The basic ideas of quantum mechanics

David Ellerman

Independent researcher, Ljubljana, Slovenia,
https://orcid.org/0000-0002-5718-618X.

Contributing authors: david@ellerman.org;

Abstract

For a century, quantum theorists have been reading the mathematical entrails of quantum mechanics (QM) to divine the nature of quantum reality. But to little avail. In this paper a different approach is taken, namely to identify and explain the basic intuitive ideas involved in QM. This does not tell one how those basic 'gears' all mesh together in the beautiful mathematics of QM. But this does give one some intuitive (*anschaulich*)) ideas about the quantum reality described in the seemingly hard-to-interpret mathematical framework.

Keywords: superposition, quantum amplitudes, Born rule, partitions, support sets, partition lattice, quantum realm of indefiniteness, information-as-distinctions, two-slit experiment, linearization, non-commutativity

Contents

1 Introduction: A new approach

For a century, quantum theorists have been reading the mathematical entrails of quantum mechanics (QM) to divine the nature of quantum reality. But to little avail. That problem of a realistic interpretation of quantum mechanics has become an open scandal [57]. New so-called "interpretations" are created all the time and no matter how bizarre, none are definitively abandoned in the "demolition derby" of interpretations. The usual circular conversation in the philosophy of quantum mechanics [e.g., [48]; [47]] typically considers the Copenhagen interpretation associated with Niels Bohr [25], together with the realistic or ontic interpretations of Bohmian mechanics [13], spontaneous localization [31], or many-worlds [59]. There is not even wide agreement on what constitutes an "interpretation" or how it should be constituted.

A new approach is needed. This paper focuses on giving the *basic ideas* needed to understand (standard von Neumann-Dirac) QM, not on how they connect together in the full-blown mathematics. That seems a reasonable place to start.

2 The basic idea of superposition

2.1 Superposition as the flip-side of abstraction

A glass half-full and a glass half-empty are the same thing viewed from different perspectives. Abstraction and superposition have that relationship ([17]; [2]). Given a set of entities with some similarities and some differences, the process of abstraction "abstracts away from the differences" to focus on the similarities. The "abstraction" is definite on the similarities and indefinite on the differences. In Figure 1, we have two similar isosceles triangles with labeled edges and angles. The process of abstraction to see only the similarities can also be seen as the process of superposition by rendering the triangles indefinite on their differences. This quantum superposition might be symbolized as: definite + definite = indefinite, or, in more detail, superposition of similarities = definite on the one hand, and superposition of differences = indefinite on the other hand.

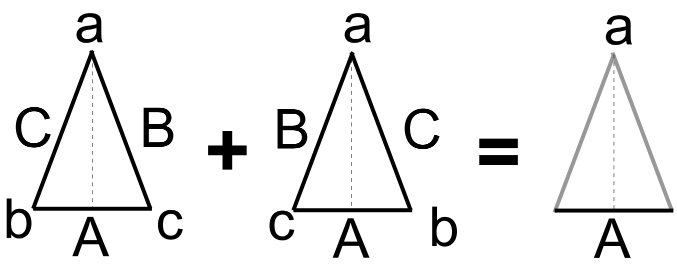

Fig. 1 Superposition of two differently labeled isosceles triangles is indefinite where they differ

A very common misunderstanding is to see quantum superposition (ontologically) as being like the classical superposition of waves like water, sound, or electromagnetic waves. The math is the same, but the interpretation should be different. The addition of two classical waves is just as definite or well-defined as the summand waves–as illustrated in Figure 2.

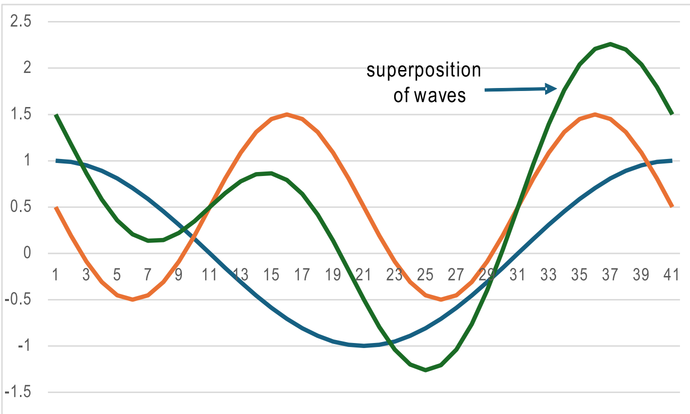

Fig. 2 Classical superposition as definite + definite = definite

That classical notion of superposition is very different from superposition seen as the flip-side of abstraction where the superposition is indefinite, e.g., indefinite between going through the two-slits in the double-slit experiment or or going through the two arms in an interferometer [5, 18].

> Such analogies have led to the name "Wave Mechanics" being sometimes given to quantum mechanics. It is important to remember, however, that the superposition that occurs in quantum mechanics is of an essentially different nature from any occurring in the classical theory, as is shown by the fact that the quantum superposition principle demands indeterminacy in the results of observations in order to be capable of a sensible physical interpretation. The analogies are thus liable to be misleading. [11, 14]

The math is the same in the two cases of classical and quantum superposition, i.e., vector addition, but the interpretation in the two cases is very different. The complex numbers are the natural math to describe waves since the polar representation of a complex number is an amplitude and a phase. But that does not mean that the wave function is a physical wave. The reality that is described is that of superposition as indefiniteness in the states of quantum particles. R.I.G. Hughes referred to quantum indefiniteness as "latency."

> The wave formalism offers a convenient mathematical representation of this latency, for not only can the mathematics of wave effects, like interference and diffraction, be expressed in terms of the addition of vectors (that is, their linear superposition; see [27, chap. 29-5], but the converse, also holds. [35, 303]

That is, the math of vector addition to describe quantum superposition as indefiniteness can always be seen in terms of classical wave effects such as interference. The wave math of classical superposition thus *tracks* (i.e., also describes) the quantum notion of superposition as indefiniteness. The math of classical wave motion (e.g., the ripple tank model of the two-slit experiment using classical water waves) also describes the math of evolution of quantum particles in superposition or indefinite states (see Figure 11 which illustrates this point in the context of the two-slit experiment).

The huge payoff from interpreting quantum superposition as creating indistinctions, i.e., rendering differences as indistinct (as in Figure 1), is that we can see that the opposite process of creating distinctions is state reduction (see later section), i.e.,

superposition = making indistinctions;
state reduction = making distinctions.

With only the classical notion of superposition, state reduction ("collapse of the wave function") appears as a mystery (the so-called "measurement problem") rather than just the inverse of quantum superposition.

2.2 Superposition accounts for quantum amplitudes and the Born rule

2.2.1 Superposition events in probability theory

There is some agreement, starting at least with Paul Dirac [11, Chapter I], that the idea of superposition is the key non-classical notion in QM. Since the superposition of quantum states is another quantum state, i.e., the linear combination of quantum states is a quantum state, the idea of superposition is responsible for the quantum states forming a linear vector space. Since our goal is conceptual understanding, not mathematical generality, we stick to the finite-dimensional case. As Hermann Weyl put it "no essential features of quantum mechanics are lost by using the finite-dimensional model." [60, 257]

When a normalized quantum state vector is expressed in the basis of eigenvectors (eigenstates) of an observable, then the Born rule states that absolute square of the coefficients is the probability of an eigenstate being the result of a measurement of that state by that observable. Steven Weinberg asked: "Where does the Born rule come from?" [62, 92] To answer that question, we might go back to a suggestion of Gian-Carlo Rota; "I will lay my cards on the table: a revision of the notion of a sample space is my ultimate concern." [50, 57]

> Behind the Feynman integral there lurks an even more enticing (and even less rigorous) concept: that of an amplitude which is meant to be the quantum-mechanical analog of probability (one gets probabilities by taking the absolute values of amplitudes and squaring them: hence the slogan "quantum mechanics is the imaginary square root of probability theory"). A concept similar to that of a sample space should be brought into existence for amplitudes and quantum mechanics should be developed starting from this concept. [49, 229]

Hence we start with the ordinary notion in finite probability theory of a *sample space* U with outcomes $u_1, ..., u_n$ which we initially assume are equiprobable. An *event* is a subset $S \subseteq U$ of outcomes. The subset S and the whole set U have no structure connecting the outcomes. The simplest possible idea for Rota's " revision" is to postulate a new type of event, a *superposition event*, symbolized ΣS, where the outcomes are "superposed" instead of having unrelated outcomes as in the classical discrete event S. And the simplest possible assumption about the probabilities is to assume they are unchanged, i.e., $\Pr(u_i|\Sigma S) := \Pr(u_i|S)$. At first this looks like a bug but it is a feature since the same thing occurs in QM where the probabilities for eigenstate outcomes in the superposition state are the same as the probabilities in the corresponding complete decomposed mixed state [5, 176]. Those two states can only be differentiated by a measurement in another basis.

2.2.2 Superposition and relation matrices

How can S and ΣS be differently represented in the same basis? An $n \times 1$ column vector $|S\rangle$ of $0,1$-entries to represent which outcomes are in the *support* of the event

S or ΣS, or even a similar vector of probabilities, will not do the job since they are the same for both S and ΣS.

One must move to an $n \times n$ matrix to represent the difference. We can begin with the simplest case of $0,1$-matrices to represent the differences. A *binary relation* R on U is a subset $R \subseteq U \times U$. The relation can be represented by an $n \times n$ *relation matrix* (also called *incidence matrix*) $\mathrm{Rel}(R)$ where $\mathrm{Rel}(R)_{jk} = 1$ if $(u_j, u_k) \in R$, else 0. Let $\Delta S \subseteq U \times U$ be the set of self-pairs of elements in S, i.e., the diagonal $\Delta S = \{(u_i, u_i) \,|u_i \in S\}$. Then the ordinary discrete event S is represented by the diagonal matrix $\mathrm{Rel}(\Delta S)$ with the diagonal being the values of the characteristic function $\chi_S : U \to \{0,1\}$, i.e., $\chi_S(u_i) = 1$ if $u_i \in S$, else 0. In contrast the superposition event ΣS is represented by $\mathrm{Rel}(S \times S)$ so that $\mathrm{Rel}(S \times S)_{jk} = 1$ if $u_j, u_k \in S$, else 0. Intuitively, in the superposition event ΣS, the elements of S are rendered indefinite on their differences which has been described as them being blurred, blobbed, smeared, or cohered together. This cohering together of superposed outcomes is then represented by the non-zero off-diagonal elements of $\mathrm{Rel}(S \times S)$. The difference between the two matrices $\mathrm{Rel}(\Delta S)$ and $\mathrm{Rel}(S \times S)$ is in those non-zero off-diagonal elements. Even though we are only dealing (at first) with $0,1$-matrices, we can already see the foreshadowing of how superposition is responsible for the non-classical aspects of QM.

> For this reason, the off-diagonal terms of a density matrix ... are often called "quantum coherences" because they are responsible for the interference effects typical of quantum mechanics that are absent in classical dynamics. [5, 177]

The 'waste case' of superposition is when S is a singleton event and accordingly in that case, $\mathrm{Rel}(\Delta S) = \mathrm{Rel}(S \times S)$ so we will only speak of a "superposition event" ΣS when $|S| \geq 2$.

Two new phenomena appear in the matrix representation $\mathrm{Rel}(S \times S)$ of a superposition event. One is the non-zero off-diagonal entries representing the cohering together of the outcomes in the superposition event ΣS. The other new result is that only $\mathrm{Rel}(S \times S)$ can be obtained as the outer (or external) product or "square" of the "square root" support $0,1$-vector $|S\rangle$ and its transpose $|S\rangle^t$, i.e.,

$$|S\rangle\,|S\rangle^t = \mathrm{Rel}(S \times S).$$

If we represent the j^{th} component of $|S\rangle$ by $\langle u_j|S\rangle$, then $\mathrm{Rel}(S \times S)_{jk} = \langle u_j|S\rangle\,\langle u_k|S\rangle$. This is a *very* important result since it shows how, in an ultra-simple matrix representation of superposition, the "square root" vector $|S\rangle$ appears that foreshadows the vector of quantum amplitudes in QM. And even the Born rule is foreshadowed in the fact that: $\langle u_i|S\rangle^2 = \mathrm{Rel}(S \times S)_{ii}$.

