $e_a \in \mathrm{Het}(Ga,a)$, as the respective adjoint correlates of 1_{Fx} and 1_{Ga} under the isomorphisms. We showed that from the birepresentation of $\mathrm{Het}(x,a)$, any chimera morphism $x \overset{f}{\Rightarrow} a$ in $\mathrm{Het}(x,a)$ would have two factorizations: $f_* h_x = f = e_a f^*$. These two factorizations are spliced together along the main diagonal $f : x \Rightarrow a$ to form the het (commutative) adjunctive square.

$$\begin{array}{ccccc} & x & \overset{f^*}{\longrightarrow} & Ga & \\ h_x & \Downarrow & \searrow^{f} & \Downarrow & e_a \\ & Fx & \overset{f_*}{\longrightarrow} & a & \end{array}$$

Het Adjunctive Square

The het adjunctive square is the diagrammatic representation of the *full* adjunction representation (i.e., with the $\mathrm{Het}(x,a)$ in the middle):

$$\mathrm{Hom}_{\mathbf{A}}(Fx,a) \cong \mathrm{Het}(x,a) \cong \mathrm{Hom}_{\mathbf{X}}(x,Ga).$$

Sometimes, the two adjoint transposes are written vertically as in a Gentzen-style rule of inference:

$$\frac{x \to Ga}{Fx \to a}$$

Gentzen-style presentation of an adjunction

This can be seen as a proto-het adjunctive square without the vertical hets—at least when the homomorphism involving the left adjoint is on the bottom.

Some of the rigmarole of the conventional treatment of adjoints (*sans* chimeras) is only necessary because of the "heterophobic" restriction to morphisms within one category or within the other. For instance, the UMP for the hom unit $\eta_x : x \to GFx$ is that given any morphism $f^* : x \to Ga$ in $\mathbf{X}$, there is a unique morphism $f_* : Fx \to a$ in the other category $\mathbf{A}$ such that a G-functorial image back in the original category $\mathbf{X}$ gives the factorization of f through the unit: $x \xrightarrow{f^*} Ga = x \xrightarrow{\eta_x} GFx \xrightarrow{Gf_*} Ga$. The UMP has to go back and forth between homomorphisms in the two categories because it avoids mention of the heteromorphisms between the categories, as shown in Figure 4.

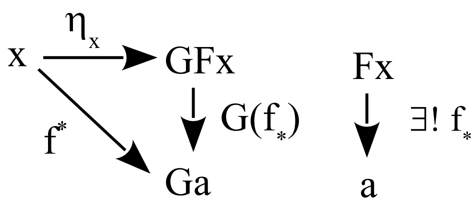

Figure 4. Over-and-back diagram for universality of η_x.

Figure 5 gives the dual diagram.

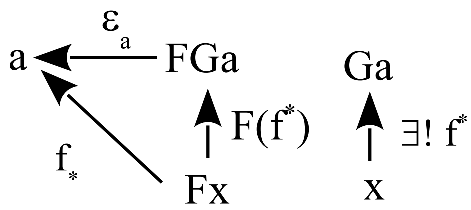

Figure 5. Over-and-back diagram for universality of ε_a.

The universal mapping property for the het unit $h_x : x \Rightarrow Fx$ is much simpler (i.e., no G and no over-and-back). Given any heteromorphism $f : x \Rightarrow a$, there is a unique homomorphism $f_* : Fx \to a$ in the codomain category $\mathbf{A}$ such that $x \overset{f}{\Rightarrow} a = x \overset{h_x}{\Rightarrow} Fx \xrightarrow{f_*} a$. And the dual universality property for the het counit $e_a : Ga \Rightarrow a$ is that given any $f : x \Rightarrow a$, there is a unique homomorphism $f^* : x \to Ga$ such that $x \overset{f}{\Rightarrow} a = x \overset{f^*}{\to} Ga \overset{e_a}{\Rightarrow} a$ (with no mention of F or any over-and-back). Both universality properties are represented in the adjunctive square diagram.

$$\begin{array}{ccccc} & x & \overset{\exists! f^*}{\longrightarrow} & Ga & \\ h_x & \Downarrow & \searrow^{\forall f} & \Downarrow & e_a \\ & Fx & \overset{\exists! f_*}{\longrightarrow} & a & \end{array}$$

For instance, in the "old days" (before category theory), one might have stated the universal mapping property of the free group Fx on a set x by saying that for any map $f : x \Rightarrow a$ from x into a group a, there is a unique group homomorphism $f_* : Fx \to a$ that preserves the action of f on the generators x, i.e., such that $x \overset{f}{\Rightarrow} a = x \hookrightarrow Fx \overset{f_*}{\to} a$ [12] (p. 69). That is, there is just the left half-adjunction or left universal part of the free-group adjunction. There is nothing sloppy or "wrong" in that old way of stating the universal mapping property.

Dually for the hom counit, given any morphism $f_* : Fx \to a$ in $\mathbf{A}$, there is a unique morphism $f^* : x \to Ga$ in the other category $\mathbf{X}$, such that the F-functorial image back in the original category $\mathbf{A}$ gives the factorization of f_* though the counit $Fx \xrightarrow{f_*} a = Fx \xrightarrow{Ff^*} FGa \xrightarrow{\varepsilon_a} a$. For the het counit, given any heteromorphism $f : x \Rightarrow a$, there is a unique homomorphism $f^* : x \to Ga$ in the domain category $\mathbf{X}$ such that $x \overset{f}{\Rightarrow} a = x \xrightarrow{f^*} Ga \overset{e_a}{\Rightarrow} a$. Putting these two het UMPs together yields the het adjunctive square diagram, just as previously putting the two hom UMPs together yielded the hom-pair adjunctive square diagram.