2.2.3 Superposition and density matrices

Dividing the relation matrices $\mathrm{Rel}(\Delta S)$ and $\mathrm{Rel}(S \times S)$ through by their trace (sum of diagonal elements) turns them into *density matrices* [61] that represent quantum states:

$$\rho(\Sigma S) = \frac{\mathrm{Rel}(S\times S)}{\mathrm{tr}[\mathrm{Rel}(S\times S)]} \text{ and } \rho(\Delta S) = \frac{\mathrm{Rel}(\Delta S)}{\mathrm{tr}[\mathrm{Rel}(\Delta S)]}.$$

They are density matrices for the equiprobable case obtained by dividing through by $\mathrm{tr}[\mathrm{Rel}(S \times S)] = \mathrm{tr}[\mathrm{Rel}(\Delta S)] = |S|$ so that:

$$\Pr(u_i|\Sigma S) := \Pr(u_i|S) = \frac{1}{|S|} \text{ if } u_i \in S, \text{ else } 0.$$

The density matrix $\rho(\Sigma S)$ is idempotent, i.e.,

$$\rho(\Sigma S)^2 = \frac{1}{|S|}\mathrm{Rel}(S\times S)\,\mathrm{Rel}(S\times S)\,\frac{1}{|S|} = \frac{1}{|S|}|S\rangle|S\rangle^t|S\rangle|S\rangle^t\frac{1}{|S|} = \frac{1}{|S|}|S\rangle|S||S\rangle^t\frac{1}{|S|} = \frac{1}{|S|}\mathrm{Rel}(S\times S) = \rho(\Sigma S).$$

Hence it is a density matrix ρ that is called a *pure state* in QM, i.e., where $\rho^2 = \rho$, while $\rho(\Delta S)$ ($|S| \geq 2$) is what is called a *mixed state.*

For example, suppose $U = \{a, b, c\}$ and $S = \{a, c\}$ with equiprobable outcomes. Then we have:

$$\mathrm{Rel}(\Delta S) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathrm{Rel}(S\times S) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Dividing through by $|S| = 2$ gives:

$$\rho(\Delta S) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \text{ and } \rho(\Sigma S) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

as well as:

$$\rho(\Sigma S)^2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}^2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \rho(\Sigma S).$$

The diagonal entries in any density matrix are non-negative reals that sum to one like probabilities. The eigenvalues of a diagonal matrix like $\rho(\Delta S)$ are its diagonal entries. But the eigenvalues of a pure state density matrix like $\rho(\Sigma S)$ are one with the value of 1 with the rest being zeros. Hence for any pure state $\rho^2 = \rho$ in QM, there is a normalized eigenvector $|s\rangle$ corresponding to the eigenvalue of 1 so the spectral decomposition of ρ as the sum of eigenvalues times the projectors to the eigenstates is: $\rho = |s\rangle\langle s|$ so that ρ is again obtained as an *outer* (or *external*) *product* or "square" of a "square root" vector $|s\rangle$ times its (conjugate) transpose. In the case of our example, the normalized eigenvector corresponding to the eigenvalue of 1 for $\rho(\Sigma S)$ is (up to sign) $|s\rangle = \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]^t$ so that:

$$|s\rangle\,|s\rangle^t = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right] = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \rho\,(\Sigma S).$$

The new item that appears for pure state density matrices ρ is this state vector $|s\rangle$ (corresponding to the eigenvalue of 1) such that $\rho = |s\rangle\,|s\rangle^t = |s\rangle\,\langle s|$ (writing $|s\rangle^t$ as $\langle s|$ in the Dirac notation). And, as in the example, the product of each entry in $|s\rangle$ with corresponding entry in the (conjugate) transpose is the diagonal entry in the density matrix, e.g., $\langle a|s\rangle^2 = \frac{1}{2} = \langle c|s\rangle^2$ and $\langle b|s\rangle^2 = 0$ in the example, which in general is the Born rule.

Thus what we have derived starting with only the notion of a superposition event ΣS in an extended probability theory smoothly generalizes to the general case in QM where transpose is the conjugate transpose and where the square $\langle u_i|s\rangle^2$ is the absolute value squared $|\langle u_i|s\rangle|^2$. The claim is that this *basic idea* of a superposition event ΣS leads by this natural logical progression to the notion of quantum amplitudes $|s\rangle$ (or "square roots") and the Born rule $\Pr(u_i|s) = |\langle u_i|s\rangle|^2$.

Of course, in the context of the full mathematics of QM, there can be many so-called "derivations" of the Born rule [56], not to mention the mathematics of the Gleason Theorem [32]. But such elegant and sophisticated results do not really answer Weinberg's question: "Where does the Born rule come from?". Our approach here is different, namely to give the *basic idea* behind the Born rule. And we have argued that the Born rule and quantum amplitudes (whose absolute squares give the probabilities) are natural consequences of just introducing the notion of a superposition event into probability theory [23].

There are a few other aspects that might be noted. The notion of "support" records only the information about a scalar as to whether it is non-zero or zero. Given a state vector $|\psi\rangle = \sum_{i=1}^{n} \alpha_i\,|u_i\rangle$, the *support set* is the set of basic vectors with non-zero coefficients: $\text{supp}\,(|\psi\rangle) = \{|u_i\rangle\,|\alpha_i \neq 0\}$, the *support vector* $|\text{supp}\,(|\psi\rangle)\rangle$ is the vector where the components $\langle u_i|\psi\rangle$ are replaced by their supports, i.e., $\langle u_i\,|\text{supp}\,(|\psi\rangle)\rangle = 1$ if $\langle u_i|\psi\rangle \neq 0$, else 0, and the *support matrix* of a matrix replaces its entries by their supports: $\text{supp}\,(\rho)_{jk} = 1$ if $\rho_{jk} \neq 0$, else 0. Then it is an easy result for any pure state density matrix $\rho = |s\rangle\,\langle s|$ in QM that:

$$\text{supp}\,(\rho) = \text{Rel}\,(\text{supp}\,(|s\rangle) \times \text{supp}\,(|s\rangle)).$$

which verifies our description of the superposition event ΣS as $\text{Rel}(S \times S)$. In other words, the pattern of non-zero entries in any pure state density matrix in QM is $S \times S$ for some subset S of the basis set in which the matrix is represented. This results follows from the fact that in any (algebraic) field from $\mathbb{Z}_2$ to $\mathbb{C}$, the product of two scalars $\langle u_j|s\rangle$ and $\langle u_k|s\rangle$ is non-zero if and only if (iff) both scalars are non-zero.

A non-trivial question is the interpretation of the state vectors $|s\rangle$ that give the corresponding pure state density matrices $\rho = |s\rangle\,\langle s|$. Many interpretations of QM take the state vectors, e.g., wave functions, as ontological entities, rather than just a computational devices to compute the probability amplitude and probabilities (via the Born rule) of possible measurement outcomes. A classical discrete event and a superposition event are not ontological entities. They are simply a mathematical part of the extended probability theory *to indicate the possible or potential outcomes*. We have derived the quantum amplitudes starting only with the notion of a superposition event with no ontological assumptions.

Moreover, if the "Schrödinger wave" is an ontic wave, then it is very unclear why the absolute squares should be probabilities rather than some notion of intensity. But our derivation shows how the "square root" state vectors $|s\rangle$ arise out of adding superposition events to *probability theory* so the probabilistic interpretation is there from the beginning.

There is another argument, that might be mentioned, as to why the " Schrödinger wave" is not a description of an ontic wave. In William Rowan Hamilton's optico-mechanical formulation of classical particle mechanics, the mathematics of waves appears [9, Sec. 7.9]. There is even a classical form of "wave-particle duality." "Both optical and mechanical phenomena can be described in wave terms as well as in particle terms." [41, 276] Certainly no one interpreted this mathematical artifact of waves as representing ontic waves *in classical particle mechanics*. But Schrödinger introduced Planck's constant h and reformulated the Hamilton-Jacobi equation over the complex numbers to obtain his famous equation.

> Schrödinger had in 1927 the original idea of going beyond the analogy between geometrical optics and mechanics, established by Hamilton's partial differential equation, and changing over from the phase function ϕ to the wave function ψ. [41, 279]

It is rather implausible to think, after these changes to obtain the Schrödinger equation, that the wave mathematics would suddenly describe ontic waves instead of the indefinite superposition states of quantum particles. And since Hamilton's geometrical mechanics is the classical limit as Planck's constant $h \to 0$, how would an ontic wave mechanics turns into particle mechanics in the limit? In short, the wave functions of QM are about the indefinite superposition states of quantum particles as opposed to the fully definite states of classical particles; it is not about physical waves.

3 The basic math of indefiniteness and definiteness

3.1 The logic of partitions (or equivalence relations)

Abstracting away from the differences between to entities u_j and u_k means that they are equivalent in what characteristics that remain. In other words, neglecting their differences means they are now in the same equivalence class of some equivalence relation. And *equivalence relation* $E \subseteq U \times U$ is a reflexive, symmetric, and transitive binary relation on U. Each element of U has an *equivalence class* of other elements equivalent to it. Those equivalence classes are non-empty disjoint subsets of U that cover all of U so they form a *partition* of U. Equivalence relations and partitions are essentially the same concept but viewed from different perspectives, concepts that Gian-Carlo Rota called "cyrptomorphic" [40, 153]. We will focus on the notion of a partition.

A *partition* π on $U = \{u_1, ..., u_n\}$ is a set of nonempty disjoint subsets $\pi = \{B_j\}_{j=1}^{m}$ whose union is all of U. A *distinction* or *dit* of π is an ordered pair (u_j, u_k) in different blocks of π so let $\operatorname{dit}(\pi) \subseteq U \times U$ be the set of distinctions of π. Then an *indistinction* or *indit* of π is an ordered pair of elements of U in the same block of π and $\operatorname{indit}(\pi) = U \times U - \operatorname{dit}(\pi)$ is the equivalence relation associated with π.

The concepts of distinctions and indistinctions, distin-guishability and indistinguishability, and definiteness and indefiniteness are the key concepts, the 'natural language,' of quantum mechanics. Partition logic is their logic. They are responsible for the widespread belief that "information," i.e., information-as-distinctions, plays a basic role in QM. The role of those concept is quite explicit in the Feynman rules.

> If several alternative subprocesses, indistinguishable within the given physical arrangement, lead from the initial state to the final (registered) result, then the amplitudes for all the indistinguishable processes must be added to get the total amplitude for their combination (quantum law of superposition of amplitudes).
>
> If several distinguishable alternative processes lead from the initial preparation to the same final result, then the probabilities for all these processes must be added to get the total probability for the final result (law of addition of probabilities). [36, 110]

This ontic role of distinctions and indistinctions, i.e., information, is quite unknown in the classical physics of fully definite particles.

If $\sigma = \{C_{j'}\}_{j'=1}^{m'}$ is another partition on U, then σ is *refined* by π, denoted $\sigma \precsim \pi$, if for any block $B_j \in \pi$ there is a block $C_{j'} \in \sigma$ containing it, i.e., $B_j \subseteq C_{j'}$. The refinement partial ordering can also be expressed as just the inclusion relation on ditsets:

$$\sigma \precsim \pi \text{ iff } \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi).$$

If $\Pi(U)$ is the set of partitions on U, then refinement is a partial order on $\Pi(U)$. Moving upward in that partial order means making more distinctions. The top or maximal partition of the partial order is the *discrete partition* $\mathbf{1}_U = \{\{u_i\}\}_{u_i \in U}$ where all the blocks are singletons. The bottom or minimal partition is the *indiscrete partition* $\mathbf{0}_U = \{U\}$ whose only block is U. Since U is the only block in $\mathbf{0}_U$, it has no distinctions: $\operatorname{dit}(\mathbf{0}_U) = \emptyset$. The discrete partition makes all possible distinctions. Since no element can be distinguished from itself, $\operatorname{indit}(\mathbf{1}_U) = \Delta$ where diagonal Δ is $\Delta = \{(u_i, u_i) | u_i \in U\}$ and $\operatorname{dit}(\mathbf{1}_U) = U \times U - \Delta$.

The *join* of π and σ, denoted $\pi \vee \sigma$, is the partition whose blocks are all the nonempty intersections $B_j \cap C_{j'}$ of the blocks of π and σ. The join is the least upper bound of π and σ in the refinement partial order, and its ditset is: $\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$. Then by DeMorgan's law, its indit set is: $\operatorname{indit}(\pi \vee \sigma) = \operatorname{indit}(\pi) \cap \operatorname{indit}(\sigma)$, so the join of partitions corresponds to the intersection of the corresponding equivalence relations. Since the intersection of two equivalence relations is always an equivalence relation, the *meet* (greatest lower bound) of π and σ, denoted $\pi \wedge \sigma$, is the partition whose corresponding equivalence relation is the smallest equivalence relation containing $\mathrm{indit}(\pi) \cup \mathrm{indit}(\sigma)$. That join and meet operation on $\Pi(U)$ make it a *lattice* which was known in the 19th century (Dedekind and Schröder).

Figure 3 gives the lattice of partitions for $U = \{a, b, c\}$ where refinement is indicated by the lines between partitions.

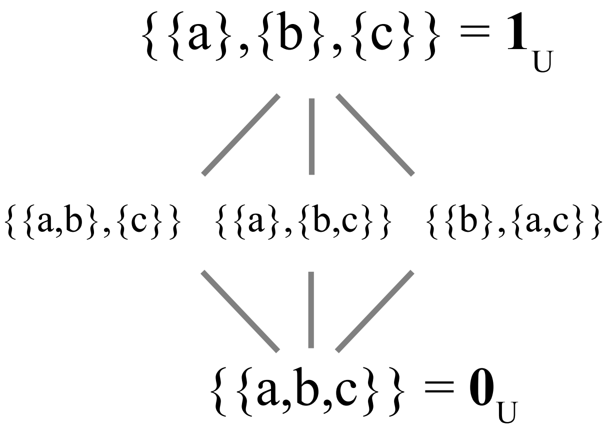

Fig. 3 Partition lattice on $U = \{a, b, c\}$

The ditset of the partition $\pi = \{\{a, b\}, \{c\}\}$ is $\mathrm{dit}(\pi) = \{(a, c), (b, c), ...\}$ where the ellipsis stands for the reversed ordered pairs. The ditset of the discrete partition $\mathbf{1}_U$ is $\mathrm{dit}(\mathbf{1}_U) = \{(a, b), (a, c), (b, c), ...\}$ so in moving up the refinement partial order from π to $\mathbf{1}_U$, means distinguishing a from b in block or equivalence class $\{a, b\}$, i.e., making the distinctions $\{(a, b), ...\} = \{(a, b), (b, a)\} = \mathrm{dit}(\mathbf{1}_U) - \mathrm{dit}(\pi)$.

Without at least the implication operation on partitions, the lattice of partitions $\Pi(U)$ is not properly called a " logic." When the implication operation (and other logical operations) were defined (in the 21st century), then the *logic* of partitions could be developed ([19]; [14]).

In category theory, there is the basic turn-around-arrows duality between subsets (or subobjects or 'parts') and partitions (or quotient objects). "The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition." [42, 85] This duality is illustrated in Figure 4 where the image of a set-function $f : X \to Y$ is subset $f(X) \subseteq Y$ of the codomain Y and the coimage (or inverse-image) $f^{-1} = \left\{f^{-1}(y)\right\}_{y \in f(X)}$ is a partition on the domain X.

Since the usual logic is the Boolean logic of subsets (usually presented in the special case of the logic of propositions), the logic of partitions is, in that sense, the dual logic to Boolean logic [14]. We will eventually see that this basic duality is reflected in the difference between classical mechanics and quantum mechanics [20]. This duality comes out if we compare the Boolean lattice $\wp(U)$ and the partition lattice $\Pi(U)$ by considering what happens in terms of substance (or matter) and form [3] moving from the bottom to the top of each lattice.

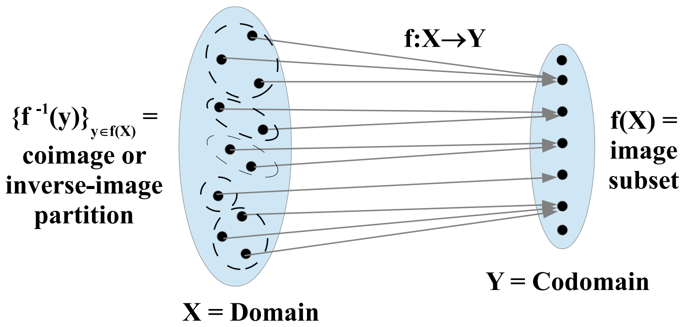

Fig. 4 Duality between subsets and partitions illustrated with a function $f : X \to Y$

The bottom of the Boolean lattice is the empty set $\emptyset$ and as we move up the lattice, new substance is created in each subset until we reach the top $U = \{a, b, c\}$. Each element in U is always fully formed. In the partition lattice, we see the opposite. At the bottom is the indiscrete partition $\mathbf{0}_U = \{U\} = \{\{a, b, c\}\}$ so all the substance is there but in a totally indefinite form with no distinctions. As we move up the lattice, there is no new substance but the elements a, b, or c become more in-formed with distinctions between the elements of U. Elsewhere we have argued that information is distinctions ([18]; [24]). Hence moving up the Boolean lattice means new substance is created but no new information (since the elements are always fully distinguished). In the dual case, moving up the partition lattice means new information-as-distinctions is created but no new substance. That illustration of the duality is pictured in Figure 5.

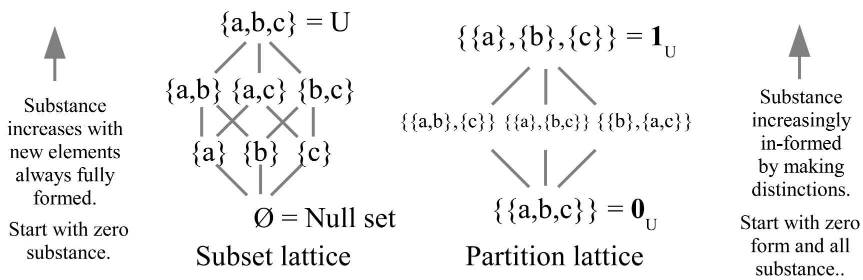

Fig. 5 Duality illustrated between the lattice of subsets and the lattice of partitions

3.2 Mapping a quantum system into a partition lattice using support sets

Think of U as the basis set of orthonormal eigenvectors for some observable. The support sets for quantum states can then be mapped into the partition lattice $\Pi(U)$. The blocks in the partitions (equivalence classes of equivalence relations) represent the support sets of quantum states. In $\mathbf{0}_U$, there is only one block U and it is the support set of any (pure) quantum state that is a superposition of all the eigenstates–like what we previously denoted as ΣU (since all the non-singleton subsets S appearing in $\Pi(U)$

represent superpositions previously denoted ΣS). At the other extreme is $\mathbf{1}_U$ whose blocks are all the singletons $\{u_i\}$ for $u_i \in U$ so $\mathbf{1}_U$ is the support set of the completely decomposed mixed state–like what we previously described as the discrete classical sample space U.

In between are the support sets for all the mixed states that could be obtained from a superposition of all the eigenstates (modeled by $\mathbf{0}_U$) by the projective measurement or state reduction by the observable, the operation which is mathematically described by the Lüders mixture operation ([44]; [30]; [5, 279]). Those mixed states are special since the supports of the states in the mixture are all disjoint, and that disjointness is inherited from the disjointness of the eigenspaces of the observable via the Lüders mixture operation representing projective measurement. And those disjoint supports cover all of U since they result from projective measurement of $\mathbf{0}_U$, so those mixed state supports make up the partitions between the top and bottom of the partition lattice.

For instance, the partition lattice $\Pi(\{a,b,c\})$ (Figure 3) represents the supports of the possible states of the quantum system representing one quantum particle in the three-dimensional Hilbert space $\mathbb{C}^3$. The striking thing is that the lattice of support sets boils the possible quantum states of the particle down to show the clear separation between the classical part $\mathbf{1}_U$ and all the other quantum states represented by partitions with at least one non-singleton block representing the support set of a superposition state. Thus the system can be represented like an iceberg [39, 7], the above-water part being the classical mixed state $\mathbf{1}_U$, and the below-water part being all the states involving at least one non-classical (i.e., indefinite) state, i.e., a superposition state, as illustrated in Figure 6.

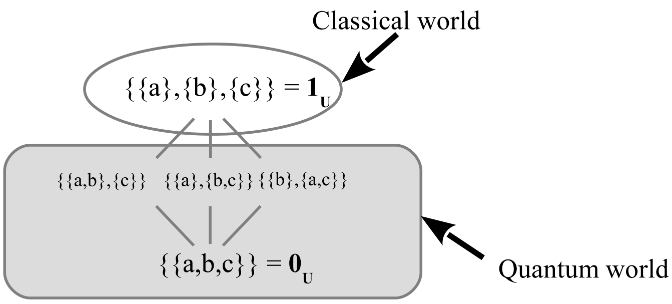

Fig. 6 Iceberg picture of possible support states of a quantum system of one particle with three eigenstates a, b, or c

The iceberg picture of the partition lattice matches up with the image of reality divided into actuality (the classical part) and " potentiality" (the quantum part) as advocated by Werner Heisenberg [34], Abner Shimony [52], Gregg Jaeger [36], Diederik Aerts [1], Ruth Kastner [38], Leonardo Chiatti [8], and many others.

> Heisenberg [34, 53] used the term "potentiality" to characterize a property which is objectively indefinite, whose value when actualized is a matter of

objective chance, and which is assigned a definite probability by an algorithm presupposing a definite mathematical structure of states and properties. Potentiality is a modality that is somehow intermediate between actuality and mere logical possibility. That properties can have this modality, and that states of physical systems are characterized partially by the potentialities they determine and not just by the catalogue of properties to which they assign definite values, are profound discoveries about the world, rather than about human knowledge. [53, 6]

It is particularly important to see that moving upward in the partition lattice, e.g., from a quantum state ("potentiality") to a classical state (actuality) means *making distinctions* (since the refinement partial order on partitions is just inclusion of ditsets). This is the process of *emergence* from the quantum world of indefiniteness into the classical world of definiteness, e.g., as illustrated in the partition lattice of Figure 5.

Any non-philosophical quantum theorist who believes a superposition state of an observable does not have a definite value prior to measurement has already implicitly acknowledged this quantum underworld of indefiniteness. The partition lattice of indefinite support sets as partition blocks (below the top of classical states) thus only adds some structure to what is commonly understood.

Heisenberg seems to have clothed his metaphysical speculations in discussions of Greek philosophy and hence his use of Aristotle's notion of "potentiality." But Shimony pointed out that this was not a felicitous choice of concepts.

> The historical reference should perhaps be dismissed, since quantum mechanical potentiality is completely devoid of teleological significance, which is central to Aristotle's conception. What it has in common with Aristotle's conception is the indefinite character of certain properties of the system. [52, 313-4]

Indeed, the notion of indefiniteness is already a sufficient description of the quantum underworld to contrast with the classical concepts of full definiteness. Rather than "actuality" and "potentiality" (or " latency" [46]; [35]), reality divides into the quantum world of indefiniteness from which, with distinctions, *emerges* into the classical world of definiteness. The admission of this quantum underworld of indefiniteness is the main ontological implication of our analysis.

The partition lattice $\Pi(U)$ adds structure (simplified to the level of support sets) to the iceberg picture. This is well-illustrated in the mapping of the two von Neumann processes in the partition lattice for the two-slit experiment (see below).

The emergence from indefinite to less indefinite or fully definite in the partition lattice matches up precisely with the case of state reduction in the Feynman rules [36, 110] due to making distinctions, i.e., distinguishing between the superposed alternative paths. Moving upward from an indefinite state to a less indefinite or even a definite state, i.e., refinement, in the partition lattice represents a state reduction–which in unnecessarily anthropocentric terms is often called a "measurement" even though there need be no human involvement or interaction with a macroscopic apparatus. A human measurement involves amplifying a quantum level state reduction to the level of human observation but such technological considerations have no role in quantum *theory*.

3.3 Join of partitions = partition lattice version of projective measurement

Taking U as an orthonormal eigenbasis for an observable (i.e., a Hermitian or self-adjoint operator), the operator assigns a real number to each eigenvector so it can be described as a numerical attribute $f : U \to \mathbb{R}$. The inverse-image $f^{-1} = \left\{f^{-1}(r)\right\}_{r\in f(U)}$ is a partition on U and taking the state to be measured as a partition π, then the result of the "projection-valued measurement" described by the Lüders mixture operation is simply the join $f^{-1} \vee \pi$. The Lüders mixture operation is defined in terms of projections and density matrices, but in the basic environment of the powerset $\wp(U)$, a projection operator P_S defined by a subset $S \subseteq U$ takes any subset T to $S \cap T$, an idempotent operation. Hence the projection-valued measurement of "mixed state" $\pi = \{B_1, ..., B_m\}$ by the "observable" represented by $f^{-1} = \left\{f^{-1}(r)\right\}_{r\in f(U)}$ takes the blocks B_j of π to their projections $f^{-1}(r) \cap B_j$, which are precisely the blocks of the join $f^{-1} \vee \pi$. The indefinite states represented by the B_j's are reduced to the more definite states $f^{-1}(r) \cap B_j$ in the mixture $f^{-1} \vee \pi$. The non-zero off-diagonal elements in $\mathrm{Rel}(B_j \times B_j)$ and $\rho(\Sigma B_j)$ represent the pairs of elements in the superposition state ΣB_j (hereafter B_j means ΣB_j unless indicated otherwise) that are blobbed or cohered together like "quantum coherences" [5, 177]. The Lüders mixture operation is an operation that transforms a density matrix, such as

$$\rho(\pi) = \sum_{B_j \in \pi} \Pr(B_j)\, \rho(B_j)$$

into the post-measurement density matrix $\widehat{\rho}(\pi)$. The non-zero entries in $\rho(\pi)$ that are zeroed in that operation $\rho(\pi) \to \widehat{\rho}(\pi)$, i.e., the pairs that are decohered in the join operation, are the pairs with different f-values. That is the basic idea of projection-valued measurement; it distinguishes superposed (i.e., indefinite) states that have different eigenvalues.

4 The pedagogical model of QM over (support) sets

4.1 The state space over $\mathbb{Z}_2$

A pedagogical (or 'toy') model of QM, called *QM/Sets* ([15]; [22]) , can be constructed by working with support vectors instead of the full state vectors over the complex numbers $\mathbb{C}$. The support vectors have only $0,1$-components so they are vectors in the vector space $\mathbb{Z}_2^n$ over the field $\mathbb{Z}_2 = \{0, 1\}$ where $1 + 1 = 0$. Of course, a lot of information is lost about the non-zero coefficients in a state vector $|\psi\rangle = \sum_{i=1}^{n} \alpha_i |u_i\rangle$, but enough is retained to illustrate in a simple *basic* manner some of the paradoxical aspects of QM (see the treatment of the two-slit experiment below).