3.4. Het Natural Transformations

One of the main motivations for category theory was to mathematically characterize the intuitive notion of naturality for homomorphisms as in the standard example of the canonical linear homomorphism with a vector space embedded into its double dual. Many heteromorphisms are rather arbitrary, but certain ones are quite canonical so we should be able to mathematically characterize that canonicity or naturality just as we do for homomorphisms. Indeed, the notion of a natural transformation is immediately generalized to functors with different codomains by taking the components to be heteromorphisms. Given functors $F : \mathbf{X} \to \mathbf{A}$ and $H : \mathbf{X} \to \mathbf{B}$ with a common domain *and* given a het-bifunctor $\mathrm{Het} : \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$, a *chimera* or *het natural transformation relative to* Het $\varphi : F \Rightarrow H$ is given by a set of heteromorphisms $\{\varphi_x \in \mathrm{Het}(Fx, Hx)\}$ indexed by the objects of $\mathbf{X}$ such that for any $j : x \to x'$, the following diagram commutes.

$$\begin{array}{ccccc} & Fx & \stackrel{\varphi_x}{\Longrightarrow} & Hx & \\ Fj & \downarrow & & \downarrow & Hj \\ & Fx' & \stackrel{\varphi_{x'}}{\Longrightarrow} & Hx' & \end{array}$$

Het natural transformation

As with any commutative diagram involving heteromorphisms, composition and commutativity are defined using the het-bifunctor (similar remarks can be applied to any ordinary commutative hom diagram where it is the hom-bifunctor behind the scenes). For instance, the above commutative squares that define het natural transformations are unpacked as the following behind-the-scenes commutative squares in **Set** for the underlying het-bifunctor.

$$\begin{array}{ccccc} & & \varphi_x & & \\ & 1 & \longrightarrow & \mathrm{Het}(Fx, Hx) & \\ \varphi_{x'} & \downarrow & & \downarrow & \mathrm{Het}(Fx, Hj) \\ & \mathrm{Het}(Fx', Hx') & \longrightarrow & \mathrm{Het}(Fx, Hx') & \\ & & \mathrm{Het}(Fj, Hx') & & \end{array}$$

The composition $Fx \stackrel{\varphi_x}{\Longrightarrow} Hx \stackrel{Hj}{\longrightarrow} Hx'$ is $\mathrm{Het}(Fx, Hj)(\varphi_x) \in \mathrm{Het}(Fx, Hx')$, the composition $Fx \stackrel{Fj}{\longrightarrow} Fx' \stackrel{\varphi_{x'}}{\Longrightarrow} Hx'$ is $\mathrm{Het}(Fj, Hx')(\varphi_{x'}) \in \mathrm{Het}(Fx, Hx')$, and commutativity means that they are the same element of $\mathrm{Het}(Fx, Hx')$. These het natural transformations do not compose like the morphisms in a functor category but they are acted upon by the natural transformations in the functor categories on each side to yield het natural transformations.

There are het natural transformations each way between any functor and the identity on its domain if the functor itself is used to define the appropriate het-bifunctor. That is, given *any* functor $F : \mathbf{X} \to \mathbf{A}$, there is a het natural transformation $1_{\mathbf{X}} \Rightarrow F$ relative to the bifunctor defined as $\mathrm{Het}(x, a) = \mathrm{Hom}_{\mathbf{A}}(Fx, a)$ as well as a het natural transformation $F \Rightarrow 1_{\mathbf{X}}$ relative to $\mathrm{Het}(a, x) = \mathrm{Hom}_{\mathbf{A}}(a, Fx)$.

Het natural transformations "in effect" already occur with reflective (or coreflective) subcategories. A subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is a *reflective subcategory* if the inclusion functor $K : \mathbf{A} \hookrightarrow \mathbf{B}$ has a left adjoint. For any such reflective adjunctions, the heteromorphisms $\mathrm{Het}(b, a)$ are the $\mathbf{B}$-morphisms with their heads in the subcategory $\mathbf{A}$ so the representation on the right $\mathrm{Het}(b, a) \cong \mathrm{Hom}_{\mathbf{B}}(b, Ka)$ is trivial. The left adjoint $F : \mathbf{B} \to \mathbf{A}$ gives the representation on the left: $\mathrm{Hom}_{\mathbf{A}}(Fb, a) \cong \mathrm{Het}(b, a) \cong \mathrm{Hom}_{\mathbf{B}}(b, Ka)$. Then, it is perfectly "natural" to see the unit of the adjunction as defining a natural transformation $\eta : 1_{\mathbf{B}} \Rightarrow F$, but this is actually a het natural transformation (since the codomain of F

is **A**). Hence, the conventional (heterophobic) treatment (e.g., [17] (p. 89)) is to define another functor R with the same domain and values on objects and morphisms as F except that its codomain is taken to be **B** so that we can then have a hom natural transformation $\eta : 1_{\mathbf{X}} \to R$ between two functors with the same codomain. Similar remarks hold for the dual coreflective case where the inclusion functor has a right adjoint and where the heteromorphisms are turned around, i.e., are **B**-morphisms with their tail in subcategory **A**.

Given any adjunction isomorphism $\mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Het}(x, a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga)$, the adjoint correlates of the identities $1_{Fx} \in \mathrm{Hom}_{\mathbf{A}}(Fx, Fx)$ are the het units $h_x \in \mathrm{Het}(x, Fx)$ and the hom units $\eta_x \in \mathrm{Hom}_{\mathbf{X}}(x, GFx)$. The het units together give the het natural transformation $h : 1_{\mathbf{X}} \Rightarrow F$, while the hom units give the hom natural transformation $\eta : 1_{\mathbf{X}} \to GF$. The adjoint correlates of the identities $1_{Ga} \in \mathrm{Hom}_{\mathbf{X}}(Ga, Ga)$ are the het counits $e_a \in \mathrm{Het}(Ga, a)$ and the hom counits $\varepsilon_a \in \mathrm{Hom}_{\mathbf{A}}(FGa, a)$. The het counits together give the het natural transformation $e : G \Rightarrow 1_{\mathbf{A}}$, while the hom counits give the hom natural transformation $\varepsilon : FG \to 1_{\mathbf{A}}$.