Since a n-ary $0,1$-vector in $\mathbb{Z}_2^n$ also defines a support subset $S \subseteq U$, the vectors can also treated as subsets in the powerset $\wp(U)$. The addition of two $0,1$-vectors in

$\mathbb{Z}_2^n$ then corresponds to the addition of two subsets $S, T \in \wp(U)$ by the *symmetric difference* operation, i.e.,

$$S + T = (S - T) \cup (T - S).$$

For instance, if $U = \{a, b, c\}$, $S = \{a, b\}$, and $T = \{b, c\}$, then $S+T = \{a, b\} + \{b, c\} = \{a, c\}$ (since the b's cancel out). With that addition operation on $\wp(U)$, there is a vector space isomorphism $\mathbb{Z}_2^n \cong \wp(U)$. As in any vector space, there are many different basis sets, three of which are given in Table 1. Each row in the table represents the same abstract vector or 'ket' represented in a different basis set. The left-most column gives the corresponding column vector in $\mathbb{Z}_2^3$ (where the superscript t represents the transpose).

$\mathbb{Z}_2^3$	$U = \{a, b, c\}$	$U' = \{a', b', c'\}$	$U'' = \{a'', b'', c''\}$
$[1, 1, 1]^t$	$\{a, b, c\}$	$\{b'\}$	$\{a'', b'', c''\}$
$[1, 1, 0]^t$	$\{a, b\}$	$\{a'\}$	$\{b''\}$
$[0, 1, 1]^t$	$\{b, c\}$	$\{c'\}$	$\{b'', c''\}$
$[1, 0, 1]^t$	$\{a, c\}$	$\{a', c'\}$	$\{c''\}$
$[1, 0, 0]^t$	$\{a\}$	$\{b', c'\}$	$\{a''\}$
$[0, 1, 0]^t$	$\{b\}$	$\{a', b', c'\}$	$\{a'', b''\}$
$[0, 0, 1]^t$	$\{c\}$	$\{a', b'\}$	$\{a'', c''\}$
$[0, 0, 0]^t$	$\emptyset$	$\emptyset$	$\emptyset$

Table 1: Ket table giving the isomorphisms $\mathbb{Z}_2^3 \cong \wp(U) \cong \wp(U') \cong \wp(U'')$

Taking U as the computational basis, it is easy to see, for example, that $\{a'\}$, $\{b'\}$, and $\{c'\}$ also form a basis, the U'-basis, since:
$\{b'\} + \{c'\} = \{b', c'\} = \{a, b, c\} + \{b, c\} = \{a\}$,
$\{a'\} + \{b'\} + \{c'\} = \{a', b', c'\} = \{a, b\} + \{a, b, c\} + \{b, c\} = \{b\}$, and
$\{a'\} + \{b'\} = \{a', b'\} = \{a, b\} + \{a, b, c\} = \{c\}$.

4.2 von Neumann's two quantum processes

John von Neumann postulated two and only two types of quantum processes: Type I are the state reductions and Type II are the evolutions according to the Schrödinger equation [58]. What is the basic idea? We have already seen that the basic idea in Type I state reduction is a process of making distinctions. Hence the natural definition of the other Type II processes would be processes that don't make distinctions. The distinctness of two (normalized) quantum states ψ and ϕ is their inner product $\langle\psi|\phi\rangle$. If the inner product is zero, they are totally distinct with no 'overlap.' If the inner product is one, then there is total overlap, i.e., they are the same state. Hence the basic idea of a Type II process is one that doesn't make distinctions so the measure of the indistinctness of distinctness of two quantum states, i.e., the inner product, is preserved–which is a unitary transformation. The connection to the solutions to the Schrödinger equation is given by Stone's theorem [54].

Another way to characterize a unitary transformation is that it transforms orthonormal basis sets into orthonormal basis sets. There are no inner products in vector spaces over finite fields like $\mathbb{Z}_2$ in our pedagogical model QM/Sets. Hence the corresponding idea in a finite vector space is a transformation that is *non-singular*, i.e., transforms basis sets into basis sets so that is the Type II process assumed in the model.

4.3 Probabilities in QM/Sets

The Dirac brackets in QM give the "overlap" between two states where the minimal overlap is $\langle\psi|\phi\rangle = 0$ for states that are orthogonal (or disjoint) and maximal overlap is $\langle\psi|\phi\rangle = 1$ when they are the same state. In QM/Sets, there is an obvious notion of overlap, the cardinality of the intersection, that takes its values in the natural numbers $\mathbb{N}$. That is, for $S, T \in \wp(U)$:

$$\langle S|_U T\rangle := |S \cap T|.$$

The ket $|T\rangle$ denotes the ket of $T \in \wp(U)$ and is in that sense basis-independent, but the 'bra' $\langle S|_U$ must be taken as basis-dependent as indicated by the subscript U since the intersection $S \cap T$ requires that both S and T be subsets of U.

The unitary transformations in QM are replaced by the non-singular transformations in the vector space $\wp(U)$ that carry a basis set to a basis set. Accordingly, that preserves the overlaps instead of the inner products. For instance in the non-singular transformation from the U-basis to the U'-basis, i.e., $\{a\} \rightsquigarrow \{a'\}$, $\{b\} \rightsquigarrow \{b'\}$, and $\{c\} \rightsquigarrow \{c'\}$, we have:

$$\langle S|_U T\rangle = \langle S'|_{U'} T'\rangle.$$

Preservation of bra-kets under non-singular transformations

A projection operator P on a vector space is an operator that is idempotent, i.e., $PP = P$. For the universe set U with the disjoint basis $\{\{u_i\}\}_{u_i \in U}$, the projection operator $\{u_i\} \cap () : \wp(U) \to \wp(U)$ takes S to $\{u_i\} \cap S$ which is $\{u_i\}$ if $u_i \in S$ and $\emptyset$ otherwise. The characteristic function $\chi_{\{u_i\}} : U \to \mathbb{Z}_2$, with value 1 at u_i, else 0, defined the same projection operator $\widehat{\chi}_{\{u_i\}} = \{u_i\} \cap () : \wp(U) \to \wp(U)$ by $\widehat{\chi}_{\{u_i\}} \{u_j\} = \chi_{\{u_i\}}(u_j) \{u_i\}$. Then the sum of these projection operators over the whole U-basis is the identity operator:

$$\sum_{u_i \in U} \{u_i\} \cap () = \sum_{u_i \in U} \widehat{\chi}_{\{u_i\}} = I() : \wp(U) \to \wp(U).$$

In QM, given an orthonormal (ON) basis $\{|u_i\rangle\}_{i=1}^{n}$ of the Hilbert space V, the characteristic function $\chi_{\{u_i\}} : U \to [0,1]$ defines the projection operator $\widehat{\chi}_{\{u_i\}} = |u_i\rangle \langle u_i| : V \to V$ and the sum of the ket-bra projection operators is also the identity operator:

$$\sum_{i=1}^{n} |u_i\rangle \langle u_i| = \sum_{i=1}^{n} \widehat{\chi}_{\{u_i\}} = I : V \to V$$

Completeness of the ket-bra sum

Now $\langle\{u_i\}|_U S\rangle = \langle S|_U\{u_i\}\rangle = |S\cap\{u_i\}| = \chi_S(u_i)$. Then any bra-ket $\langle S|_U T\rangle$ can be resolved using the ket-bra sum:

$$\sum_{u_i\in U}\langle S|_U\{u_i\}\rangle\langle\{u_i\}|_U T\rangle = \sum_{u_i\in U}\chi_S(u_i)\chi_T(u_i) = |S\cap T| = \langle S|_U T\rangle$$

which is the QM/Sets version of the QM:

$$\langle\psi|\varphi\rangle = \sum_i\langle\psi|u_i\rangle\langle u_i|\varphi\rangle$$
Resolution of unity by ket-bra sum

In QM, the magnitude or norm of a vector ψ is often denoted $|\psi| = \sqrt{\langle\psi|\psi\rangle}$ but that conflicts with our notation $|S|$ for cardinality, so we will use $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$ for the norm in QM and the corresponding norm in QM/Sets is:

$$\|S\|_U = \sqrt{\langle S|_U S\rangle} = \sqrt{|S|}$$
Norm in QM/Sets

which takes values in the real numbers $\mathbb{R}$. Applied to the resolution of unity:

$$\|S\|_U^2 = \langle S|_U S\rangle = \sum_{u\in U}\langle S|_U\{u_i\}\rangle\langle\{u_i\}|_U S\rangle = |S|$$

which in QM is:

$$\|\psi\|^2 = \langle\psi|\psi\rangle = \sum_i\langle\psi|u_i\rangle\langle u_i|\psi\rangle = \sum_i\langle u_i|\psi\rangle^*\langle u_i|\psi\rangle$$

where $\langle u_i|\psi\rangle^* = \langle\psi|u_i\rangle$ is the complex conjugate of $\langle u_i|\psi\rangle$.

Since the non-zero amplitudes are replaced by ones in the move to support vectors, the outcomes are assumed equiprobable in QM/Sets. In QM, a vector can be normalized at any time, but in QM/Sets, normalization is only done when probabilities are computed, so to better draw out the analogies, we will not necessarily assume a vector ψ is normalized. When a state ψ is measured in the measurement basis $\{|u_i\rangle\}$, then the probability of obtaining u_i is given by the Born Rule:

$$\Pr(u_i|\psi) = \frac{\|\langle u_i|\psi\rangle\|^2}{\|\psi\|^2}$$

and the corresponding Born Rule formula in QM/Sets is:

$$\Pr(u_i|_U S) = \frac{\|\langle\{u_i\}|_U S\rangle\|_U^2}{\|S\|_U^2} = \frac{|\{u_i\}\cap S|}{|S|} = \begin{cases} 1/|S| \text{ if } u_i\in S \\ 0 \text{ if } u_i\notin S \end{cases}.$$

And given a numerical attribute $f: U\to\mathbb{R}$, then $f^{-1}(r)\cap() : \wp(U)\to\wp(U)$ is a projection operator and the probability of getting $r\in f(U)$ when measuring S is:

$$\Pr(r|S) = \frac{\left\|f^{-1}(r)\cap S\right\|^2}{\|S\|} = \frac{|f^{-1}(r)\cap S|}{|S|}.$$

In QM, if P_r is the projection operator to the eigenspace of the eigenvalue r, then the probability of getting that eigenvalue when measuring $|\psi\rangle$ is:

$$\Pr(r||\psi\rangle) = \frac{\|P_r|\psi\rangle\|^2}{\||\psi\rangle\|^2}.$$

In this manner, QM/Sets produces a simplified version of QM over support sets. The crucial application is the two-slit experiment.

4.4 The two-slit experiment: the setup

The two-slit experiment is the best example to illustrate the basic ideas of QM. Indeed, according to Richard Feynman, it contains "the only mystery" [29, Sec. 1.1]. The pedagogical model greatly simplifies the model but reproduces "the only mystery" in the form of the question: "With no detection at the slits, how does the particle get from the two-slit screen to the detection wall without going through one of the slits– in which case there would be no interference effects?". The mystery is often covered up with a bit of magic or legerdemain called "wave-particle complementarity." With no detection at the slits, the particle suddenly turns into a wave which, in a certain sense, "goes through both slits" like in the classroom ripple-tank demonstration [43] using classical waves. But a wave cannot register at just one point on the detection wall so the wave thoughtfully turns back into a particle after the interference effects. The actual explanation can be easily seen in the pedagogical model without any such magic.

The assumed discrete dynamics is that of the non-singular transformation where in each time period, the U-basis turns into the U'-basis, i.e., $\{a\} \rightsquigarrow \{a'\} = \{a, b\}$, $\{b\} \rightsquigarrow \{b'\} = \{a, b, c\}$, and $\{c\} \rightsquigarrow \{c'\} = \{b, c\}$. The three states $\{a, b, c\}$ of the one particle system represent vertical distance positions with the particle emitter at $\{b\}$ and the two slits on the screen at $\{a\}$ and $\{c\}$ as illustrated in Figure 7.

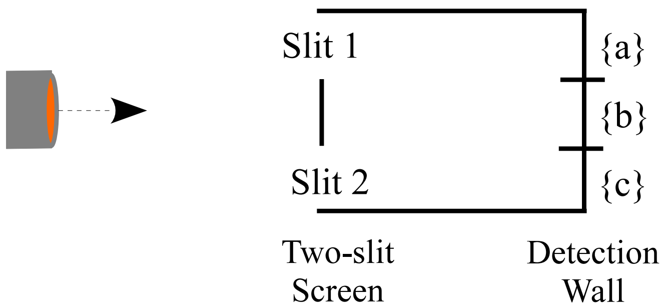

Fig. 7 Setup for the two-slit experiment

In the first time period, the particle moves from the particle emitter at $\{b\}$ to the screen at $\{b'\} = \{a, b, c\}$. Then the first state reduction takes place where the particle either hits the screen at $\{b\}$ with probability $\Pr(\{b\}|_U \{a, b, c\}) = \frac{|\{b\} \cap U|}{|U|} = \frac{1}{3}$.

Otherwise, the particle arrives at the screen in the superposition state $\{a, c\}$ with probability $\Pr(\{a, c\} |_U \{a.b.c\}) = \frac{|\{a,c\} \cap \{U\}|}{|U|} = \frac{2}{3}$. Then we have the two cases of Case 1 of detectors at the slits or Case 2 of no detectors at the slits.

4.5 Case 1: Detection at the slits

The detectors *distinguish* between the two states in superposition at the screen: $\{a, c\} = \{$Going through slit 1, Going through slit 2$\}$ so the superposition is reduced to one of the components. If it is reduced to $\{a\} = \{$Going through slit 1$\}$, then it evolves to the superposition $\{a, b\}$ at the detection wall which *distinguishes* between the two components so the superposition $\{a'\} = \{a, b\}$ reduces to $\{a\}$ or to $\{b\}$ with probability $\frac{1}{2}$ each. Similarly, if the $\{a, c\}$ superposition at the screen reduces to $\{c\} = \{$Going through slit 2$\}$, then it evolves to the superposition $\{c'\} = \{b, c\}$ at the detection wall. And then the wall *distinguishes* between those two components so they reduce to $\{b\}$ or $\{c\}$ with $\frac{1}{2}$ probability each. The probabilities multiply along the paths (Feynman Rule 3.5 law of multiplication of probabilities [36, 111]), so we have at $\{a\}$ and $\{c\}$ on the wall:

$$\Pr(a|\text{wall}) = \frac{2}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{6} = \Pr(c|\text{wall}).$$

There are two paths for the particle to reach $\{b\}$ at the wall, so (Feynman rule 3.3 law of addition of probabilities [36, 110]) those probabilities add so that:

$$\Pr(b|\text{wall}) = \frac{2}{3}\frac{1}{2}\frac{1}{2} + \frac{2}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

The bar graph of the Case 1 probabilities is illustrated in Figure 8.

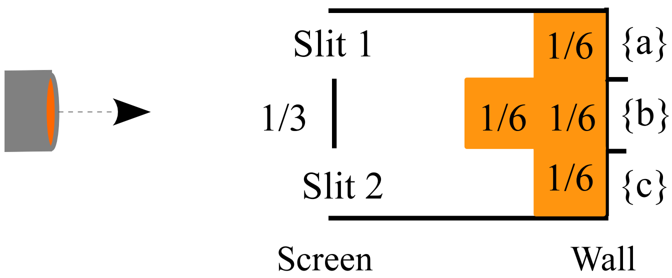

Fig. 8 Probabilities of the particle (starting at the emitter) of hitting the detection wall

4.6 Case 2: No detection at the slits

In Case 2, there is no state reduction in the superposition $\{a, c\}$ at the screen so that superposition state, which is below-the-water in the iceberg picture, i.e., in the quantum world, continues to evolve according to the dynamics:

$$\text{At the screen } \{a,c\} = \{a\} + \{c\} \rightsquigarrow \{a'\} + \{c'\} = \{a,b\} + \{b,c\} = \{a,c\} \text{ at the wall.}$$

Then the detection wall *distinguishes* between the components of $\{a,c\}$ at the wall so $\{a\}$ or $\{c\}$ occur with $\frac{1}{2}$ probability each. Then the probabilities at the wall are:

$$\Pr(a|\text{wall}) = \frac{2}{3}\frac{1}{2} = \frac{1}{3} = \Pr(c|\text{wall}) \text{ and } \Pr(b|\text{wall}) = 0.$$

The bar graph of the Case 2 probabilities is illustrated in Figure 9.

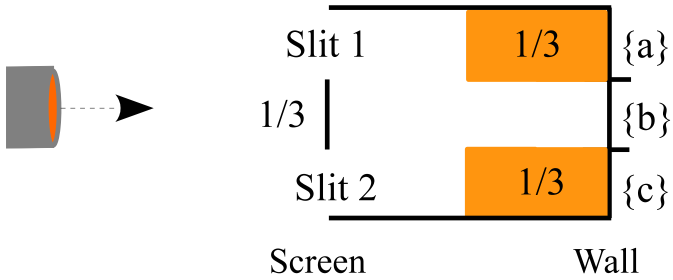

Fig. 9 Probabilities of the particle (starting at the emitter) of hitting at the detection wall

Immediately we see the striped pattern due to the interference–in this case the destructive interference at b in the evolution:

$$\text{At the screen } \{a,c\} \rightsquigarrow \{a,b\} + \{b,c\} = \{a,c\} \text{ at the wall.}$$

4.7 Analysis of the two-slit experiment

This is the analysis in the simplified pedagogical model of QM/Sets that illustrates the basic ideas involved in the two-slit experiment. To complete the explanation, we need to bring in the iceberg/partition-lattice picture to illustrate the two cases using the partition lattices. At the screen, the two states of $\{a\} = \{\text{Going through slit 1}\}$ and $\{c\} = \{\text{Going through slit 2}\}$ are *classical* above-the-water states, neither of which occurs in Case 2 of no detection at the slits. As Feynman put it: "We must conclude that when both holes are open it is *not true* that the particle goes through one hole or the other." [26, 536] Hence the question: "In Case 2, which slit does the particle go through?" falsely assumes that one of those two classical events occurs. This is an example of trying to fit quantum (below-the-water) events into a classical (above-the-water) framework of thinking so it appears to be a mystery how the particle can get to the detection wall without one of the classical states of going through one of the slits occurring. The dead end of the reasoning using classical states is what prompts the magic of "particle-wave complementarity" to picture the quantum particle as suddenly turning into a classical wave "going through both slits." But that is not what happens. Since we interpreted superposition non-classically as indefiniteness and since the particle retains the indefinite state $\{a,c\}$ in Case 2, the particle evolves in the below-the-water quantum world of indefiniteness as illustrated in Figure 10.

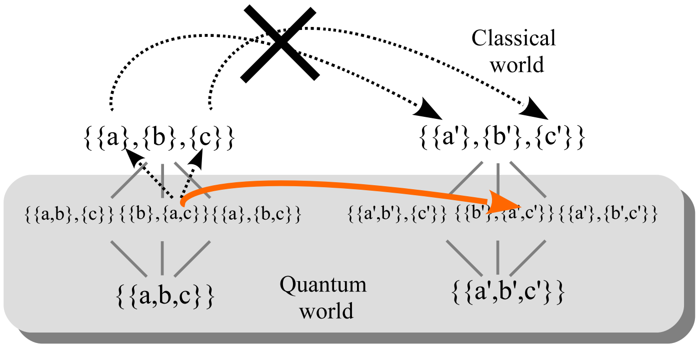

Fig. 10 Case 2 evolution of the particle in the superposition state $\{a, c\}$ which is in the quantum world

The partition lattice is a super-simplified picture of reality which consists of the classical reality of fully definite states and the quantum world of indefinite superposition mixed and pure states.

1. Since von Neumann's Type I state reductions result from making distinctions, they are always represented by upward arrows (from indefinite to more definite states) in the partition lattice.
2. The von Neumann Type II evolutions are represented by the non-upward arrow (horizontal or downward) arrows in the partition lattice.

Figure 11 illustrates both the (under-water or quantum world) evolution (dotted horizontal arrows) from screen to the wall in Case 2 and then the state reductions (upward solid arrows) at the wall.

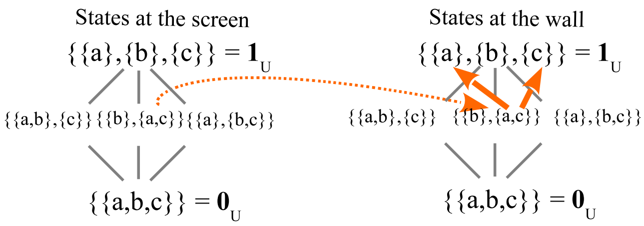

Fig. 11 Two von Neumann processes in Case 2

The partition lattices can also be used to represent the state reductions and evolutions in Case 1. Again in Figure 12, the dotted (non-upward) arrows are Type II evolutions and the solid upward arrows are state reductions.

In both Case 1 and Case 2, there was the same evolution from the emitter to the screen and the same state reduction at the screen as illustrated in Figure 13.

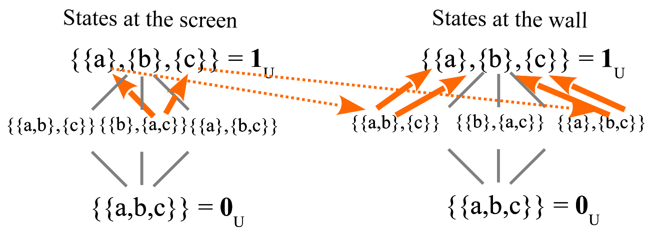

Fig. 12 Two von Neumann processes in Case 1

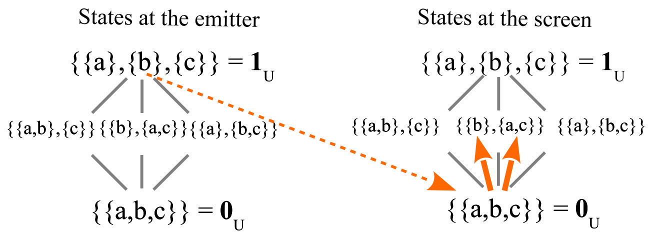

Fig. 13 Evolution from emitter to screen and state reduction at the screen

5 Linearization: Two-way street between sets and vector spaces

5.1 The Yoga of Linearization

There is an extensive two-way connection between set concepts and vector space concepts. It gives a translation dictionary between sets and vector spaces. In enumerative combinatorics, set concepts are correlated with the corresponding concepts in finite vector spaces of order $q = p^n$ (where p is a prime) so the vector space concept is called the "q-analog" [33]. But we are concerned with the dictionary relating set concepts to the corresponding vector space concepts in the Hilbert spaces of QM so we will call them "QM-analogs." This is an important tool to illustrate the basic ideas of QM by distilling them down to the corresponding (support) set concepts as we have seen with QM/Sets and the analysis of the two-slit experiment.

Yoga of Linearization
Given a basis set of a vector space, apply the set concept to the basis set and then what is linearly generated is the corresponding or QM-analog vector space concept.

We start by taking U to be a basis set of a finite-dimensional Hilbert space V, e.g., an orthonormal eigenbasis for an observable. Then the set notion of a subset $S \subseteq U$ linearly generates a subspace $[S] \subseteq V$. The cardinality $|S|$ of S equals the dimension $\dim([S])$ of $[S]$. A *real-valued numerical attribute on* U is a function $f : U \to \mathbb{R}$. This set concept generates a Hermitian operator $\widehat{f} : V \to V$ by the equation $\widehat{f} u_i = f(u_i) u_i$

(where we ignore the difference between set element u_i and basis vector $|u_i\rangle$). When appropriate, the QM-analog is indicated by the $\widehat{hat}$ on the set version as in $\widehat{f}$ and f. Each value r of $f : U \to \mathbb{R}$ is an eigenvalue of $\widehat{f}$, each subset $S \subseteq f^{-1}(r)$ is a constant set of f so the set version of the eigenvector-eigenvalue equation $\widehat{f}u_i = ru_i$ is $f \upharpoonright S = rS$, which means f restricted to S is a constant set of f with value r.

Each block B_j in a set partition π on U generates a subspace $[B_j]$ and the set of those subspaces $\widehat{\pi} = \{[B_j]\}_{B_j \in \pi}$ is a *direct sum decomposition* (DSD) of V as we would expect from the q-analog: "Direct-sum decompositions are a q-analog of partitions of a finite set." [6, 764]

A set partition $\pi = \{B_1, ..., B_m\}$ on U could have been defined as a set of non-empty subsets so that each non-empty subset $S \subseteq U$ is uniquely expressed as the union of subsets of the B_j's, i.e., $S = \cup_{B_j \cap S \neq \emptyset} B_j \cap S$. Similarly, a DSD can be defined as a set of non-trivial subspaces $\{V_j\}_{j=1}^{m}$ of V such that every non-zero vector in V is uniquely expressed as the sum of vectors from the V_j's. When the partition on U is $f^{-1} = \left\{f^{-1}(r)\right\}_{r \in f(U)}$, then the QM-analog DSD corresponding to f^{-1} is the DSD $\widehat{f^{-1}} = \left\{\left[f^{-1}(r)\right]\right\}_{r \in f(U)}$ of eigenspaces of the Hermitian operator $\widehat{f} : V \to V$. If f is just an attribute defined by a characteristic function $\chi_S : U \to \{0,1\}$, then the QM-analog operator $\widehat{\chi}_{[S]} u_i = \chi_S(u_i) u_i$ is the projection operator projecting to the space $[S]$. The spectral decomposition of $\widehat{f}$ is:

$$\widehat{f} = \sum_{r \in f(U)} r \widehat{\chi}_{[f^{-1}(r)]}.$$

Then we can work backwards to see that the set-version of the spectral decomposition for the numerical attribute f is obtained by "taking off the (operator) hats":

$$f = \sum_{r \in f(U)} r \chi_{f^{-1}(r)}.$$

An important example of the correlations starts with a number of numerical attributes $f, g, ..., h : U \to \mathbb{R}$ all defined on the same basis set U. Then the QM-analog Hermitian operators $\widehat{f}, \widehat{g}, ..., \widehat{h}$ are all *commuting* operators (since U provides a common basis of simultaneous eigenvectors). Moreover, if the join $f^{-1} \vee g^{-1} \vee ... \vee h^{-1} = \mathbf{1}_U$, i.e., has all blocks of cardinality one, then each element $u_i \in U$ is uniquely specified by the ordered-tuple of attribute values $(f(u_i), g(u_i), ..., h(u_i))$. We might say that numerical attributes defined on the same set are *compatible* and if their join is the discrete partition $\mathbf{1}_U$, then they are a Complete Set of Compatible Attributes, a CSCA.