4. Results

4.1. The Product Adjunction for Sets

Let **X** be the category **Set** of sets and let **A** be the category $\mathbf{Set}^2 = \mathbf{Set} \times \mathbf{Set}$ of ordered pairs of sets. A heteromorphism from a set to a pair of sets is a pair of set maps with a common domain $(f_1, f_2) : W \Rightarrow (X, Y)$, which is called a *cone*. The het-bifunctor is given by $\mathrm{Het}(W, (X, Y)) = \{W \Rightarrow (X, Y)\}$, the set of all cones from W to (X, Y). To construct a representation on the right, suppose we are given a pair of sets $(X, Y) \in \mathbf{Set}^2$. How could one construct a set, to be denoted $X \times Y$, such that all cones $W \Rightarrow (X, Y)$ from any set W could be represented by set functions (morphisms within **Set**) $W \to X \times Y$? In the "atomic" case of $W = 1$ (the one-element set), a 1-cone $1 \Rightarrow (X, Y)$ would just pick out an ordered pair (x, y) of elements, the first from X and the second from Y. Any cone $W \Rightarrow (X, Y)$ would just pick out a set of pairs of elements. Hence, the universal object would have to be the set $\{(x, y) : x \in X, y \in Y\}$ of *all* such pairs, which yields the Cartesian product of sets $X \times Y$. The assignment of that set to each pair of sets gives the right adjoint $G : \mathbf{Set}^2 \to \mathbf{Set}$, where $G((X, Y)) = X \times Y$ (and similarly for morphisms). The het counit $e_{(X,Y)} : X \times Y \Rightarrow (X, Y)$ canonically takes each ordered pair (x, y) as a single element in $X \times Y$ to that pair of elements in (X, Y). The universal mapping property of the Cartesian product $X \times Y$ then holds; given any set W and a cone $(f_1, f_2) : W \Rightarrow (X, Y)$, there is a unique set function $\langle f_1, f_2 \rangle : W \to X \times Y$ defined by $\langle f_1, f_2 \rangle(w) = (f_1(w), f_2(w))$ that factors the cone through the het counit:

$$\begin{array}{cccc} W & \xrightarrow{\langle f_1, f_2 \rangle} & X \times Y & \\ (f_1, f_2) & \searrow\!\!\searrow & \Downarrow & e_{(X,Y)} \\ & & (X, Y) & \end{array}$$

Right half-adjunction of the product adjunction.

=For the product adjunction, the right half-adjunction is the important one.

Fixing W in **Set**, how could we find a universal object in $\mathbf{Set}^2$ so that all heteromorphisms $(f_1, f_2) : W \Rightarrow (X, Y)$ could be uniquely factored through it? The obvious suggestion is the pair (W, W), which defines a functor $F : \mathbf{Set} \to \mathbf{Set}^2$ and where the het unit $h_W : W \Rightarrow (W, W)$ is just the pair of identity maps $h_W = (1_W, 1_W)$. Then, for each cone $(f_1, f_2) : W \Rightarrow (X, Y)$, there is a unique pair of maps, also denoted as $(f_1, f_2) : (W, W) \to (X, Y)$, which are a morphism in $\mathbf{Set}^2$ and which factors the cone through the het unit:

$$\begin{array}{cccc} & W & & \\ h_W & \Downarrow & \searrow & (f_1, f_2) \\ & (W,W) & \overset{(f_1,f_2)}{\longrightarrow} & (X,Y) \end{array}$$

Left half-adjunction of the product adjunction.

Splicing the two half-adjunctions along the diagonal gives the following:

$$\begin{array}{ccccc} & W & \overset{\langle f_1,f_2\rangle}{\longrightarrow} & X \times Y & \\ h_W & \Downarrow & \searrow^{(f_1,f_2)} & \Downarrow & e_{(X,Y)} \\ & (W,W) & \overset{(f_1,f_2)}{\longrightarrow} & (X,Y) & \end{array}$$

Het adjunctive square for the product adjunction.

The two factor maps on the top and bottom are uniquely associated with the diagonal cones, and the isomorphism is natural so that we have natural isomorphisms between the hom-bifunctors and the het-bifunctor:

$$\mathrm{Hom}_{\mathbf{Set}^2}((W,W),(X,Y)) \cong \mathrm{Het}(W,(X,Y)) \cong \mathrm{Hom}_{\mathbf{Set}}(W, X \times Y).$$

4.2. The Coproduct Adjunction for Sets

The construction dual to the product is the coproduct, which for the category of sets is the disjoint union of sets. Let **A** be the category **Set** of sets and let **X** be the category $\mathbf{Set}^2 = \mathbf{Set} \times \mathbf{Set}$ of ordered pairs of sets. A heteromorphism from a pair of sets to a set is a pair of set maps with a common codomain $(g_1, g_2) : (X,Y) \Rightarrow Z$, which is called a *cocone*. The het-bifunctor is given by $\mathrm{Het}((X,Y),Z) = \{(X,Y) \Rightarrow Z\}$, the set of all cocones from (X,Y) to Z. To construct a representation on the left, suppose we are given a fixed pair of sets $(X,Y) \in \mathbf{Set}^2$. The *coproduct* or *disjoint union* is the set $F((X,Y)) = X + Y$ such that all cocones $(X,Y) \Rightarrow Z$ to any set Z could be represented by set functions (morphisms within **Set**) $X + Y \to Z$. The het unit $h_{(X,Y)}$ is a canonical "injection" cocone $h_{(X,Y)} = (i_X, i_Y) : (X,Y) \Rightarrow X+Y$. Here, for $x \in X$, $i_X(x)$ is the copy of x in $X+Y$, and similarly, for $y \in Y$, $i_Y(y)$ is the copy of y in $X+Y$. Given any cocone $(g_1, g_2) : (X,Y) \Rightarrow Z$, there is a unique set map $\{g_1, g_2\} : X+Y \to Z$ [which takes x in the copy of X in $X+Y$ to $g_1(x)$ and takes y in the copy of Y to $g_2(y)$] such that $(X,Y) \overset{(i_X,i_Y)}{\Rightarrow} X+Y \overset{\{g_1,g_2\}}{\to} Z = (X,Y) \overset{(g_1,g_2)}{\Rightarrow} Z$.

$$\begin{array}{cccc} & (X,Y) & & \\ (i_X,i_Y) & \Downarrow & \searrow & (g_1,g_2) \\ & X+Y & \overset{\{g_1,g_2\}}{\longrightarrow} & Z \end{array}$$

Left half-adjunction for coproduct adjunction.

The left half-adjunction is the important one for the coproduct adjunction. The factor map $\{g_1, g_2\}$ represents within **Set** the action of the cocone $(g_1, g_2) : (X,Y) \Rightarrow Z$, which is a heteromorphism from an object in $\mathbf{Set}^2$ to an object in **Set**. That representation gives the natural isomorphism:

$$\mathrm{Hom}_{\mathbf{Set}}(X+Y, Z) \cong \mathrm{Het}((X,Y),Z).$$

Now, fix the object $Z \in \mathbf{Set}$ and find the representation on the right of any heteromorphism from any object $(X,Y) \in \mathbf{Set}^2$ to Z. We need a universal object in Set^2 and a universal morphism from that object to Z so that any cocone $(g_1, g_2) : (X,Y) \Rightarrow Z$ can be uniquely factored through the universal. The obvious choice of the object in $\mathbf{Set}^2$ to represent Z is $G(Z) = (Z,Z)$ and the obvious universal cocone $(Z,Z) \Rightarrow Z$ is the het counit $e_Z = (1_Z, 1_Z)$, a pair of identity maps. The unique factorization is so trivial that we will use the same notation (g_1, g_2) for both the heteromorphism $(X,Y) \Rightarrow Z$ and the homomorphism $(X,Y) \to (Z,Z)$ within the category $\mathbf{Set}^2$.

$$\begin{array}{ccccc} (X,Y) & \overset{(g_1,g_2)}{\longrightarrow} & (Z,Z) & & \\ (g_1,g_2) & \searrow & \Downarrow & (1_Z,1_Z) \\ & & Z & & \end{array}$$

Right half-adjunction of the coproduct adjunction.

The correlation between the het (g_1, g_2) and the hom (g_1, g_2) gives the representation on the right:

$$\text{Het}((X,Y),Z) \cong \text{Hom}_{\mathbf{Set}^2}((X,Y),(Z,Z)).$$

Splicing the two half-adjunctions along the diagonal gives the following:

$$\begin{array}{ccccc} & (X,Y) & \overset{(g_1,g_2)}{\longrightarrow} & (Z,Z) & \\ h_{(X,Y)} & \Downarrow & \searrow^{(g_1,g_2)} & \Downarrow & e_Z \\ & X+Y & \overset{\{g_1,g_2\}}{\longrightarrow} & Z & \end{array}$$

Het adjunctive square for the coproduct adjunction.

Combining the left and right representations gives the usual characterization of an adjunction as a natural isomorphism of two hom-functors (ignoring the het-bifunctor middle term):

$$\text{Hom}_{\mathbf{Set}}(X+Y,Z) \cong \text{Het}((X,Y),Z) \cong \text{Hom}_{\mathbf{Set}^2}((X,Y),(Z,Z)).$$

4.3. Adjoints to Forgetful Functors

Perhaps the most accessible adjunctions are the free forgetful adjunctions between $\mathbf{X} = \mathbf{Set}$ and a category of algebras such as the category of groups $\mathbf{A} = \mathbf{Grps}$. The right adjoint $G : \mathbf{A} \to \mathbf{X}$ forgets the group structure to give the underlying set GA of a group A. The left adjoint $F : \mathbf{X} \to \mathbf{A}$ gives the free group FX generated by a set X.

For this adjunction, the heteromorphisms are any set functions $X \overset{f}{\Rightarrow} A$ (with the codomain being a group A), and the het-bifunctor is given by such functions: $\text{Het}(X, A) = \{X \Rightarrow A\}$ (with the obvious morphisms). A heteromorphism $f : X \Rightarrow A$ determines a set map $f^* : X \to GA$ trivially and it determines a group homomorphism $f_* : FX \to A$ by mapping the generators $x \in X$ to their images $f(x) \in A$ and then mapping the other elements of FX as they must be mapped in order for f_* to be a group homomorphism. The het unit $h_X : X \Rightarrow FX$ is the insertion of the generators into the free group, and the het counit $e_A : GA \Rightarrow A$ is just the retracting of the elements of the underlying set back to the group. These factor maps f^* and f_* uniquely complete the usual two half-adjunction triangles, which together give the following:

$$\begin{array}{ccccc} & X & \overset{f^*}{\longrightarrow} & GA & \\ h_X & \Downarrow & \searrow^{f} & \Downarrow & e_A \\ & FX & \overset{f_*}{\longrightarrow} & A & \end{array}$$

Het adjunctive square for the free group adjunction.