Similarly, each inverse-image partition $f^{-1} = \left\{f^{-1}(r)\right\}_{r \in f(U)}$ defines the QM-analog DSD $\widehat{f^{-1}} = \left\{\left[f^{-1}(r)\right]\right\}_{r \in f(U)}$. The *join of DSDs of commuting operators* $\widehat{f^{-1}} \vee \widehat{g^{-1}} \vee ... \vee \widehat{h^{-1}}$ is defined as the DSD of non-zero subspaces obtained by the intersections of the eigenspaces of the operators. If the join $\widehat{f^{-1}} \vee \widehat{g^{-1}} \vee ... \vee \widehat{h^{-1}}$ of those DSDs is the DSD $\widehat{\mathbf{1}_U}$ of subspaces of cardinality one or rays (instead of blocks of cardinality one), then the set of operators is said to be *complete*, i.e., a Complete Set of Commuting Operators, a CSCO [11, 57]. And, as in the set case, each eigenvector u_i in the basis set of simultaneous eigenvectors U is then uniquely characterized by the ordered-tuple of eigenvalues $(f(u_i), g(u_i), ..., h(u_i))$. That set version of a CSCA shows the basic idea behind Dirac's CSCOs.

These correlations (using $\widehat{hat}$ notation) are summarized in Table 2.

Set math	QM-analogs
Subset $S \subseteq U = \{u_1, ..., u_n\}$	Subspace $[S] \subseteq V$
Numerical attribute $f : U \to \mathbb{R}$	Hermitian op. $\widehat{f} u_i = f(u_i) u_i$
Constant set $f \restriction S = rS$, $r \in f(U)$	Eigenvector $\widehat{f} v = rv$
Value r of f	Eigenvalue r of $\widehat{f}$
Set of constant r-sets $\wp(f^{-1}(r))$	Eigenspace of r, $[f^{-1}(r)]$
Partition: $f^{-1} = \{f^{-1}(r)\}_{r \in f(U)}$	Add $\widehat{hat}$ for DSD: $\widehat{f^{-1}} = \{[f^{-1}(r)]\}_{r \in f(U)}$
χ-function $\chi_{f^{-1}(r)} : U \to \{0,1\}$	Projection op. $\widehat{\chi}_{f^{-1}(r)} u_i = \chi_{f^{-1}(r)}(u_i) u_i$
Spectral decomp. $f = \sum_{r \in f(U)} r \chi_{f^{-1}(r)}$	Add $\widehat{hats}$: $\widehat{f} = \sum_{r \in f(U)} r \widehat{\chi}_{f^{-1}(r)}$
Num. attrib. $f, g, ..., h : U \to \mathbb{R}$ same U	Add $\widehat{hats}$ for commuting op.: $\widehat{f}, \widehat{g}, ..., \widehat{h}$
$U =$ same domain of $f, g, ..., h$	$U =$ Simultaneous eigenvectors $\widehat{f}, \widehat{g}, ..., \widehat{h}$
$f^{-1} \vee g^{-1} = \{f^{-1}(r) \cap g^{-1}(s) \neq \emptyset\}$	$\widehat{f^{-1}} \vee \widehat{g^{-1}} = \{[f^{-1}(r)] \cap [g^{-1}(s)] \neq \{0\}\}$
$f^{-1} \vee g^{-1} \vee ... \vee h^{-1} = \mathbf{1}_U$ $u_i \leftrightarrow (f(u_i), g(u_i), ..., h(u_i))$ Set version CSCA	$\widehat{f^{-1}} \vee \widehat{g^{-1}} \vee ... \vee \widehat{h^{-1}} = \widehat{\mathbf{1}_U}$ $u_i \leftrightarrow (f(u_i), g(u_i), ..., h(u_i))$ Dirac's CSCO

Table 2: Correlations between set concepts and QM-analog concepts

5.2 Non-commutativity

In the early days of QM, the non-commutativity of observables seemed like a key characteristic of QM as opposed to classical mechanics, e.g., Dirac's q-numbers (linear operators) versus the classical c-numbers [12]. But this seems to put the emphasis in the wrong place. After all, the non-commutativity of matrix multiplication is a feature of vector spaces *per se* so the emphasis should be put on the quantum states forming a vector space in the first place–which puts the emphasis back on superposition rather than non-commutativity.

The vector space version of a set partition is a direct sum decomposition or DSD. What is usually taken as "quantum logic" is the logic of (closed) subspaces of a Hilbert space [7]. Since subspaces are the vector space version of subsets, that quantum logic of subspaces is the quantum version of the Boolean logic of subsets. Since partitions are category-theoretically dual to subsets and since DSDs are the vector space version of partitions, there is another dual *quantum logic of direct sum decompositions* [16], which could also be viewed as the quantum logic of observables (since the observable differs from its DSD of eigenspaces by including the eigenvalues associated with the eigenspaces–just as a numerical attribute $f : U \to \mathbb{R}$ differs from the partition $\{f^{-1}(r)\}_{r \in f(U)}$ only by including the numbers assigned to the blocks).

In forming the join of two partitions on the same U, we take the blocks of the join to be the non-empty intersections of the blocks from the partitions. But given two partitions on different universe sets U and U', the intersection is completely undefined.

But that changes in the vector space version of partitions, DSDs. A DSD is a basis-free notion. It is a set of subspaces and subspaces always have intersections as subspaces. That is the basic idea that creates the new possibilities of commuting DSDs, non-commuting DSDs, and even conjugate DSDs.

Let $\{V_j\}_{j=1}^{m}$ and $\{W_{j'}\}_{j'=1}^{m'}$ be two DSDs of a vector space V. We take the set of non-zero intersections $V_j \cap W_{j'} \neq \{0\}$ in a join-like operation. If the DSDs were the DSDs of eigenspace DSDs of two observables $F, G : V \to V$, then the non-zero vectors in the blocks would be simultaneous eigenvectors for the two observables. But the big difference is that the vectors in those blocks need not span the whole space. The join-like operation is only a "proto-join."

Let $\mathcal{SE}$ be the vector space spanned by the vectors in those non-zero intersections $V_j \cap W_{j'}$. If those two DSDs were the eigenspace DSDs of two observable operators F and G, then it is a theorem [21, 68] that: $\mathcal{SE}$ is the kernel (i.e., subspace of elements mapped to zero) of the commutator $[F, G] = FG - GF$ of the operators. Now F and G *commute* if the commutator is the zero operator, i.e., if its kernel $\mathcal{SE}$ is the whole space V in which case the proto-join of the two DSDs is a proper join, a join we have already seen in the analysis of Dirac's CSCOs. If $\mathcal{SE} \neq V$, then the two DSDs are *non-commuting* and if $\mathcal{SE} = \{0\}$, then the DSDs are *conjugate*. In the transition from an observable operator to its eigenspace DSD, the information that is lost is the distinct eigenvalues associated with eigenspaces. But that information is irrelevant for the definitions of commuting, not commuting, and conjugate so those are, as we have seen, properties of the DSDs.

This distinction between linear operators and DSDs is more pronounced when we move to other vector spaces. In the spaces over $\mathbb{Z}_2$, the only linear operators are projection operators but far more general are the DSDs $\left\{\left[f^{-1}(r)\right]\right\}_{r \in f(U)}$ in $\mathbb{Z}_2^{|U|}$ resulting from numerical attributes $f : U \to \mathbb{R}$.

Commutativity. To illustrate this analysis, let $U = \{u_1, ..., u_n\}$ in QM/Sets. Then any two numerical attributes $f, g : U \to \mathbb{R}$ defined on that same U with have inverse-image partitions $f^{-1} = \left\{f^{-1}(r)\right\}_{r \in f(U)}$ and $g^{-1} = \left\{g^{-1}(s)\right\}_{s \in g(U)}$ on U and the blocks of the two partitions will generate two DSDs on $\mathbb{Z}_2^n \cong \wp(U)$. The subspaces in the join-like operation on those two DSDs will be the subspaces generated by the blocks of $f^{-1} \vee g^{-1}$ which contain all basis elements $\{u_1\}, ..., \{u_n\}$ of $\wp(U)$ so the proto-join of the DSDs spans the whole space and is thus a join of DSDs. That is an example of commuting DSDs.

Non-commutativity. To consider an example of non-commutativity, consider the U-basis and the U'-basis of $\mathbb{Z}_2^3$ given in Table 1 and used in the two-slit example. Consider the numerical attribute $f : U = \{a, b, c\} \to \mathbb{R}$ of $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$. On the U'-basis, consider the numerical attribute $g : U' = \{a', b', c'\} \to \mathbb{R}$ where $g(a') = 1 = g(b')$ and $g(c') = 2$. Then the DSD determined by f is the set of three subspaces: $\{\{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{c\}\}\}$. The DSD determined by g is the set of two subspaces: $\{\{\emptyset, \{a'\}, \{b'\}, \{a', b'\}\}, \{\emptyset, \{c'\}\}\}$. We may then express the g-DSD in the computational U-basis as: $\{\{\emptyset, \{a, b\}, \{a, b, c\}, \{c\}\}, \{\emptyset, \{b, c\}\}\}$. Then when we take the join-like operation or proto-join by taking all the intersections of subspaces, then many are just the zero space such as $\{\emptyset, \{a\}\} \cap \{\emptyset, \{a, b\}, \{a, b, c\}, \{c\}\} = \{\emptyset\}$, but only one intersection is non-trivial: $\{\emptyset, \{c\}\} \cap \{\emptyset, \{a, b\}, \{a, b, c\}, \{c\}\} = \{\emptyset, \{c\}\} = \mathcal{SE}$.

However, the vectors in this subspace hardly span the whole space so those two DSDs are non-commuting but not conjugate.

Conjugacy. The vector spaces $\mathbb{Z}_2^{2m}$ for $m > 1$ have a special structure much like the Fourier transformation between conjugate variables in the full math of QM. The simplest such space for $m = 2$ is $\mathbb{Z}_2^4 \cong \wp(U)$ for $U = \{a, b, c, d\}$. From the U-basis, we canonically construct the $\hat{U}$-basis for $\hat{U} = \left\{\hat{a}, \hat{b}, \hat{c}, \hat{d}\right\}$ where the circumflex operation (unrelated to our previous $\widehat{hat}$ notation) just leaves out the element under the circumflex. Thus $\{\hat{a}\} = \{b, c, d\}$, $\left\{\hat{b}\right\} = \{a, c, d\}$, $\{\hat{c}\} = \{a, b, c\}$. and $\left\{\hat{d}\right\} = \{a, b, c\}$. And, as in the Fourier transformation, the reverse operation on the $\hat{U}$-basis gives back the U-basis. Thus $\left\{\hat{b}, \hat{c}, \hat{d}\right\} = \{a, c, d\} + \{a, b, d\} + \{a, b, c\} = \{a\}$ since all the elements but a occurs an even number of times so they cancel out in the addition mod(2) while a occurs an odd number of times. And it works similarly for the other elements, e.g., $\{b\} = \left\{\hat{a}, \hat{c}, \hat{d}\right\}$ and so forth. For $\mathbb{Z}_2^{2m}$ of even dimension, the circumflex-vectors form a basis but not for vector spaces over $\mathbb{Z}_2$ of odd dimension. And for $U = \{a, b\}$, $\{\hat{a}\} = \{b\}$ and $\left\{\hat{b}\right\} = \{a\}$, so $\hat{U}$-basis in that case is the same as the U-basis. That is why the Fourier-like transform is for vector spaces over $\mathbb{Z}_2$ of even dimension greater than two.

In QM, the Fourier transformation gives conjugate bases which gives the conjugacy between the quantum variables such as position and momentum [10, Sec. 4.1.2]. In our example of $\mathbb{Z}_2^{2m}$ $(m > 1)$, any U-basis has a conjugate $\hat{U}$-basis. As in the quantum case, let us assign different values to the different basis vectors so the DSD coming from the U-basis is: $\{\{\emptyset, \{u_1\}\}, ..., \{\emptyset, \{u_{2m}\}\}\}$ and from the $\hat{U}$-basis, the DSD is $\{\{\emptyset, \{\hat{u}_1\}\}, ..., \{\emptyset, \{\hat{u}_{2m}\}\}\}$. When the DSD from the $\hat{U}$-basis is expressed in the computational U-basis, then it is clear that all the intersections of the subspaces from the two DSDs are the zero space $\{\emptyset\}$ so those DSDs are conjugate.

6 The basic idea in state reduction ("measurement")

6.1 State reduction = Superposition^{-1}

Superposition adds together definite (eigen) states to give an indefinite state. State reduction results from an interaction that makes distinctions between the indefinite states of a superposition. Thus state reduction "undoes" what superposition "does." The state reduction takes place wherever such an interaction occurs which is almost certainly at the quantum level. Then it must be detected and amplified to the human level to form a 'measurement' in the anthropomorphic sense.

In the iceberg/partition-lattice representations using support sets, state reduction is indicated by upward arrows taking an indefinite state to a more definite state by making distinctions. Support sets get smaller or remain the same. When superposition is misinterpreted as being like classical wave superposition (definite + definite = definite), then it is indeed unclear what state reduction is. But when superposition is interpreted as making an indefinite state out of definite states (definite + definite = indefinite), then it is easy to see what state reduction does; it makes distinctions by partly or wholly distinguishing the states in the superposition state. And distinguishing or not distinguishing between the superposed alternative states is precisely the content of the Feynman rules.

> If you could, in principle, distinguish the alternative final states (even though you do not bother to do so), the total, final probability is obtained by calculating the probability for each state (not the amplitude) and then adding them together. If you cannot distinguish the final states even in principle, then the probability amplitudes must be summed before taking the absolute square to find the actual probability. [29, Sec. 3.16]

The distinctions or distinguishings between the alternatives has nothing to do with human observations.

> In other words, the superposition of amplitudes ... is only valid if there is no way to know, even in principle, which path the particle took. It is important to realize that this does not imply that an observer actually takes note of what happens. It is sufficient to destroy the interference pattern, if the path information is accessible in principle from the experiment or even if it is dispersed in the environment and beyond any technical possibility to be recovered, but in principle still "out there." The absence of any such information is the essential criterion for quantum interference to appear. [63, 484]

The "absence of any such information" means the absence of distinctions or distinguishings as in the notion of information-as-distinctions [18].

In his textbooks, Feynman [29, Sec. 3.3] always gave examples at the quantum level of the two cases: distinguishing or not between the superposed alternative paths. Consider a neutron that is scattering off the nuclei of atoms in a crystal. If there is no distinguishing which nuclei was scattered off of, e.g., the nuclei have no spin, then the amplitude for the neutron to be scattered to some point would be the addition of the scattering amplitudes off the various nuclei. Since there is no distinguishing physical event to distinguish between scattering off one nucleus or another, there is no state reduction in the superposition of the states of scattering off different nuclei so the amplitudes add.

But if all the nuclei had spin in, say, the down direction while the neutron had spin up, then in the scattering interaction, one of the nuclei might flip its spin which would be the quantum level physical event to distinguish that trajectory. Then the probability of the neutron arriving at the given point with its spin reversed (indicating that a spin flip had occurred) would be the sum of the probabilities (not the amplitudes) for those distinguished trajectories over all the nuclei. In that case, the superposition was reduced (the indefinite became definite) and the nucleus with its spin flipped plays the role of a detector registering a hit. The spin-state of the nuclei served as a quantum-level measuring apparatus to distinguish ("measure") which scattering trajectory was taken by the neutron to reach the detector. No macroscopic apparatus was involved in the state reduction (unlike in the 'decoherence' analysis [64]).

6.2 Illustrating state reduction with Weyl's "pasta machine" and Feynman's rules

Hermann Weyl approvingly quoted [60, 255] Arthur Eddington who said that a relativity theorist carries a measuring rod while a quantum theorist carries a sieve–which Weyl called a "grating." Weyl started with a numerical attribute, e.g., $f : U \to \mathbb{R}$, which defined an inverse-image partition or " grating" or "aggregate [which] is used in the sense of 'set of elements with equivalence relation." [60, 239]. Then he, in effect, used the yoga of linearization so an "aggregate of n states has to be replaced by an n-dimensional Euclidean vector space" [60, 256] in QM. The notion of a vector space partition or "grating" in QM is a "splitting of the total vector space into mutually orthogonal subspaces" so that "each vector $\vec{x}$ splits into r component vectors lying in the several subspaces" [60, 256], i.e., a direct-sum decomposition of the space. After referring to a partition and its vector space counterpart, a DSD, as a sieve or grating, Weyl says that "Measurement means application of a sieve or grating" [60, 259], i.e., the making of distinctions according to which hole in the grating the particle went through.

Weyl's imagery can be illustrated with a "pasta machine" where a ball of pasta (a quantum particle in a superposition state) has the interaction of going through different holes with various shapes. The pasta ball can be thought as the indefinite superposition of the distinct pasta shapes. The two cases of Feynman's rule are illustrated in Figure 14. The left side is the case of the interaction with the pasta grating distinguishing between the different shapes superposed in the pasta ball.

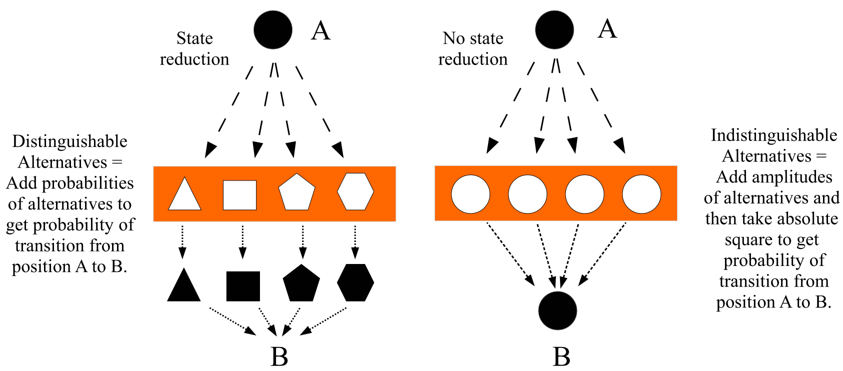

Fig. 14 Two cases in Feynman's rule illustrated with Weyl's pasta-machine grating imagery

On the right side is the null grating that makes no distinctions between the alternative paths from A to B so the amplitudes add and the absolute square gives the probability of the pasta-ball-particle going from A to B. The pasta-machine imagery gives the basic idea in Feynman's rule where distinguishable alternative paths implies state reductions and indistinguishable paths describes unitary evolution by adding the path amplitudes to get from A to B (in Feynman's path integral formulation of QM [28]).

The distinguishings have nothing to do with humans. The distinguishings are the making of distinctions. It has long been suspected that " information" has a fundamental role in QM. This analysis of state reduction and the definition of information as the quantitative measure of distinctions, called "logical entropy" ([45], [18], [24]), verifies that idea and illustrates it with upward arrows in the iceberg/partition-lattice diagrams.

7 The basic idea behind fermions and bosons

Leibniz and Kant both spelled out a basic idea of classical metaphysics. For Leibniz, it was the Principle of Identity of Indistinguishables (PII) [4, 22]. If objects are fully definite, then two distinct objects must have some attribute that one has but not the other. Given two allegedly different objects, by going down far enough, there must be a distinguishing attribute, otherwise they are identical. Kant expressed the same idea as the Principle of Complete Determination.

> Every thing, however, as to its possibility, further stands under the principle of thoroughgoing determination; according to which, among all possible predicates of things, insofar as they are compared with their opposites, one must apply to it. [37, B600]

In modern terms, classical reality was "definite all the way down" (to paraphrase the joke about "turtles all the way down"). In the partition lattice, the discrete partition $\mathbf{1}_U$ represents the classical world so it satisfies the *partition version of Leibniz's PII*:

$$\text{If } u \text{ and } u' \text{ are indistinguishable by } \mathbf{1}_U\text{, i.e., } (u,u') \in \operatorname{indit}(\mathbf{1}_U)\text{, then } u = u'.$$

That is only true for the classical-level partition $\mathbf{1}_U$; all other partitions contain at least one block with two or more elements which is the support set of a superposition state and thus non-classical. However, Figure 5 emphasizes that in the Boolean lattice of subsets, there is definiteness all the way down; the elements a, b, and c are always fully definite. In contrast, for the partition lattice, full classical definiteness exists only at the top level in the discrete partition $\mathbf{1}_U$.

Quantum reality is different from classical reality; it is not definite all the way down.

> In quantum mechanics, however, identical particles are truly indistinguishable. This is because we cannot specify more than a complete set of commuting observables for each of the particles; in particular, we cannot label the particle by coloring it blue. [51, 446]

Since quantum reality is not definite all the way down, this creates the possibility of two types of particles:

1. Fermions: the type where the existing level of definiteness was sufficient to uniquely determine the particle, and
2. Bosons: the type of particle where that limited level of definiteness is insufficient to uniquely determine the particle so there could be many particles of that type with the same complete description.

That is the basic idea behind the two types of particles. That basic idea can be modeled with symmetric and anti-symmetric wave functions, but we are concerned with the basic idea. There is a sophisticated theorem, the spin-statistics theorem in quantum field theory [55], that relates the two types of particles to spin, but our purpose is again to give the *basic idea* behind having two types of particles in a reality that is not definite all the way down.

Leibniz's PII says that a complete description uniquely determines an entity. The Pauli Exclusion Principle says that a complete CSCO description uniquely determines a fermion. Weyl emphasizes that the Pauli Principle is just the application of the Leibniz PII in a reality that is not definite all the way down.

> The upshot of it all is that the electrons satisfy Leibniz's *principium identitatis indiscernibilium*, or that the electronic gas is a "monomial aggregate" (Fermi-Dirac statistics). ... As to the Leibniz-Pauli exclusion principle, it is found to hold for electrons but not for photons. [60, 247]

For a metaphor, consider postal package addresses that were only definite down to the street number, i.e., country, state, city, postal code, and street number. In a neighborhood zoned for single family dwellings, i.e., a " fermionic" neighborhood, the street-number would have a single family or it would be a vacant lot. In a neighborhood zoned for multifamily dwellings such as apartment houses, i.e., a " bosonic" neighborhood, the street-number address would be insufficient to determine the recipient. There could be many recipients fitting that same street-number address. That difference is the simple result of the limited addresses. Within the mathematical machinery of QM, the difference is between anti-symmetric and symmetric wave functions and between half-integer spin and integer spin, but our goal was to give the basic idea behind that difference, i.e., in quantum state descriptions not being "definite all the way down."

The schematic argument that a complete state description uniquely determines a particle leads to an antisymmetric state vector for a system of indistinguishable fermions is summarized as follows as illustrated in Figure 15 There are two numer-

Fig. 15 Setup for argument that fermion systems have antisymmtric state vectors

ically distinct but indistinguishable fermions, e.g., electrons, 1 and 2, and there are two ways they could go to states a or c. The amplitude $1 \to a$ and $2 \to c$ is $\langle a|1\rangle \langle c|2\rangle$. If the particles were permuted, then the amplitude is $Perm_{1,2} \langle a|1\rangle \langle c|2\rangle$. Since those two arrangements are indistinguishable, the amplitude for that end state is the sum of the two amplitudes: $\langle a|1\rangle \langle c|2\rangle + Perm_{1,2} \langle a|1\rangle \langle c|2\rangle$. But if we then try to put the two particles in the same end state b, then by assumption that is impossible, i.e., has amplitude 0, so $Perm_{1,2} \langle b|1\rangle \langle b|2\rangle = -\langle b|1\rangle \langle b|2\rangle$, and thus in general $Perm_{1,2} \langle a|1\rangle \langle c|2\rangle = -\langle a|1\rangle \langle c|2\rangle$. Those are the basic ideas behind fermions, as opposed to bosons, and the Pauli exclusion principle.

8 Concluding remarks

The aim of this paper has been to try to explain in intuitive (*anschaulich*) terms the basic ideas behind the more puzzling aspects of quantum mechanics. In summary, here are some of those basic ideas.

1. The *basic idea* of superposition:
   (a) as being the flip-side of abstraction–the combination of entities with some similarities and some differences by abstracting away from the differences (making them indefinite) and being definite only on the similarities,
   (b) as being the quantum notion (definite + definite = indefinite) unlike the classical notion of superposition (definite + definite = definite), and
   (c) as being the key feature of quantum states responsible for them forming a vector space.
2. The *basic idea* behind quantum amplitudes and the Born rule is shown by the simple extension of finite probability theory by adding superposition events ΣS in addition to the usual discrete events S. The $n \times n$ matrix representation of superposition events (unlike classical discrete events) has a vector "square root" of quantum amplitudes whose square gives the Born rule probabilities for the outcomes of the superposition event.
3. The *basic idea* of constructing a pedagogical or toy model of QM, QM/Sets, by simplifying quantum pure state vectors down to their support sets in a vector space over $\mathbb{Z}_2$.
4. The *basic idea* of constructing a partition lattice based on a pure state whose bottom or indiscrete partition has the only block as the support set of the pure state, whose top or discrete partition is the support of the corresponding completely decomposed mixed state, and where the intermediate partitions consists of the support sets of the mixed states resulting from projective measurements of the pure state.
5. The *basic idea* of a projective measurement resulting in a mixed state given by the Lüders mixture operation is shown to be the join operation in the partition lattice of support sets.
6. The *basic idea* of a vN Type I process as making distinctions and a vN Type II processes as not making distinctions so the measure of the indistinctness and distinctness of two quantum states, i.e., the inner product, is preserved.
7. The *basic idea* of the partition lattice of a pure state allows intuitive pictures of:

(a) The iceberg/partition-lattice picture of the division between the fully definite classical states in the discrete partition at the top and the indefinite states of the quantum world represented by the rest of the lattice below the top;
(b) Upward moves in the lattice correspond to projective measurements (Lüders mixture operations) of making distinctions;
(c) Non-upward (horizontal or downward) non-singular moves in the lattice correspond to unitary evolution (not making distinctions); and
(d) The classical world emerges from the quantum world by the making of distinctions.