These associations also give us the following two representations:

$$\text{Hom}(FX, A) \cong \text{Het}(X, A) \cong \text{Hom}(X, GA).$$

In general, the existence of a left adjoint to $U : \mathbf{A} \to \mathbf{Set}$ (i.e., a left representation of $\text{Het}(X, A) = \{X \Rightarrow A\}$) will depend on whether or not there is an $\mathbf{A}$-object FX with the least or minimal structure so that every chimera $X \overset{f}{\Rightarrow} A$ will determine a unique representing $\mathbf{A}$-morphism $f_* : FX \to A$.

The existence of a *right* adjoint to U will depend on whether or not for any set X, there is an **A**-object IX with the greatest or maximum structure so that any chimera $A \Rightarrow X$ can be represented by an **A**-morphism $A \to IX$.

Consider the underlying set functor $U : \mathbf{Pos} \to \mathbf{Set}$ from the category of partially ordered sets (an ordering that is reflexive, transitive, and anti-symmetric) with order-preserving maps to the category of sets. It has a left adjoint since each set has a least partial order on it, namely the discrete ordering. Hence, any chimera function $X \overset{f}{\Rightarrow} A$ from a set X to a partially ordered set or poset A could be represented as a set function $X \xrightarrow{f^*} UA$ or as an order-preserving function $DX \xrightarrow{f_*} A$, where DX gives the discrete ordering on X. The functor giving the discrete partial ordering on a set is the left half-adjoint to the underlying the set function.

In the other direction, one could take as a chimera any function $A \overset{f}{\Rightarrow} X$ (from a poset A to a set X), and it is represented on the left by the ordinary set function $UA \xrightarrow{f_*} X$ so the left half-adjunction trivially exists:

$$\begin{array}{cccc} & A & \xrightarrow{?} & IX? \\ h_A & \Downarrow & \searrow^{f} & \downarrow ? \\ & UA & \xrightarrow{f_*} & X \end{array}$$

Left half-adjunction (with no right half-adjunction).

But the underlying set functor U does not have a right adjoint since there is no maximal partial order IX on X so that any chimera $A \overset{c}{\Rightarrow} X$ could be represented as an order-preserving function $f(c) : A \to IX$. To receive all the possible orderings, the ordering relation would have to go both ways between any two points, which would then be identified by the anti-symmetry condition so that IX would collapse to a single point and the factorization of c through IX would fail. (Thanks to Vaughn Pratt for the example). Thus, poset-to-set chimera $A \Rightarrow X$ can only be represented on the left. This is a case of where a het between two categories is only represented inside one of the categories.

In relaxing the anti-symmetry condition, let $U : \mathbf{Ord} \to \mathbf{Set}$ be the underlying set functor from the category of preordered sets (reflexive and transitive orderings) to the category of sets. The discrete ordering again gives a left adjoint. But now there is also a maximal ordering on a set X, namely the "indiscrete" ordering IX on X (the "indiscriminate" or "chaotic" preorder on X), which has the ordering relation both ways between any two points. Then, a preorder-to-set chimera morphism $f : A \Rightarrow X$ (just a set function ignoring the ordering) can be represented on the left as a set function $UA \xrightarrow{f_*} X$ and on the right as an order-preserving function $A \xrightarrow{f^*} IX$ so that U also has a right adjoint I, and we have the following:

$$\begin{array}{ccccc} & A & \xrightarrow{f^*} & IX & \\ h_A & \Downarrow & \searrow^{f} & \Downarrow & e_X \\ & UA & \xrightarrow{f_*} & X & \end{array}$$

Het adjunctive square for the indiscrete underlying adjunction on preorders.

4.4. Reflective Subcategories

Suppose that **A** is a subcategory of **X** with $G : \mathbf{A} \hookrightarrow \mathbf{X}$ the inclusion functor and suppose that it has a left adjoint $F : \mathbf{X} \to \mathbf{A}$. Then, **A** is said to be a *reflective subcategory* of **X**, the left adjoint F is the *reflector*, and the adjunction is called a *reflection*: $\mathrm{Hom}_{\mathbf{A}}(Fx, a) \cong \mathrm{Hom}_{\mathbf{X}}(x, Ga)$. For all reflections, the chimera morphisms are the morphisms $x \Rightarrow a$ in the ambient category **X** with their heads in the reflective subcategory **A**. Hence, the het-bifunctor would be

$$\mathrm{Het}(x, a) = \mathrm{Hom}_{\mathbf{XA}}(x, a)$$

where the **XA** subscript indicates that x can be any object in **X** but that a is any element of the subcategory **A**. Note the two ways of seeing any $f \in \text{Het}(x,a) = \text{Hom}_{\mathbf{XA}}(x,a)$. From one viewpoint, $f \in \text{Hom}_{\mathbf{XA}}(x,a) \subseteq \text{Hom}_{\mathbf{X}}(x,a)$ so that f is just a morphism inside the category **X**, but we also view it as a chimera with its tail in **X** and head in **A**. Since G is the inclusion functor, it just takes a as an element of **A** to itself as an element of **X** and similarly for morphisms. Thus, we insert $\text{Het}(x,a)$ in the middle to obtain the two representation isomorphisms:

$$\text{Hom}_{\mathbf{A}}(Fx,a) \cong \text{Het}(x,a) \cong \text{Hom}_{\mathbf{X}}(x,Ga).$$

There is also the dual case of a *coreflective subcategory*, where the inclusion functor has a right adjoint and where the chimera morphisms are turned around (tail in the subcategory and head in the ambient category). This case will be used in the next section, but here, I will focus on reflective subcategories.