8. The *basic ideas* of the two-slit experiment as shown by its formulation in QM/Sets, e.g., how to answer the question: "With no detection at the slits, how does the particle get from the two-slit screen to the detection wall on the other side of the screen without going through slit 1 or slit 2?". The table-top ripple-tank demonstration of Case 2 (no detection at the slits) or evocation of the wave-particle complementary magic are attempts to use the (misleading) *classical* (definite + definite = definite) notion of superposition of waves instead of the (below-the-water) same-math evolution of the quantum (definite + definite = indefinite) superposition of states.
9. The *basic idea* of linearization to establish a dictionary relating concepts of sets math, e.g., real-valued numerical attributes f or partitions f^{-1}, and the corresponding Hilbert space QM-analogs, e.g., Hermitian operators $\widehat{f}$ or direct sum decompositions $\widehat{f^{-1}}$.
10. The *basic idea* of non-commutativity of operators in QM was analyzed showing that this was more an aspect of matrix math in vector spaces so it is not a unique characteristic of QM and this was illustrated by giving a case of conjugacy arising from Fourier-like transforms in QM/Sets for $\mathbb{Z}_2^{2m}$ with $m > 1$.
11. The *basic idea* of state reduction (" measurement") as being the inverse of superposition where superposition arises by making indefinite the differences between eigenstates and state reduction results from an interaction that distinguishes between the superposed states (or paths)–where the two vN cases of a distinguishing and a non-distinguishing interaction were intuitively illustrated by Weyl's "pasta machine" and the Feynman rules.
12. The *basic idea* at the logical level is distinctions versus indistinctions, differences versus similarities, distinguishings versus indistinguishings–all represented at the logical level in the logic of partitions [19].
13. The *basic idea* of information-as-distinctions is the quantitative version of the logic of partitions [18]. Since indistinctions (i.e., superpositions) and distinctions (state reduction = superposition^{-1}) have an ontic role in QM, this verifies the old idea that the quantitative version of distinctions, i.e., information, has an ontic role in QM.
14. The *basic idea* of the two types of particles, fermions and bosons, arises from the contrast between classical reality as being definite all the way down and quantum reality as being definite only down to a certain level, i.e., as given by a complete set of commuting observables (CSCO), so some particles will be uniquely identified by that limited degree of definiteness (fermions) and other numerically distinct particles can all have the same limited state description (bosons).

In the full Hilbert space machinery of QM, all these 'gears' mesh together beautifully to make our most highly verified physical theory. In that sense, the theory is not the problem; the problem is how to intuitively conceptualize the underlying physical reality. Our approach to understanding that underlying reality has been to break down the machinery into basic ideas that can be understood in an intuitive manner.

Statements and Declarations

- Funding: None.
- Conflict of interest/Competing interests: None.

References

[1] Aerts, Diederik. 2010. A Potentiality and Conceptuality Interpretation of Quantum Physics. *Philosophica* 83: 15–52.

[2] Aerts, Diederik, and Bart D'Hooghe. 2012. Potentiality States: Quantum versus Classical Emergence. *ArXiv.Org*. https://arxiv.org/pdf/1212.0104.

[3] Ainsworth, Thomas. 2020. Form vs. Matter. In *The Stanford Encyclopedia of Philosophy (Summer 2020 Edition)*, edited by Edward N. Zalta. https://plato.stanford.edu/archives/sum2020/entries/form-matter/.

[4] Ariew, Roger, ed. 2000. *G. W. Leibniz and Samuel Clarke: Correspondence.* Hackett.

[5] Auletta, Gennaro, Mauro Fortunato, and Giorgio Parisi. 2009. *Quantum Mechanics.* Cambridge University Press.

[6] Bender, Edward A., and Jay R. Goldman. 1971. Enumerative Uses of Generating Functions. *Indiana University Mathematics Journal* 20 (8): 753–65.

[7] Birkhoff, Garrett, and John Von Neumann. 1936. The Logic of Quantum Mechanics. *Annals of Mathematics* 37 (4): 823–43.

[8] Chiatti, Leonardo. 2022. A Neo-Aristotelic (Maybe) Approach to Quantum Reality. *Biocosmology - Neo-Aristotelism* 12 (172): 309–25.

[9] Coopersmith, Jennifer. 2017. *The Lazy Universe: An Introduction to the Principle of Least Action.* Oxford University Press.

[10] De Haro, Sebastian, and Jeremy Butterfield. 2025. *The Philosophy and Physics of Duality.* Oxford University Press.

[11] Dirac, Paul A. M. 1958. *The Principles of Quantum Mechanics (4th ed.).* Clarendon.

[12] Dirac, Paul A. M. 1973. Development of the Physicist's Conception of Nature. In *The Physicist's Conception of Nature*, edited by J. Mehra, 1–15. Reidel.

[13] Dürr, Detlef, Sheldon Goldstein, and Nino Zangi. 2013. *Quantum Physics without Quantum Philosophy.* Springer.

[14] Ellerman, David 2010. The Logic of Partitions: Introduction to the Dual of the Logic of Subsets. *Review of Symbolic Logic.* 3 (2 June): 287-350.

[15] Ellerman, David. 2017. Quantum Mechanics over Sets: A Pedagogical Model with Non-Commutative finite Probability Theory as Its Quantum Probability Calculus. *Synthese* 194 (12): 4863–96.

[16] Ellerman, David. 2018. The Quantum Logic of Direct-Sum Decompositions: The Dual to the Quantum Logic of Subspaces. *Logic Journal of the IGPL* 26 (1 (January)): 1–13. https://doi.org/10.1093/jigpal/jzx026.

[17] Ellerman, David. 2021. On Abstraction in Mathematics and Indefiniteness in Quantum Mechanics. *Journal of Philosophical Logic.* https://doi.org/10.1007/s10992-020-09586-1.

[18] Ellerman, David. 2021. *New Foundations for Information Theory: Logical Entropy and Shannon Entropy.* SpringerNature.

[19] Ellerman, David. 2023. *The Logic of Partitions: With Two Major Applications. Studies in Logic 101.* College Publications, 2023. https://www.collegepublications.co.uk/logic/?00052.

[20] Ellerman, David. 2024. A Fundamental Duality in the Exact Sciences: The Application to Quantum Mechanics. *Foundations* 4 (2): 175–204. https://doi.org/10.3390/foundations4020013.

[21] Ellerman, David. 2024. *Partitions, Objective Indefiniteness, and Quantum Reality: The Objective Indefiniteness Interpretation of Quantum Mechanics.* Springer Nature. https://doi.org/10.1007/978-3-031-61786-7_2.

[22] Ellerman, David. 2024. A New Approach to Understanding Quantum Mechanics: Illustrated Using a Pedagogical Model over $\mathbb{Z}_2$. *AppliedMath* 4 (2): 468–94. https://doi.org/10.3390/appliedmath4020025.

[23] Ellerman, David. 2025. Born Again! The Born Rule as a Feature of Superposition. *International Journal of Quantum Foundations* 11: 57–72. https://ijqf.org/archives/7372.

[24] Ellerman, David. 2025. A New Logical Measure for Quantum Information. *Quantum Information and Computation* 25:81–95. https://doi.org/10.2478/qic-2024-0000.

[25] Faye, Jan. 2024. Copenhagen Interpretation of Quantum Mechanics. In *Stanford Encyclopedia of Philosophy.* 2024. http://plato.stanford.edu/entries/qm-copenhagen/.

[26] Feynman, Richard P. 1951. The Concept of Probability in Quantum Mechanics. In *Second Berkeley Symposium on Mathematical Statistics and Probability*, 533–41. University of California Press. https://projecteuclid.org/euclid.bsmsp/1200500252.

[27] Feynman, Richard P., Robert B. Leighton, and Matthew Sands. 1963. *The Feynman Lectures on Physics: Mainly Mechanics, Radiation, and Heat Vol. I.* Addison-Wesley.

[28] Feynman, Richard P., and Albert R. Hibbs. 1965. *Quantum Mechanics and Path Integrals*. McGraw-Hill.

[29] Feynman, Richard P., Robert B. Leighton, and Matthew Sands. 2010. *The Feynman Lectures on Physics: Quantum Mechanics Vol. III (New Millennium Ed.)*. Addison-Wesley.

[30] Furry, Wendell H. 1966. Some Aspects of the Quantum Theory of Measurement. In *Lectures in Theoretical Physics Vol. 8A*, edited by Wesley E. Brittin, 1–64. University of Colorado Press.

[31] Ghirardi, Gian Carlo, A. Rimini, and T. Weber. 1986. Unified Dynamics for Microscopic and Macroscopic Systems. *Physical Review D* 34:470–91.

[32] Gleason, A. M. 1957. Measures on the Closed Subspaces of a Hilbert Space. *Journal of Mathematics and Mechanics* 6 (6): 885–93.

[33] Goldman, Jay R., and Gian-Carlo Rota. 1970. On the Foundations of Combinatorial Theory IV: Finite Vector Spaces and Eulerian Generating Functions. *Studies in Applied Mathematics* XLIX (3 Sept.): 239–58.

[34] Heisenberg, Werner 1958. *Physics & Philosophy: The Revolution in Modern Science*. Harper Torchbooks.

[35] Hughes, R. I. G. 1989. *The Structure and Interpretation of Quantum Mechanics.* Harvard University Press.

[36] Jaeger, Gregg. 2014. *Quantum Objects: Non-Local Correlation, Causality and Objective Indefiniteness in the Quantum World*. Springer.

[37] Kant, Immanuel. 1998. *Critique of Pure Reason. The Cambridge Edition of the Works of Immanuel Kant*. Translated by Paul Guyer and Allen W. Wood. Cambridge University Press.

[38] Kastner, Ruth E. 2013. *The Transactional Interpretation of Quantum Mechanics: The Reality of Possibility*. Cambridge University Press.

[39] Kastner, Ruth E. 2019. *Adventures in Quantumland: Exploring Our Unseen Reality*. World Scientific.

[40] Kung, Joseph P. S., Gian-Carlo Rota and Catherine H. Yan 2009. *Combinatorics: The Rota Way*. Cambridge University Press.

[41] Lanczos, Cornelius. 1949. *The Variational Principles of Mechanics*. University of Toronto Press.

[42] Lawvere, F. William, and Robert Rosebrugh. 2003. *Sets for Mathematics*. Cambridge University Press.

[43] Llowarch, W. 1961. *Ripple Tank Studies of Wave Motion*. Oxford University Press.

[44] Lüders, Gerhart. 1951. Über Die Zustandsänderung Durch Meßprozeß. *Annalen Der Physik* 8 (6): 322–28.

[45] Manfredi, Giovanni, ed. 2022. ?Logical Entropy – Special Issue.? *4Open*, no. 5: E1. https://doi.org/10.1051/fopen/2022005.

[46] Margenau, Henry. 1954. Advantages and Disadvantages of Various Interpretations of the Quantum Theory. *Physics Today* 7 (10): 6–13. http://dx.doi.org/10.1063/1.3061432.

[47] Maudlin, Tim. 2019. *Philosophy of Physics: Quantum Theory*. Princeton University Press.

[48] Norsen, Travis. 2017. *Foundations of Quantum Mechanics*. Cham, Switzerland: Springer International.

[49] Rota, Gian-Carlo. 1997. *Indiscrete Thoughts*. Birkhäuser.

[50] Rota, Gian-Carlo. 2001. Twelve Problems in Probability No One Likes to Bring up. In *Algebraic Combinatorics and Computer Science: A Tribute to Gian-Carlo Rota*, edited by Henry Crapo and Domenico Senato, 57–93. Springer.

[51] Sakurai, J.J., and J. Napolitano. 2011. *Modern Quantum Mechanics. 2nd ed.* Addison-Wesley.

[52] Shimony, Abner. 1993. *Search for a Naturalistic Worldview. II Natural Science and Metaphysics*. Cambridge University Press.

[53] Shimony, Abner. 1999. Philosophical and Experimental Perspectives on Quantum Physics. In *Philosophical and Experimental Perspectives on Quantum Physics:*

Vienna Circle Institute Yearbook 7, 1–18. Springer Science+Business Media.

[54] Stone, M. H. 1932. On one-parameter unitary groups in Hilbert Space. *Annals of Mathematics.* 33 (3): 643–648.

[55] Streater, R. F., and A. S. Wightman. 2000. *PCT, Spin and Statistics, and All That.* Princeton University Press.

[56] Vaidman, Lev. 2020. Derivations of the Born Rule. In *Quantum, Probability, Logic*, edited by Meir Hemmo and Orly Shenker, 567–84. Cham, Switzerland: SpringerNature. https://doi.org/10.1007/978-3-030-34316-3.

[57] Kampen, N.G. van. 2008. The Scandal of Quantum Mechanics. *American Journal of Physics* 76: 989–90.

[58] von Neumann, John 1955. *Mathematical Foundations of Quantum Mechanics.* Robert T. Beyer trans., Princeton University Press.

[59] Wallace, David. 2012. *The Emergent Multiverse: Quantum Theory According to the Everett Interpretation.* Oxford University Press.

[60] Weyl, Hermann 1949. *Philosophy of Mathematics and Natural Science.* Princeton University Press.

[61] Weinberg, Steven. 2014. Quantum Mechanics without State Vectors. *Physical Review A* 90: 042102. https://doi.org/10.1103/PhysRevA.90.042102.

[62] Weinberg, Steven. 2015. *Lectures on Quantum Mechanics 2nd Ed.* Cambridge University Press.

[63] Zeilinger, Anton. 1999. Experiment and the Foundations of Quantum Physics. In *More Things in Heaven and Earth*, edited by B. Bederson, 482–98. Springer-Verlag.

[64] Zurek, Wojciech Hubert. 2003. Decoherence, Einselection, and the Quantum Origins of the Classical. *Review of Modern Physics* 75 (3 July): 715–75.