For an interesting example of a reflector dating back five centuries, we use the modern mathematical formulation of double-entry bookkeeping [24]. Let **Ab** be the category of abelian (i.e., commutative) groups where the group operation is written as addition. Thus 0 is the identity element, $a + a' = a' + a$, and for each element a, there is an element $-a$ such that $a + (-a) = 0$. Let **CMon** be the category of commutative monoids so the addition operation has the identity 0 but does not necessarily have an inverse. Let $G : \mathbf{Ab} \hookrightarrow \mathbf{CMon}$ be the inclusion functor.

In 1494, the mathematician Luca Pacioli published an accounting technique that had been developed in practice during the 1400s and which became known as *double-entry bookkeeping* [25]. Pacioli acknowledged having seen an earlier unpublished description of double-entry booking by Benedetto Cotrugli, which has now been published [26]. In essence, the idea was to carry out additive arithmetic with additive inverses using ordered pairs $[x \,//\, x']$ of non-negative numbers called *T-accounts*. (The double-slash separator was suggested by Pacioli. "At the beginning of each entry, we always provide 'per', because, first, the debtor must be given, and immediately after the creditor, the one separated from the other by two little slanting parallels (virgolette), thus, //, ..." [25] (p. 43)). The number on the left side was called the *debit entry*, and the number on the right, the *credit entry*. T-accounts were added by adding the corresponding entries: $[x \,//\, x'] + [y \,//\, y'] = [x + y \,//\, x' + y']$. Two T-accounts were deemed equal if their cross-sums were equal (the additive version of the equal cross-multiples was used to define the equality of multiplicative ordered pairs or fractions). Thus,

$$[x \,//\, x'] = [y \,//\, y'] \text{ if } x + y' = x' + y.$$

Hence, the additive inverse was obtained by "reversing the entries" (as accountants say):

$$[x \,//\, x'] + [x' \,//\, x] = [x + x' \,//\, x' + x] = [0 \,//\, 0].$$

To obtain the reflector or left adjoint $F : \mathbf{CMon} \to \mathbf{Ab}$ to G, we need only note that Pacioli was implicitly using the fact that the normal addition of numbers is *cancellative* in the sense that $x + z = y + z$ implies $x = y$. Since commutative monoids do not in general have that property, we need only to tweak the definition of the equality of T-accounts [13] (p. 17):

$$[x \,//\, x'] = [y \,//\, y'] \text{ if there is a } z \text{ such that } x + y' + z = x' + y + z.$$

This construction with the induced maps then yields a functor $F : \mathbf{CMon} \to \mathbf{Ab}$ that takes a commutative monoid m to a commutative group $Fm = P(m)$. The group $P(m)$ is usually called the "group of differences" or "inverse completion", and, in algebraic geometry, its generalization is called the "Grothendieck group". However, due to about a half-millennium of priority, we will call the additive group of differences the *Pacioli–Cotrugli* *group* of the commutative monoid m. For any such m, the het unit $h_m : m \Rightarrow Fm = P(m)$, which takes an element x to the T-account $[0 \ // \ x]$ with that credit balance (the debit balance mapping would do just as well).

For this adjunction, a heteromorphism $f : m \Rightarrow a$ is any *monoid homomorphism* from a commutative monoid m to any abelian group a (being only a monoid homomorphism, it does not need to preserve any inverses that might exist in m). The Pacioli group has the following universality property: for any heteromorphism $f : m \Rightarrow a$, there is a unique group homomorphism $f_* : Fm \to a$ such that $m \overset{h_m}{\Rightarrow} Fm \overset{f_*}{\to} a = m \overset{f}{\Rightarrow} a$. The group homomorphism factor map is $f_*([x' \ // \ x]) = c(x) + (-c(x'))$. The right adjoint G just takes a commutative group to its underlying commutative monoid. This establishes the other representation isomorphism of the adjunction:

$$\mathrm{Hom}_{\mathbf{Ab}}(Fm, a) \cong \mathrm{Het}(m, a) \cong \mathrm{Hom}_{\mathbf{CMon}}(m, Ga).$$

5. Discussion and Concluding Remarks

This paper inevitably has three themes:

- (1) The first and logically prior theme is showing that heteromorphisms can be rigorously treated as part of category theory—rather than just as stray chimeras roaming in the wilds of mathematical practice.
- (2) The main theme is showing how adjoint functors arise from the representations within two categories of the heteromorphisms between the categories.
- (3) The third theme is the question of whether or not adjunctions are really the basic concept, since the heteromorphic treatment, unlike the standard heterophobic treatment, shows that an adjunction can be broken into two half-adjunctions or universal constructions that involve only the notion of representable functors.

In the first theme, category theory has always been presented as embodying the idea of grouping mathematical objects of a certain sort together with their appropriate morphisms in a "category". In some respects, this homomorphic theme became the leading theme just as in Felix Klein's Erlanger Program, where geometries were characterized by the invariants of a specified class of transformations. Indeed, in their founding paper, Eilenberg and Mac Lane noted that category theory "may be regarded as a continuation of the Klein Erlanger Program, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings" [4] (p. 237). Hence, the whole concept of a "heteromorphism" between objects of different categories has seemed like a cross-species hybrid that is out-of-place and running against the spirit of the enterprise. Functors were defined to handle all the external relations between categories so object-to-object inter-category morphisms had no "official" role.

At the outset of this paper, a number of testimonials were quoted about the centrality of adjunctions in category theory and thus as a foundational lens with which to see what is important, universal, and natural in mathematics. Once heteromorphisms were rigorously treated using het-bifunctors (in analogy to treating homomorphisms with hom-bifunctors), it quickly became clear that an adjunction between two categories was closely related to the heteromorphisms between the objects of the two categories. Our main theme about adjoints is that left and right adjoints arise as the left and right representations within the categories of the heteromorphisms between the categories. Given the importance of adjoints, this makes an argument for taking heteromorphisms "out of the closet" and recognizing them as part of the conceptual family of category theory.

In our third theme, we have seen that all adjunctions arise as birepresentations of het-bifunctors, where the birepresentation can be split into a left and a right half-adjunction. Moreover, there can be either a left or right representation without the other (e.g., the partial order/underlying set was a simple example but the tensor product is another example). Often, one of a pair of adjoints gives the important concept, and the other adjoint functions as an auxiliary device to fill out the het-free notion of an adjunction. In the wilds of mathematical practice, the important left or right representation is routinely used along with the necessary hets without the auxiliary devices. Hence, there are grounds to conclude that it is the concept of a universal mapping property (which is naturally formulated as a representation of hets) that is the "most important concept in category theory" and that adjunctions arise as the special case of *bi*-representations of hets. (See [16] for an extensive development of this viewpoint.)

The idea of taking representable functors and universals as the conceptual lens (rather than adjoint functors) is not new. The treatment of UMPs is based on the notion of a representable functor associated with Alexander Grothendieck [27] (representable functors are defined in the first section of the first Chapter 0), and it helps to clear up another mystery:

> As we can see by looking at his [Grothendieck's] lectures in the *Séminaire Bourbaki* from 1957 until 1962, the notion of representable functors became one of the main tools he used. . . . It is far from clear why Grothendieck decided to use this notion instead of, say, adjoint functors, ... It is also clear from the various seminars that Grothendieck thought in terms of universal "problems", that is he tried to formulate the problems he was working on in terms of a universal morphism: finding a solution to the given problem amounted to finding a universal morphism in the situation. Grothendieck saw that the latter notion was subsumed under the notion of representable functor [28] (pp. 102–103).

Grothendieck took the notion of a representable functor as fundamental (as solving a universal problem), where adjoints arise as the special case of a particularly nice birepresentation.

In conclusion, I think the case can be made that the universal constructions [16] of category theory are the fundamental notions, rather than the very important special case of adjunctions, to focus the conceptual lenses on what is foundational in mathematics and, perhaps, in other sciences–not to mention in philosophy (see Appendix A).

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Appendix A. Philosophical Applications of Universals

Appendix A.1. Non-Self-Predicative and Self-Predicative Universals

The purpose of this appendix is to briefly outline some more philosophical or speculative applications of the category-theoretic universals.

An object u_F is a *universal* for a property $F(x)$ if every a such that $F(a)$ and only those a's have a "participation" ($\mu\epsilon\theta\epsilon\xi\iota\zeta$ or *methexis*) relation μ with the universal u_F:

$$a \; \mu \; u_F \text{ iff } F(a).$$

The best-known universals are just the sets $\{x|F(x)\}$ (for suitable predicates $F(x)$ in set theory) where participation is set membership $\mu = \in$:

$$a \in \{x|F(x)\} \text{ iff } F(a).$$

In axiomatic set theory, these universals have to be of higher rank or type than their members so they might be called *abstract* universals, but for our purposes, they are also *non-self-predicative* universals since a set cannot be a member of itself.

For the universals of category theory, the participation relation is that it *uniquely factors* and each universal factors through itself through the identity morphism, so the universal also has the property for which it is universal. Hence, the category theory universals could be called *concrete universals* (in contrast to the abstract universals of set theory), and they are *self-predicative* universals.

Example 1: For the Cartesian product construction $X \times Y$ in *Sets*, the property is that of being a cone of set functions $f_1 : W \to X$ and $f_2 : W \to Y$ from a single set W to the sets X and Y. The concrete and self-predicative universal for that property is the cone of projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. Then, a cone (f_1, f_2) uniquely factors through the projections (π_X, π_Y) iff the cone has the property of "being a cone of set functions $f_1 : W \to X$ and $f_2 : W \to Y$ from a single set W to the sets X and Y". That is, given any set W and a cone $(f_1, f_2) : W \Rightarrow (X, Y)$, there is a unique set function $\langle f_1, f_2 \rangle : W \to X \times Y$ defined by $\langle f_1, f_2 \rangle(w) = (f_1(w), f_2(w))$ that factors the cone through het counit $e_{(X,Y)} = (\pi_X, \pi_Y)$ so that the following diagram commutes with $f = (f_1, f_2)$.

$$\begin{array}{ccccc} W & \stackrel{\langle f_1, f_2 \rangle}{\longrightarrow} & X \times Y & & \\ & \searrow^{f} & \Downarrow & & e_{(X,Y)} \\ & & (X, Y) & & \end{array}$$

Example 2: For the free-group $F(x)$, the property is "being a set-to-group het $f : x \Rightarrow g$ from the set x to any group g", and the universal for the property is the het unit $h_x : x \Rightarrow F(x)$. Thus, for any $f : x \Rightarrow g$, there is a unique group homomorphism $g(f) : F(x) \to g$ that factors f through the het unit $h_x : x \Rightarrow F(x)$, i.e., that makes the following diagram commute.

$$\begin{array}{cccc} & x & & \\ h_X & \Downarrow & \searrow^{f} & \\ & F(x) & \stackrel{g(f)}{\longrightarrow} & g \end{array}$$

Starting with Plato, there is a rich history of the notion of universals in Western philosophy. Most of their development has focused on notions of non-self-predicative universals, which were formalized in set theory. The notion of a self-predicative universal has received relatively little attention, but that notion has now also been formalized in category theory. And even in category theory, the heteromorphic analysis shows that the fundamental notion is not the special case of the birepresentations that are adjunctions but universal constructions or UMPs.

A more extensive analysis of non-self-predicative and self-predicative universals has been developed elsewhere [3,29].

Appendix A.2. Another Way to Combine Universals

Universals are represented by triangular het diagrams. We have seen how two universal het triangular diagrams can be joined together along the diagonal to form a het adjunctive square diagram. Since adjunctions have a direction, we could take one category as the "sending category" and the other one as the "receiving category". Then, a het from an object in the sending category to an object in the receiving category is uniquely "internalized" in both categories, as illustrated in the adjunctive square diagram of Figure A1.

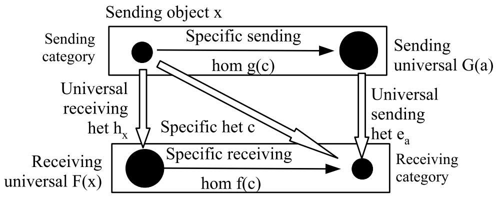

Figure A1. Adjunctive square diagram with sending and receiving category.

In this case, the sending and receiving universals are in different categories.

But there are many examples where the same entity can both receive and send, like a recorder that can both record and play back, a file program that can both save files and retrieve files, a coder that can both encode and decode messages, and, last but not least, the human brain that can both receive information from the environment and act upon the same environment.

And there is another way that two triangular diagrams can be joined together at the (right angle) corner to form a "butterfly" diagram (the two triangles being the wings of the butterfly), that is, another way for category theory to model these "universals" that can both receive and send messages or information with an "environment".

The following diagram is for an encoder–decoder example where the Cartesian coordinate system for a plane encodes a given point P in the geometrical plane with its Cartesian coordinates (x_P, y_P), and given a pair of Cartesian coordinates (x_P, y_P), it sends or plots the point P on the geometrical plane. The diagonal hets just give an association of a geometrical point P with a set of coordinates (x_P, y_P). The universal encoder–decoder operates with the universal encoding isomorphism "select coordinates of P" and the universal decoding isomorphism "plot coordinates of (x_P, y_P)" as shown in Figure A2.

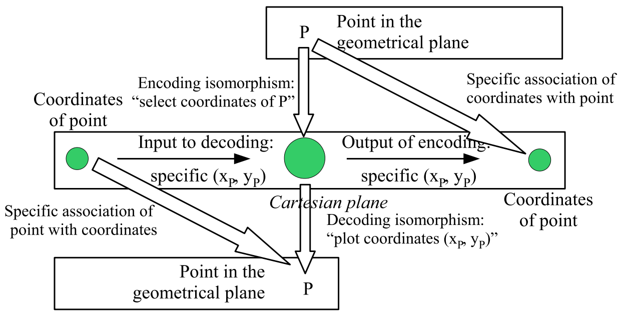

Figure A2. Universal receiving/sending example.

For a strictly category-theoretic model, the object in the center has to be both a left and right universal. For instance, a finite product of modules $M = (m_1, \ldots, m_n)$ over a given ring R is both a direct product and a coproduct (i.e., a biproduct) in Mod_R [14] (pp. 173–174). The universal het h_M is the cocone of injections of the ordered n-tuple of modules M in the n-fold product Mod_R^n to the coproduct $\bigoplus_{i=1}^{n} M_i$, and the universal het e_M is the cone of projections from $\sum_{i=1}^{n} m_i = \bigoplus_{i=1}^{n} m_i$ to the n-tuple M in Mod_R^n. The upper diagonal het is a cocone of module homomorphisms from the n-tuple M to any module m, and the lower diagonal het is a cone from any module m to the n-tuple M. Then, the cocone and cone both factor uniquely through the het h_M and the het e_M, respectively, as illustrated in the following diagram.

$$\begin{array}{ccccccc} & & & M & & & \\ & & & h_M \Downarrow & & \searrow & cocone \\ m & \longrightarrow & \sum_{i=1}^{n} m_i & = & \bigoplus_{i=1}^{n} m_i & \longrightarrow & m \\ cone & \searrow & & \Downarrow e_M & & & \\ & & & M & & & \end{array}$$

The speculative application of a left and right universal is to model the human brain. For instance, let us take the language faculty in the brain as an example. Then, there is an interesting interpretation of an auditory signal from the environment being internally represented as being "understood" or "recognized" as having a certain meaning. This is the intentionality of perception. For visual perception, it is not just "seeing" some visual inputs but "seeing them as" some recognized object. For auditory perception, there can be meaningless auditory input such as just noise or words in a foreign language, but the internal representation means the recognition of the input's meaning.

Just as we can see the internal factor maps as internalizing or recognizing some external input, on the sending side, we can see some internal "speech act" as being externalized as auditory output to the environment. There can also be auditory outputs such as random mumblings (e.g., snoring), which are not the externalization of some internal speech act. When an internal speech act is externalized, that is an example of the intentionality of action, deliberate speech action in this case. If we are going to only represent the left and right universal mapping properties of the language faculty, then we have to assume that the external auditory inputs are intelligible and that the external auditory outputs are deliberate as illustrated in Figure A3.

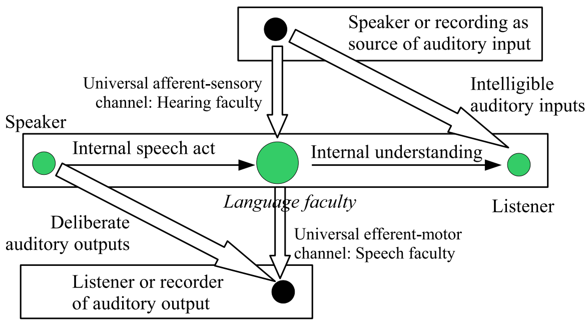

Figure A3. Butterfly diagram for language faculty.

Once we have isolated category theoretic universals, we have seen how left and right universals can be combined not only one way as an adjunction but also another way as a *left–right universal* or *receiving–sending universal*, which seems to have interesting applications [30].

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